Upper bound on the Atiyah–Singer index from tadpole cancellation

Hajime Otsuka (Kyushu University)

Reference :

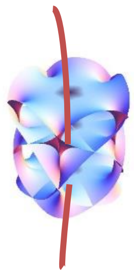
T. Kai, K. Ishiguro (KEK), S. Nishimura, H.O., M. Takeuchi (Kobe U), work in progress

String phenomenology 2023, July. 4th, 2023

Introduction — Atiyah–Singer index

- Huge number of 4D stable vacua (string landscape)

– $O(10^{500})$ Type IIB flux vacua *Ashok-Douglas ('04)*
– $O(10^{662})$ MSSM-like models in Heterotic on CYs *Constantin-He-Lukas ('18)*



- 6D compactification → Degrees of freedom
  --- fluxes (VEVs of gauge fields)
  --- branes (wrapping sub-manifolds)

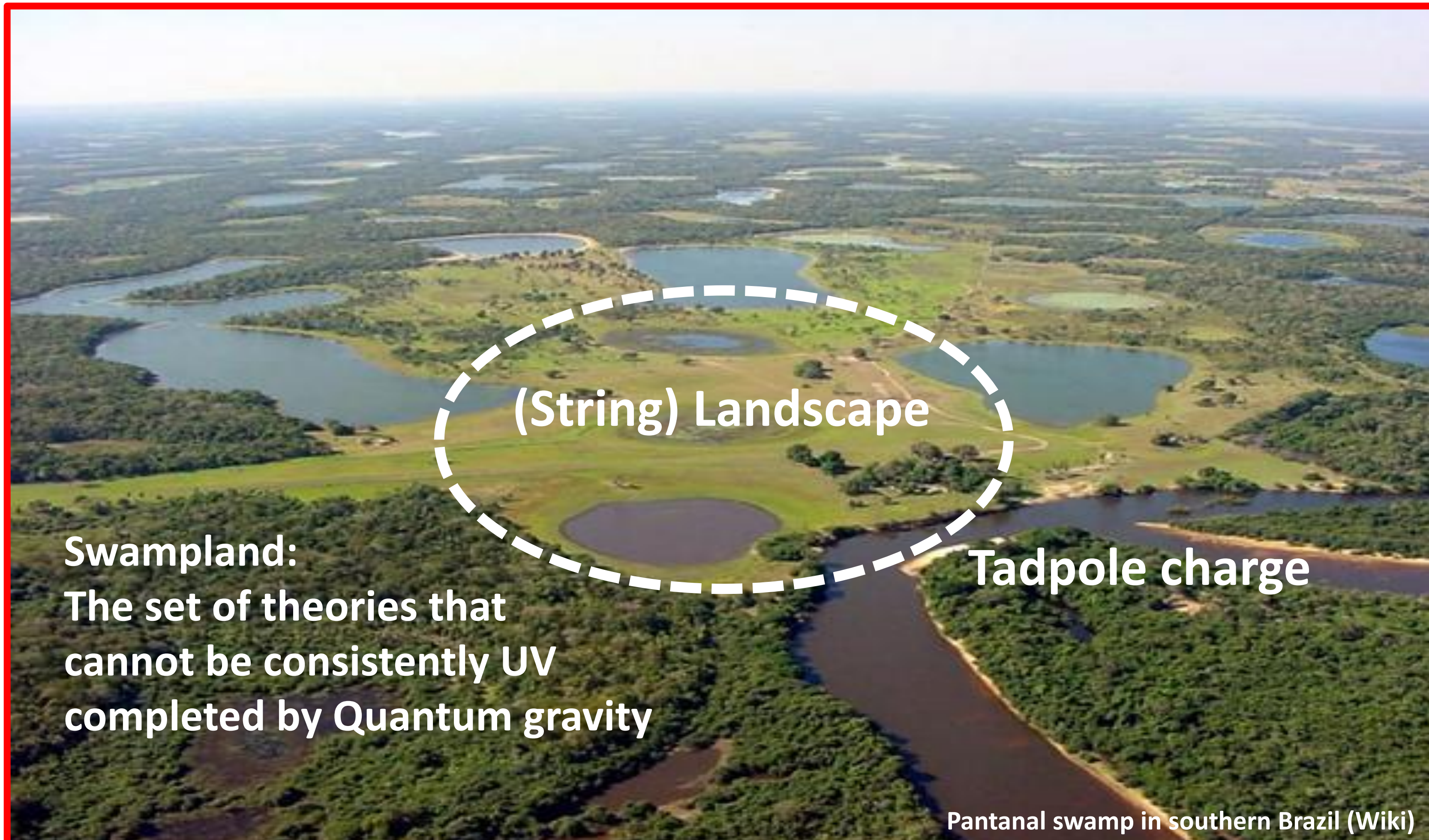
Question :

Generation number of quarks/leptons in the string landscape?

--- Counted by the Atiyah-Singer index : $\chi(M,V) = \int_M \mathrm{ch}(V) \wedge \mathrm{Td}(TM)$

Short summary

Tadpole cancellation condition
--- constrains the Atiyah-Singer Index



Outline

✓ **Introduction/Short summary**

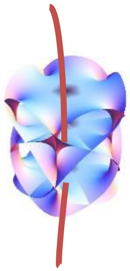
- **Heterotic string theory with line bundles**
  - SU(5) GUT in $E_8 \times E_8$ heterotic string
  - Pati-Salam in $SO(32)$ heterotic string
  - Direct flux breaking (hypercharge flux) in $SO(32)$ heterotic string
- **Conclusions and Discussions**

$E_8 \times E_8$ Heterotic Line Bundle Models on CY threefolds

Blumenhagen-Honecker-Weigand ('05), Anderson-Gray-Lukas-Palti ('11),....

- Multiple line bundles $V = \oplus_a L_a$ lead to semi-realistic SM spectra:

$$c_1(L_a) = \sum_{i=1}^{h^{1,1}} m_a^i w_i$$



-- Gauge group

$$E_8 \to SU(5) \times \Pi_a U(1)_a$$

-- Chiral zero-modes

$$248 \to \oplus_p (R_p, C_p)$$

-- Index of SU(5) matter multiplets

$$N_{\text{gen}} = -\text{ind}\,(V) = \frac{1}{2}\int_M c_3(V)$$

due to $c_1(V) = 0$ ("K-theory condition")

$E_8 \times E_8$ Heterotic Line Bundle Models on CY threefolds

- Upper bound on the Atiyah-Singer index:

$$|N_{\text{gen}}| = \left| \sum_{a,i} \frac{d_{ijk} m_a^i m_a^j m_a^k}{6} \right| \leq \frac{|m_{max}| \, ||c_2(V)||}{3} \leq \frac{|m_{max}| \, ||c_2(TX)||}{3}$$

$$|m_a^i| \leq |m_{max}|$$

Tadpole cancellation

$$c_2(V) \leq c_2(TX)$$

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- Upper bound on flux quanta m_a^i :

Constantin-Lukas-Mishra ('15)

$$0 < \sum_a m_a^i G_{ij} m_a^j = \frac{1}{\mathcal{V}} t^i c_{2i}(V) \leq \frac{1}{\mathcal{V}} ||\boldsymbol{t}| \times |c_2(TX)||$$

$$\tilde{G}_{ij} := \frac{\mathcal{V}}{||\boldsymbol{t}||} G_{ij}$$

G_{ij} : Moduli metric

$$0 < \sum_a m_a^i \tilde{G}_{ij} m_a^j \leq ||c_2(TX)||$$

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$$\text{Eigen}(\tilde{G}_{ij})|_{\min} = \lambda_{\min}$$

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--- using the Cauchy-Schwarz inequality with $\sum_{a=1}^{n} m_a^i = 0$

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--- using the Cauchy-Schwarz inequality with $\sum_{a=1}^{n} m_a^i = 0$

$$\sum_{i} (m_a^i)^2 \leq \frac{n-1}{n} \frac{||c_2(TX)||}{\lambda_{\min}}$$

n : # of U(1)

If $m_a^i \neq 0$ for $\forall a, i$

$$(m_{\max})^2 \leq \frac{n-1}{n} \frac{||c_2(TX)||}{\lambda_{\min}} - (h^{1,1} - 1)$$

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$c_2(V) \leq c_2(TX)$

- Upper bound on flux quanta m_a^i :

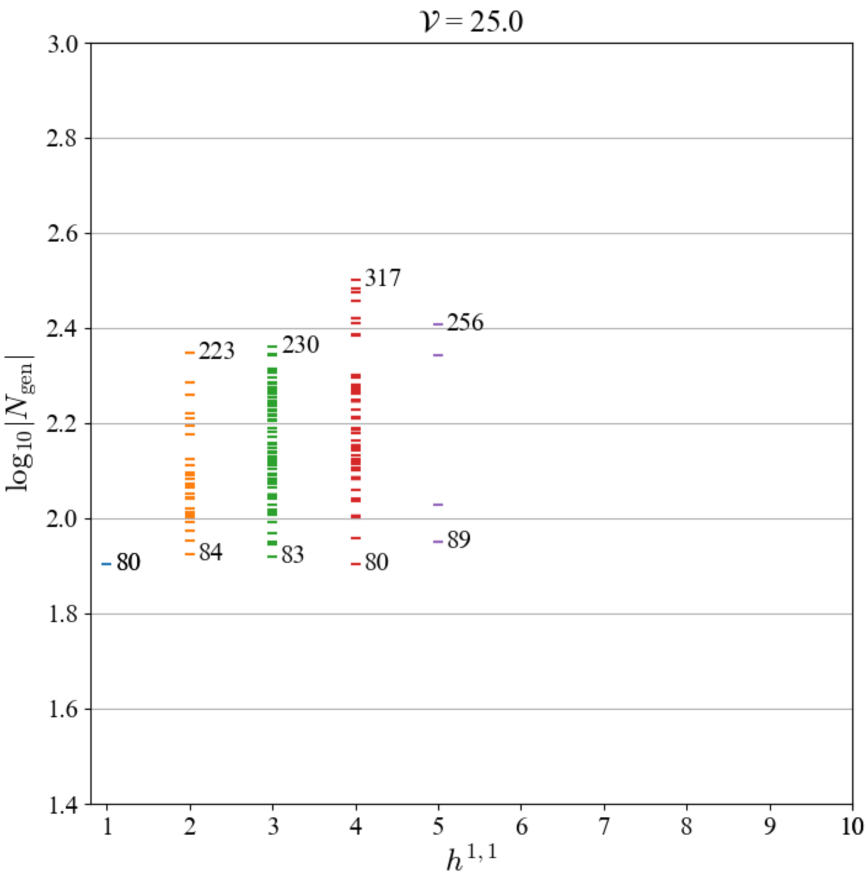
$$(m_{\rm max})^2 \leq \frac{n-1}{n} \frac{||c_2(TX)||}{\lambda_{\rm min}} - (h^{1,1} - 1)$$

We estimate the bound on “favorable” complete intersection CYs (CICYs).

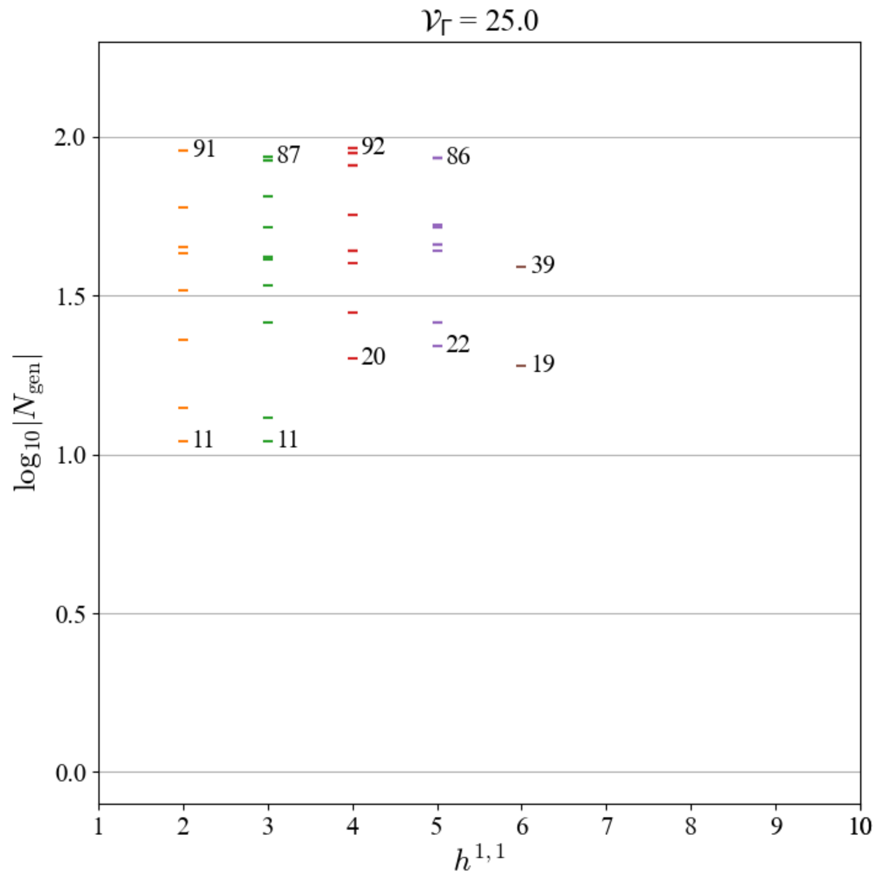
Candelas-Dale-Lutken-Schimmrigk (‘88)
Candelas-Lutken-Schimmrigk (‘88)

Upper bound on Atiyah-Singer Index in SU(5) GUT

CICYs ($\mathcal{M}$) with $\mathcal{V}$ = 25



Quotient CICYs ($\mathcal{M}/\Gamma$) with $\mathcal{V}_\Gamma$=25



Note that the CY volume is upper bounded by $\mathcal{V} = g_s^2 \alpha_{\rm GUT}^{-1} \leq 25$
$(\sum_{i,j,k} d_{ijk} < \sum_{i,j,k} d_{ijk} t^i t^j t^k = 6\mathcal{V} \leq 150)$

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Otsuka ('18),….

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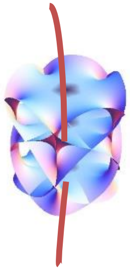

-- Gauge group

$$SO(32) \to SU(4)_C \times SU(2)_L \times SU(2)_R \times \Pi_a U(1)_a \times SO(16)_{\rm hid}$$

-- Chiral zero-modes

$$496 \to \oplus_p (R_p\,, C_p)$$

-- Index of matter multiplets

$$N_{\rm gen} = \frac{1}{2}\int_M {\rm ch}_3(C_p) + \frac{1}{12} c_2(TM) c_1(C_p)$$

$(SU(4)_C \times SU(2)_L \times SU(2)_R \times SO(16))_{U(1)_1,U(1)_2,U(1)_4,U(1)_5}$	$(SU(3)_C \times SU(2)_L \times SO(16))_{U(1)_1,U(1)_2,U(1)_3,U(1)_4,U(1)_5}$	Matter	Index
Visible sector			
$(4,2,1,1)_{1,1,0,0}$	$(3,2,1)_{1,1,1,0,0}$	Q_1	$\chi(\mathcal{M}, L_1 \otimes L_2)$
	$(1,2,1)_{1,1,-3,0,0}$	L_1	
$(4,2,1,1)_{-1,1,0,0}$	$(3,2,1)_{-1,1,1,0,0}$	Q_2	$\chi(\mathcal{M}, L_1^{-1} \otimes L_2)$
	$(1,2,1)_{-1,1,-3,0,0}$	L_2	
$(15,1,1,1)_{0,0,0,0}$	$(\bar{3},1,1)_{0,0,-4,0,0}$	$u^c_{R_1}$	$\chi(\mathcal{M}, \mathcal{O}_\mathcal{M})$
$(6,1,1,1)_{0,2,0,0}$	$(\bar{3},1,1)_{0,2,2,0,0}$	$d^c_{R_1}$	$\chi(\mathcal{M}, L_2^2)$
	$(3,1,1)_{0,2,-2,0,0}$	$\bar{d}^c_{R_2}$	
$(1,1,1,1)_{2,0,0,0}$	$(1,1,1)_{2,0,0,0,0}$	n_1	$\chi(\mathcal{M}, L_1^2)$
$(\bar{4},1,2,1)_{0,-1,-1,0}$	$(\bar{3},1,1)_{0,-1,-1,-1,0}$	$u^{c\,4}_{R_2}$	$\chi(\mathcal{M}, L_2^{-1} \otimes L_4^{-1})$
	$(1,1,1)_{0,-1,3,0,1}$	$e^{c\,5}_{R_1}$	
	$(1,1,1)_{0,-1,3,-1,0}$	$n^{c\,4}_2$	
	$(\bar{3},1,1)_{0,-1,-1,0,1}$	$d^{c\,5}_{R_3}$	
$(\bar{4},1,2,1)_{0,-1,1,0}$	$(\bar{3},1,1)_{0,-1,-1,1,0}$	$d^{c\,4}_{R_3}$	$\chi(\mathcal{M}, L_2^{-1} \otimes L_4)$
	$(1,1,1)_{0,-1,3,1,0}$	$e^{c\,4}_{R_1}$	
	$(1,1,1)_{0,-1,3,0,-1}$	$n^{c\,5}_2$	
	$(\bar{3},1,1)_{0,-1,-1,0,-1}$	$u^{c\,5}_{R_2}$	
$(1,2,2,1)_{1,0,-1,0}$	$(1,2,1)_{1,0,0,-1,0}$	L_3^4	$\chi(\mathcal{M}, L_1 \otimes L_4^{-1})$
	$(1,2,1)_{1,0,0,0,1}$	$\bar{L}_4^5$	
$(1,2,2,1)_{1,0,1,0}$	$(1,2,1)_{1,0,0,0,-1}$	L_3^5	$\chi(\mathcal{M}, L_1 \otimes L_4)$
	$(1,2,1)_{1,0,0,1,0}$	$\bar{L}_4^4$	
$(1,1,3,1)_{0,0,0,0}$	$(1,1,1)_{0,0,0,1,1}$	$e^{c\,45}_{R_2}$	$\chi(\mathcal{M}, \mathcal{O}_\mathcal{M})$
$(1,1,1,1)_{0,0,2,0}$	$(1,1,1)_{0,0,0,1,-1}$	$n^{c\,45}_3$	$\chi(\mathcal{M}, L_4^2)$
Hidden sector			
$(4,1,1,16)_{0,-1,0,0}$	$(3,1,16)_{0,-1,-1,0,0}$	–	$\chi(\mathcal{M}, L_2^{-1})$
	$(1,1,16)_{0,-1,3,0}$	–	
$(1,2,1,16)_{1,0,0,0}$	$(1,2,16)_{1,0,0,0,0}$	–	$\chi(\mathcal{M}, L_1)$
$(1,1,2,16)_{0,0,1,0}$	$(1,1,16)_{0,0,0,1,0}$	–	$\chi(\mathcal{M}, L_4)$
	$(1,1,16)_{0,0,0,0,-1}$	–	
$(1,1,1,120)_{0,0,0,0}$	$(1,1,120)_{0,0,0,0,0}$	–	$\chi(\mathcal{M}, \mathcal{O}_\mathcal{M})$

$(SU(4)_C \times SU(2)_L \times SU(2)_R \times SO(16))_{U(1)_1,U(1)_2,U(1)_4,U(1)_5}$	$(SU(3)_C \times SU(2)_L \times SO(16))_{U(1)_1,U(1)_2,U(1)_3,U(1)_4,U(1)_5}$	Matter	Index
Visible sector			
$(4,2,1,1)_{1,1,0,0}$	$(3,2,1)_{1,1,1,0,0}$ $(1,2,1)_{1,1,-3,0,0}$	Q_1 L_1	$\chi(\mathcal{M}, L_1 \otimes L_2)$
$(4,2,1,1)_{-1,1,0,0}$	$(3,2,1)_{-1,1,1,0,0}$ $(1,2,1)_{-1,1,-3,0,0}$	Q_2 L_2	$\chi(\mathcal{M}, L_1^{-1} \otimes L_2)$
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$(1,1,1,1)_{2,0,0,0}$	$(1,1,1)_{2,0,0,0,0}$	n_1	$\chi(\mathcal{M}, L_1^2)$
$(\bar{4},1,2,1)_{0,-1,-1,0}$	$(\bar{3},1,1)_{0,-1,-1,-1,0}$ $(1,1,1)_{0,-1,3,0,1}$ $(1,1,1)_{0,-1,3,-1,0}$ $(\bar{3},1,1)_{0,-1,-1,0,1}$	$u^{c\,4}_{R_2}$ $e^{c\,5}_{R_1}$ $n^{c\,4}_2$ $d^{c\,5}_{R_3}$	$\chi(\mathcal{M}, L_2^{-1} \otimes L_4^{-1})$
$(\bar{4},1,2,1)_{0,-1,1,0}$	$(\bar{3},1,1)_{0,-1,-1,1,0}$ $(1,1,1)_{0,-1,3,1,0}$ $(1,1,1)_{0,-1,3,0,-1}$ $(\bar{3},1,1)_{0,-1,-1,0,-1}$	$d^{c\,4}_{R_3}$ $e^{c\,4}_{R_1}$ $n^{c\,5}_2$ $u^{c\,5}_{R_2}$	$\chi(\mathcal{M}, L_2^{-1} \otimes L_4)$
$(1,2,2,1)_{1,0,-1,0}$	$(1,2,1)_{1,0,0,-1,0}$ $(1,2,1)_{1,0,0,0,1}$	L_3^4 $\bar{L}_4^5$	$\chi(\mathcal{M}, L_1 \otimes L_4^{-1})$
$(1,2,2,1)_{1,0,1,0}$	$(1,2,1)_{1,0,0,0,-1}$ $(1,2,1)_{1,0,0,1,0}$	L_3^5 $\bar{L}_4^4$	$\chi(\mathcal{M}, L_1 \otimes L_4)$
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Hidden sector			

Index of hidden sector = 0

$(SU(4)_C \times SU(2)_L \times SU(2)_R \times SO(16))_{U(1)_1,U(1)_2,U(1)_4,U(1)_5}$	$(SU(3)_C \times SU(2)_L \times SO(16))_{U(1)_1,U(1)_2,U(1)_3,U(1)_4,U(1)_5}$	Matter	Index
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Hidden sector			

Vector-like quarks :

$$|N_{\rm gen}^{\rm (vec)}| \leq \frac{|m_{\rm max}||c_2(V)|}{4} \leq \frac{|m_{\rm max}||c_2(TX)|}{4}$$

Tadpole cancellation

$c_2(V) \leq c_2(TX)$

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Higgs + Quarks(Leptons) :

$$|N_{\rm gen}(H) + 2N_{\rm gen}^{(\rm quark)}| \leq \frac{|m_{\rm max}||c_2(V)|}{2} \leq \frac{|m_{\rm max}||c_2(TX)|}{2}$$

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$c_2(V) \leq c_2(TX)$

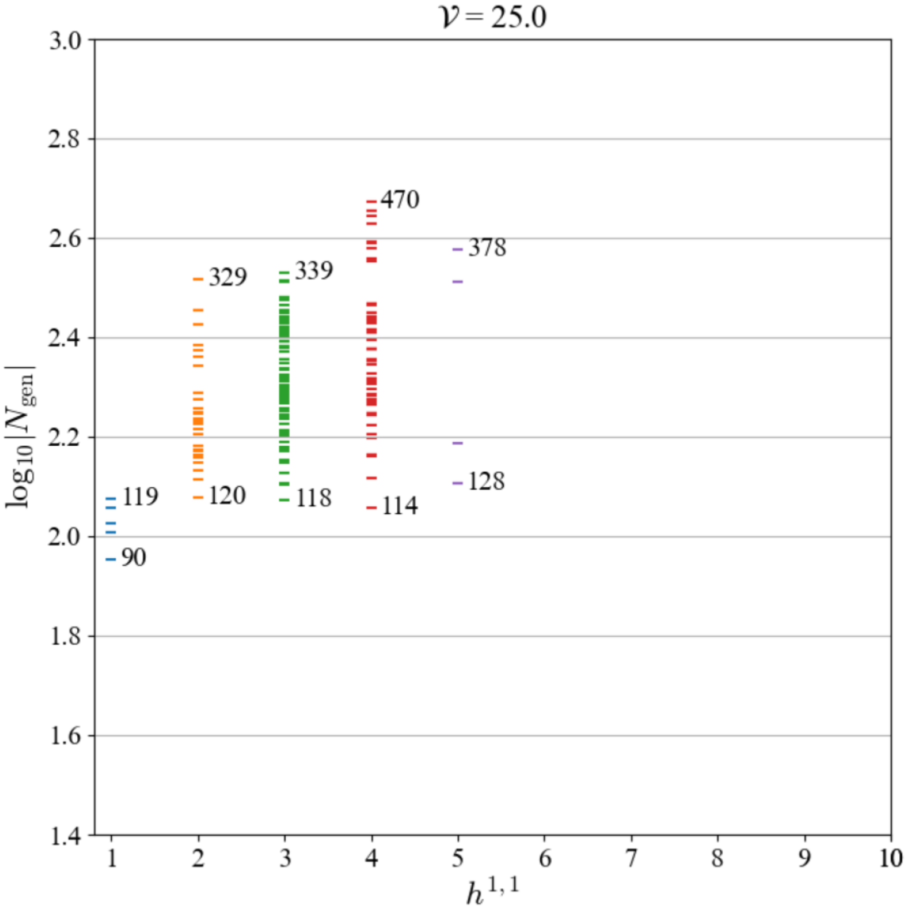
Higgs :

If $N_{\rm gen}^{(\rm quark)} = -3$

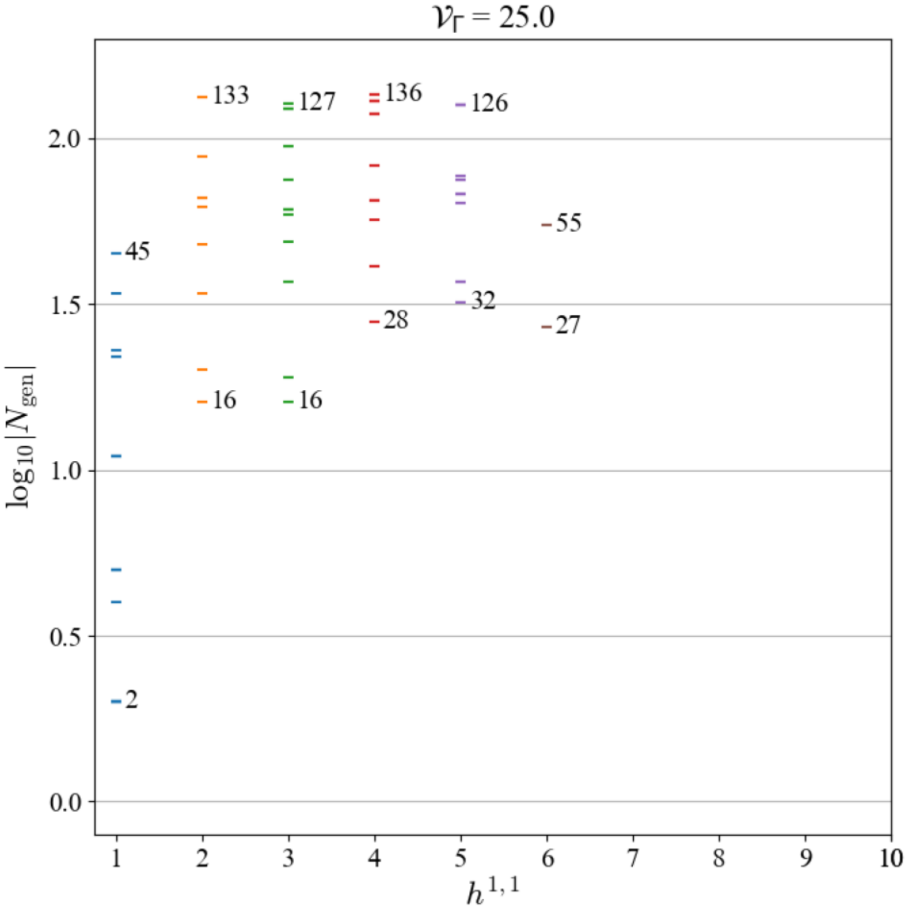
$$|N_{\rm gen}(H)| \leq \frac{|m_{\rm max}||c_2(TX)|}{2} - 6$$

Upper bound on #Higgs in Pati-Salam

CICYs ($\mathcal{M}$) with $\mathcal{V}$ = 25



Quotient CICYs ($\mathcal{M}/\Gamma$) with $\mathcal{V}_\Gamma$=25



Note that the CY volume is upper bounded by $\mathcal{V} = g_s^2 \alpha_{\text{GUT}}^{-1} \leq 25$
$(\sum_{i,j,k} d_{ijk} < \sum_{i,j,k} d_{ijk} t^i t^j t^k = 6\mathcal{V} \leq 150)$

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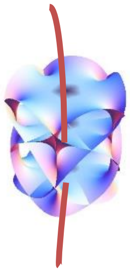

-- Gauge group

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-- Chiral zero-modes

$$496 \to \oplus_p (R_p, C_p)$$

-- Index of quarks/leptons

$$|N_{\rm gen}| \leq \frac{|m_{\rm max}||c_2(V)|}{6} \leq \frac{|m_{\rm max}||c_2(TX)|}{6}$$

Tadpole cancellation
$c_2(V) \leq c_2(TX)$

Other CY threefolds ?

- For the CY threefolds in the Kreuzer-Skarke database

Kreuzer-Skarke ('00)
Demirtas-Long-McAllister-Stillman ('18),...

For $h^{1,1} \geq 25$

$$\mathcal{V} \geq (h^{1,1})^{-\frac{1}{2}} ||\boldsymbol{t}||^3$$

$$G_{ij} \simeq (h^{1,1})^{-\frac{1}{2}} ||\boldsymbol{t}||$$

G_{ij} : Moduli metric

$$||c_2(TX)|| \geq \sum_{a,i} m_a^i \tilde{G}_{ij} m_a^j \geq \alpha (h^{1,1})^{\frac{1}{2}}$$

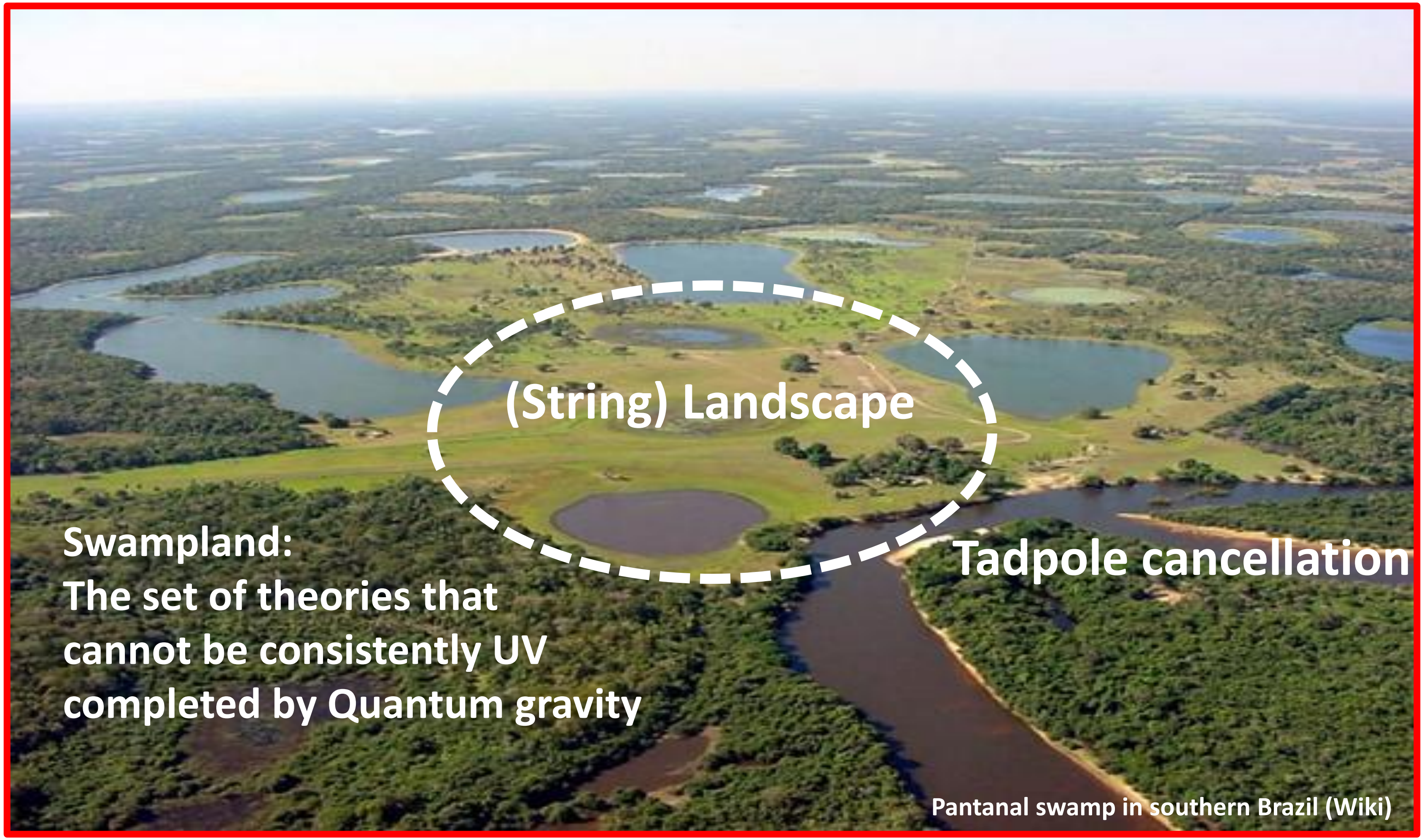
α : O(1) coefficients

$$\tilde{G}_{ij} := \frac{\mathcal{V}}{||\boldsymbol{t}||} G_{ij}$$

Tadpole cancellation
$c_2(TX) \geq c_2(V)$

Conclusion and Discussions

- We propose an upper bound on the Atiyah-Singer index in heterotic string theories on CY with line bundles



Conclusion and Discussions

- We propose an upper bound on the Atiyah-Singer index in heterotic string theories on CY with line bundles

- Generation number of quarks/leptons, Higgs are constrained by the tadpole cancellation :

$$|N_{\rm gen}| \leq \frac{|m_{\rm max}||c_2(TX)|}{4}$$

$$|m_{\rm max}| \leq \frac{n-1}{n}\frac{||c_2(TX)||}{\lambda_{\rm min}} - (h^{1,1}-1)$$

n : # of U(1)

- For $h^{1,1} \geq 25$

$$||c_2(TX)|| \geq \alpha (h^{1,1})^{\frac{1}{2}}$$

α : O(1) coefficients

- Similar upper bound in Type IIB string theory (work in progress)