Blow-up of magnetized T^2/Z_N and index theorem

Maki Takeuchi(Kobe Univ)

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1. Introduction

Question: Why are there 3 generations of quarks and leptons?

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Accidental? ↙ ↘ Has physical meaning ?

Standard Model	Extra dimension model

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Standard Model

I	II	III
$\begin{pmatrix} u_\mathrm{L} \\ d_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} c_\mathrm{L} \\ s_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} t_\mathrm{L} \\ b_\mathrm{L} \end{pmatrix}$
u_R	c_R	t_R
d_R	s_R	b_R
$\begin{pmatrix} \nu_{e\mathrm{L}} \\ e_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} \nu_{\mu\mathrm{L}} \\ \mu_\mathrm{L} \end{pmatrix}$	$\begin{pmatrix} \nu_{\tau\mathrm{L}} \\ \tau_\mathrm{L} \end{pmatrix}$
e_R	μ_R	τ_R

Extra dimension model

6d chiral fermion

$\begin{pmatrix} U(x,y) \\ D(x,y) \end{pmatrix}$
$U'(x,y)$
$D'(x,y)$
$\begin{pmatrix} N(x,y) \\ E(x,y) \end{pmatrix}$
$E'(x,y)$

Zero modes=generation

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e_R	μ_R	τ_R

Degenerate

1. Introduction

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Accidental? → **Standard Model**

Has physical meaning? → **Extra dimension model**

Standard Model

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$$U'(x,y)$$

$$D'(x,y)$$

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Zero modes

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u_R	c_R	t_R
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	μ_R	τ_R

Decided by topology of extra dimension

Degenerate

1. Introduction

Atiyah-Singer index theorem (2 dimension)

$$n_+ - n_- \propto \int_{\mathscr{M}} F$$

$n_\pm$: chiral zero modes # $\mathscr{M}$: a smooth manifold

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Atiyah-Singer index theorem (2 dimension)

$$n_+ - n_- \propto \int_{\mathscr{M}} F$$

Contribution of flux

$n_\pm$: chiral zero modes # $\mathscr{M}$: a smooth manifold

AS index theorem on magnetized T^2

$$n_+ - n_- = \int_{T^2} \frac{F}{2\pi} = M$$

Chiral zero modes #   flux quantization#

We get three generations if we choose $M = 3$.

2. Purpose of my talk

In previous paper, we obtain the following formula on T^2/Z_N orbifolds

$$n_+ - n_- = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number at fixed points

Makoto Sakamoto, **Maki Takeuchi**, Yoshiyuki Tatsuta, **Phys. Rev. D** 102 (2020) 025008

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What's physical meaning?

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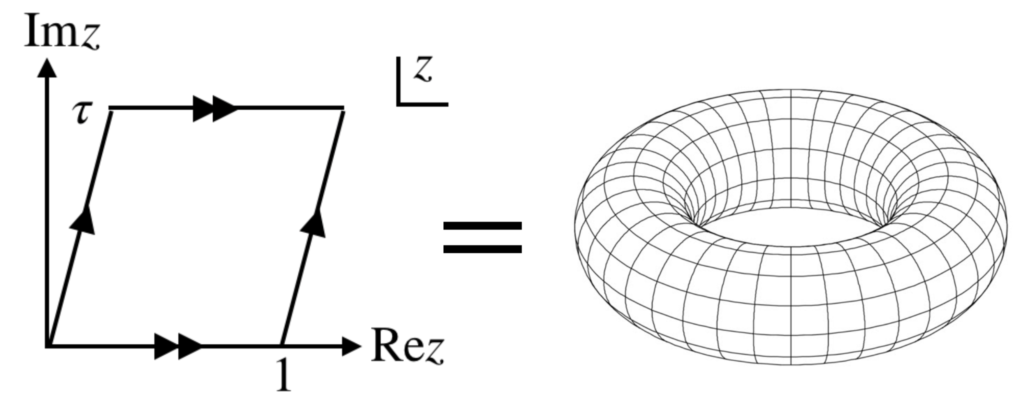
3. T^2/Z_N orbifold model

- $\mathscr{M}^4 \times T^2/Z_N$ **in flux background**

$$\frac{T^2 : z \sim z+1 \sim z+\tau}{Z_N : z \sim \rho z \quad (\rho = e^{i\frac{2\pi}{N}})} =$$

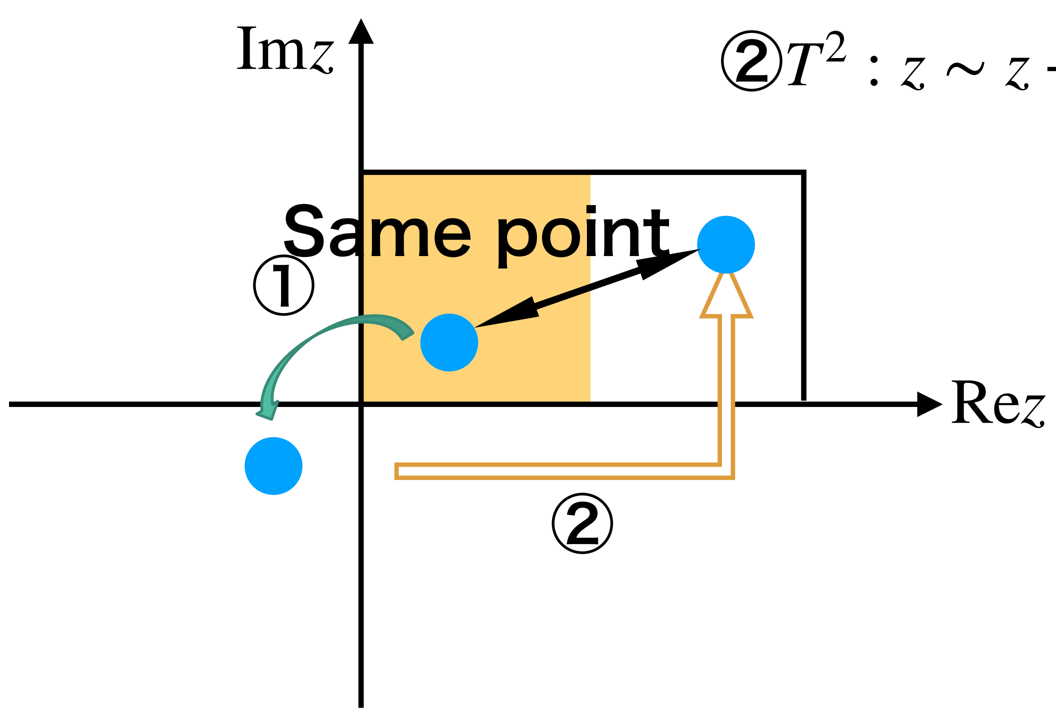


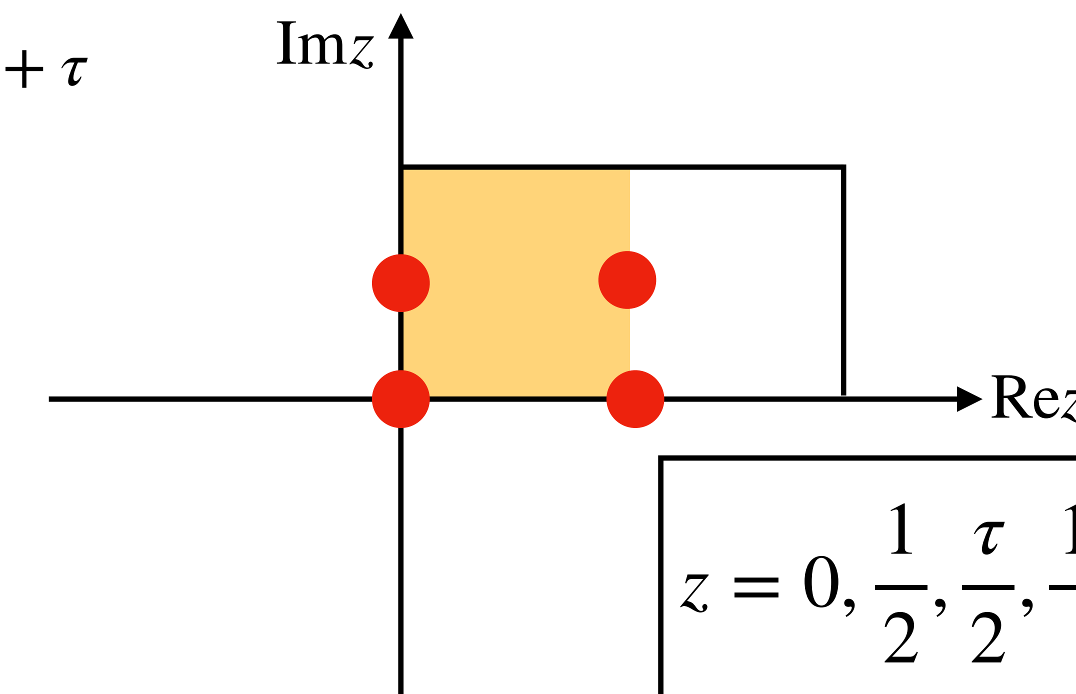
Z_N

Ex) $T^2/Z_2\,(\tau = i)$

①$Z_2 : z \sim -z$

②$T^2 : z \sim z+1 \sim z+\tau$





$$z = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}$$

Independent region is $1/N$ of T^2

Fixed point

3. T^2/Z_N orbifold model

Z_N eigen function $\psi^m_{T^2/Z_N\pm,n,j}(z)$

Boundary condition

$$\psi^m_{T^2/Z_N+,n,j}(z+1) = U_1(z)\,\psi^m_{T^2/Z_N+,n,j}(z)$$

$$\psi^m_{T^2/Z_N+,n,j}(z+\tau) = U_2(z)\,\psi^m_{T^2/Z_N+,n,j}(z)$$

$$U_1(z) = e^{iq\Lambda_1(z)}, U_2(z) = e^{iq\Lambda_2(z)}$$

$$\psi^m_{T^2/Z_N+,n,j}(\rho z) = \rho^m \psi^m_{T^2/Z_N+,n,j}(z)$$

$$\rho = e^{i\frac{2\pi}{N}}$$

Winding number

Define winding number of Z_N eigen function $\psi^m_{T^2/Z_N+,n,j}(z)$ at fixed point z^f_j

$$\psi^m_{T^2/Z_N+,n,j}(\rho z + z^f_j) = \rho^{\chi_+}\psi^m_{T^2/Z_N+,n,j}(z + z^f_j)$$

$\Rightarrow$ **winding # :** χ_+

In case of $z^f_j = 0$, $\chi_+ = m$

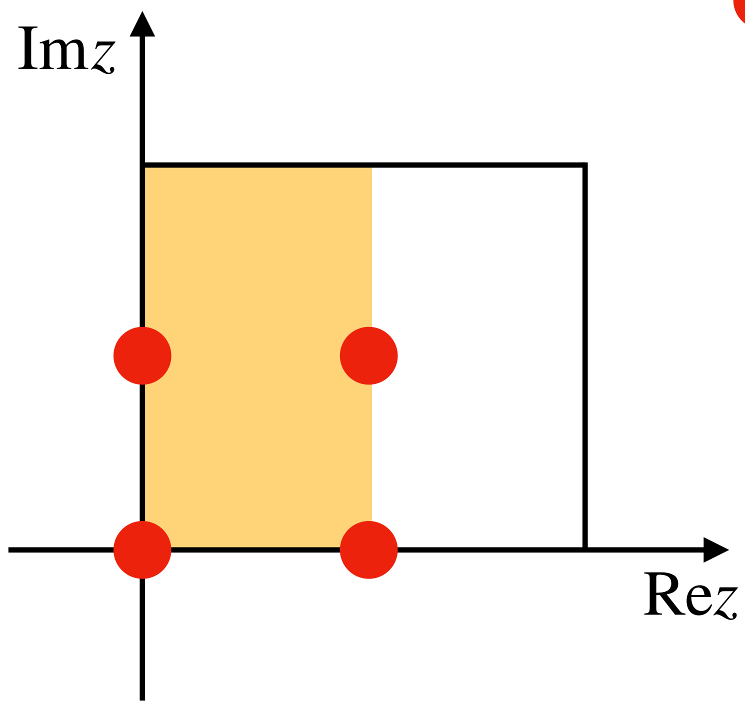
3. T^2/Z_N orbifold model

Atiyah-Singer index theorem on T^2/Z_N orbifold

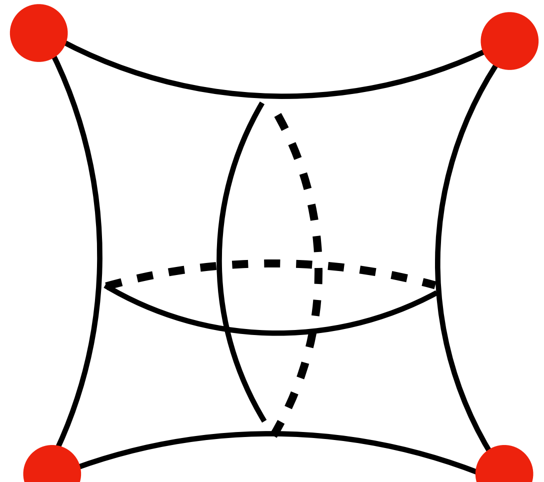
It is difficult to apply AS index theorem because fixed points are singular points.

➡ **We consider blow-up manifold without singular points.**

Ex) T^2/Z_2



● **Singular point**



T. Kobayashi, H. Otsuka, H. Uchida., JHEP 08, 104 (2019),046

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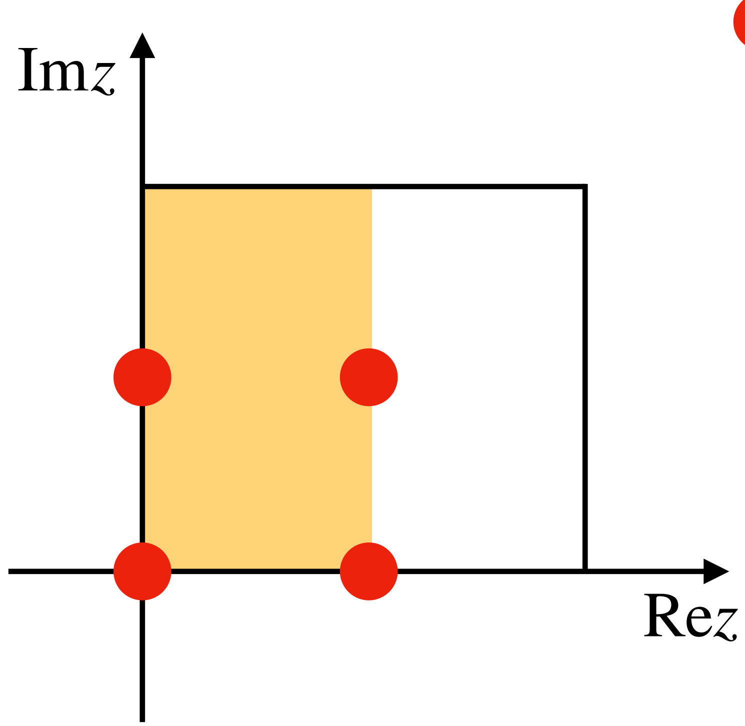

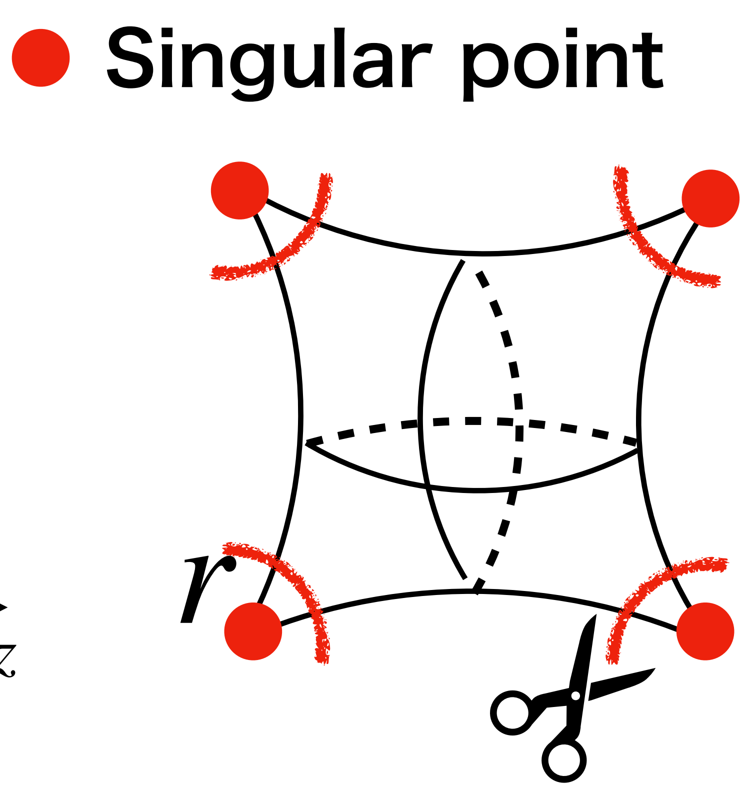

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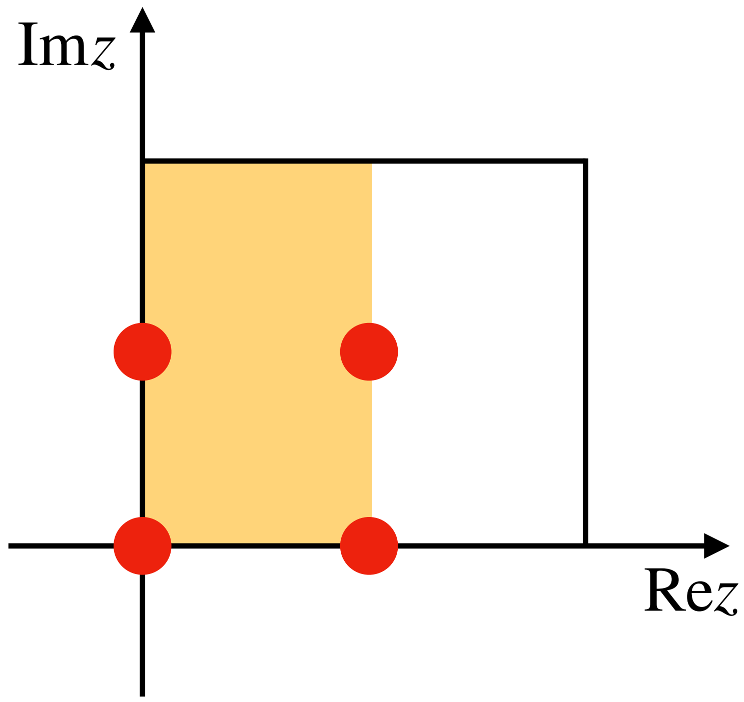
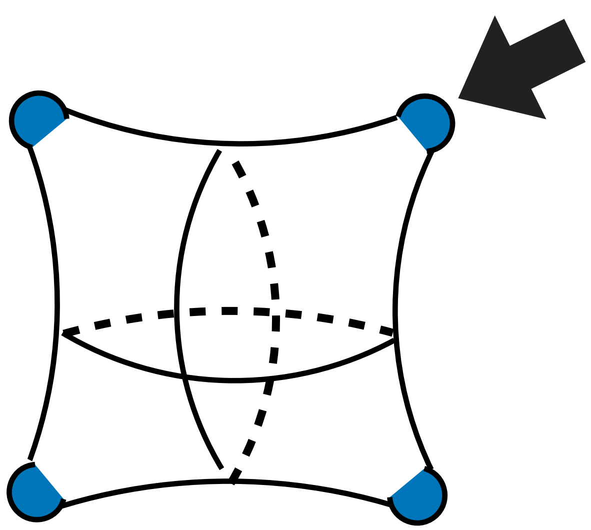


Blow-up manifold

A part of magnetized S^2



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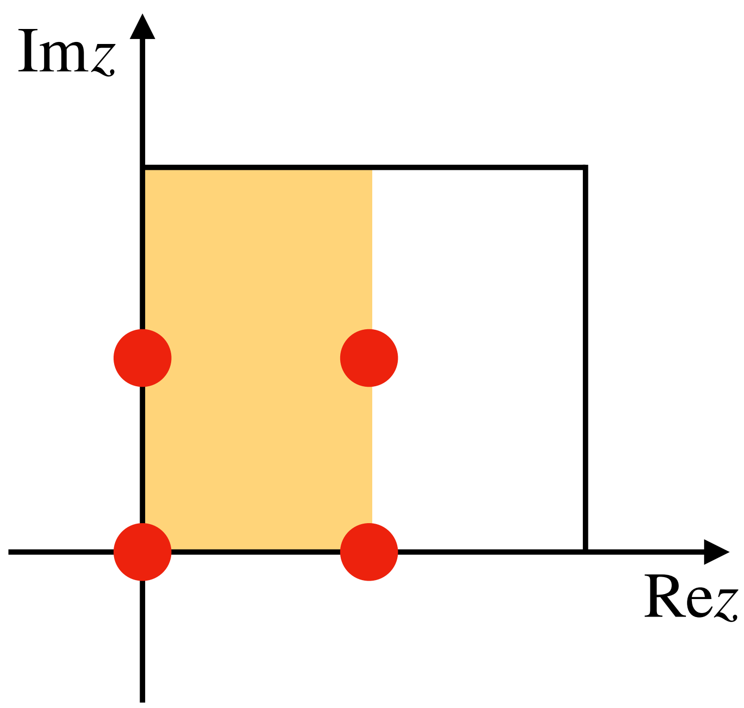

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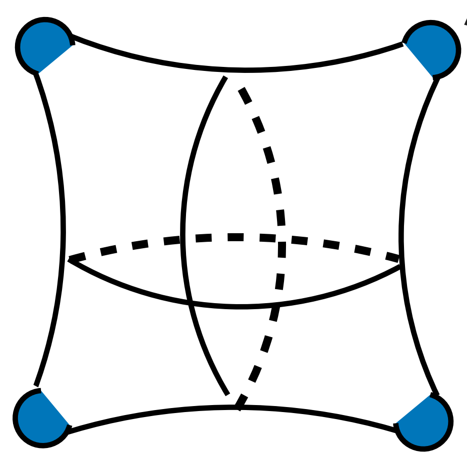


A part of magnetized S^2

We can apply AS index theorem to blow-up manifold without singularity.

T. Kobayashi, H. Otsuka, H. Uchida., JHEP 08, 104 (2019),046

4. Blow-up of T^2/Z_N orbifold

$\psi^0_{T^2/Z_N,+}(z)$ **with winding number cannot connect to** $\psi^0_{S^2,+}(z')$ **,**

because the boundary conditions are different at $z \sim z_j^f$

4. Blow-up of T^2/Z_N orbifold

Purpose : To remove winding number

We consider in case of fixed point $z_j^f = 0$ ($m \to \chi_+$ in case $z_j^f \neq 0$)

$$\psi^n_{T^2/Z_N,+}(\rho z) = \rho^m \psi^n_{T^2/Z_N,+}(z) \quad \Rightarrow \quad \tilde{\psi}^n_{T^2/Z_N,+}(\rho z) = \tilde{\psi}^n_{T^2/Z_N,+}(z)$$

``Singular" gauge transformation

$$\tilde{\psi}_{T^2/Z_N,\pm}(z) = U_{\xi^F} U_{\xi^R} \psi_{T^2/Z_N,\pm}(z) \qquad U_{\xi^F} \propto \left(\frac{z}{\bar{z}}\right)^{\frac{\xi^F}{2}}, \; U_{\xi^R} \propto \left(\frac{z}{\bar{z}}\right)^{\frac{\xi^R}{4}}$$

ξ^F : localized flux at fixed point, ξ^R : localized curvature at fixed point

$$\tilde{\psi}_{T^2/Z_N,+}(\rho z) = \rho^{\xi^F - \frac{\xi^R}{2} + m} \tilde{\psi}_{T^2/Z_N,+}(z)$$

$$\boxed{\rho^{\xi^F - \frac{\xi^R}{2} + m} = 1}$$

However this gauge transformation changes flux,
this is called ``singular gauge transformation".

4. Blow-up of T^2/Z_N orbifold

Localized flux ξ^F

$$\rho^{\xi^F-\frac{\xi^R}{2}+m}=1$$

$$\xi^F=\frac{\xi^R}{2}-m+lN \quad (l\in\mathbb{Z})$$

Degree of freedom by mod N of $\rho=e^{i2\pi/N}$

The information of winding numbers are replaced by localized flux ξ^F and localized curvature ξ^R.

Since $\tilde{\psi}_{T^2/Z_N,\pm}(z)$ have no winding numbers, these can be connected to $\psi^0_{s^2,+}(z')$.

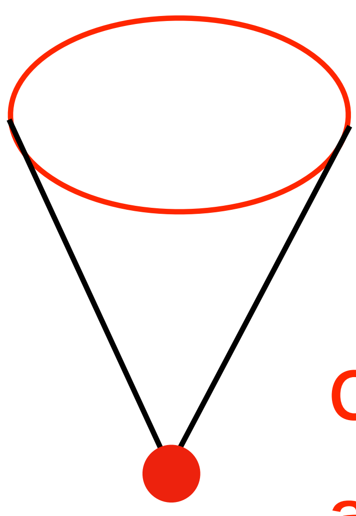
4. Blow-up of T^2/Z_N orbifold

Connection condition $\tilde{\psi}^0_{T^2/Z_N,\pm}(z)$ and $\psi^0_{S^2,+}(z')$

$$\frac{\xi^F}{N} = \frac{N-1}{2N} M'$$

ξ^F : localized flux. M' : Total flux of S^2 , $\frac{N-1}{2N}$: embedded area of S^2

Physical meaning of connection condition

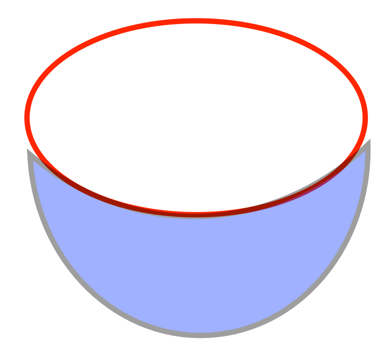


$$\frac{\xi^F}{N} = \frac{N-1}{2N} M'$$

Cut out flux of T^2/Z_N

at fixed point

Embedded flux of S^2



5. Conclusion

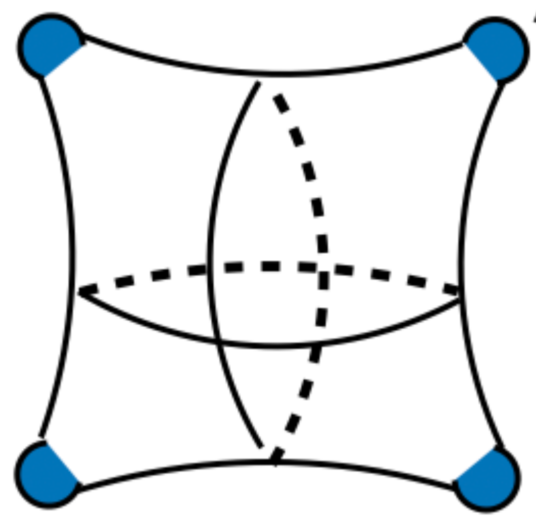
Index theorem on blow-up manifold of T^2/Z_N orbifold

$$n_+ - n_- = \int_{blow-up} \frac{F}{2\pi} = \frac{M}{N} + \sum_{z_j^f} \frac{N-1}{2N} M' = \frac{M}{N} + \sum_{z_j^f} \frac{\xi^F}{N}$$

$n_\pm$: chiral zero modes number , M' : Total flux of S^2 ,

$\frac{N-1}{2N}$: embedded area of S^2 , ξ^F : localized flux at fixed point

5. Conclusion



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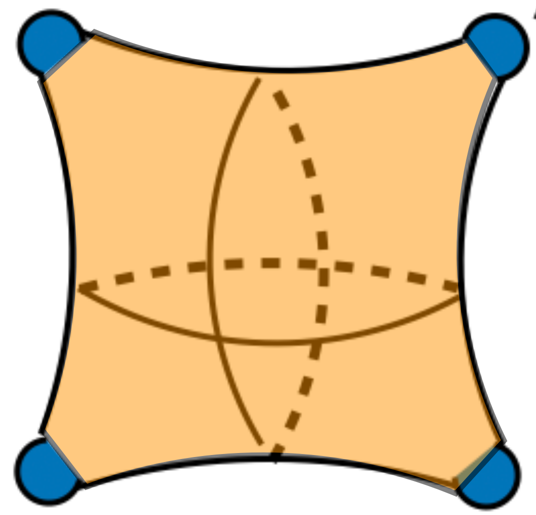

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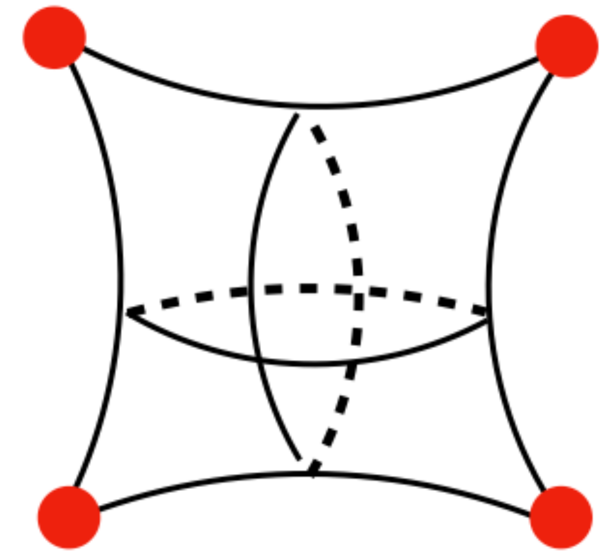

Index theorem on blow-up manifold d

$$n_+ - n_- = \int_{blow-up} \frac{F}{2\pi} = \frac{M}{N} + \sum_{z_j^f} \frac{N-1}{2N} M' = \frac{M}{N} + \sum_{z_j^f} \frac{\xi^F}{N}$$

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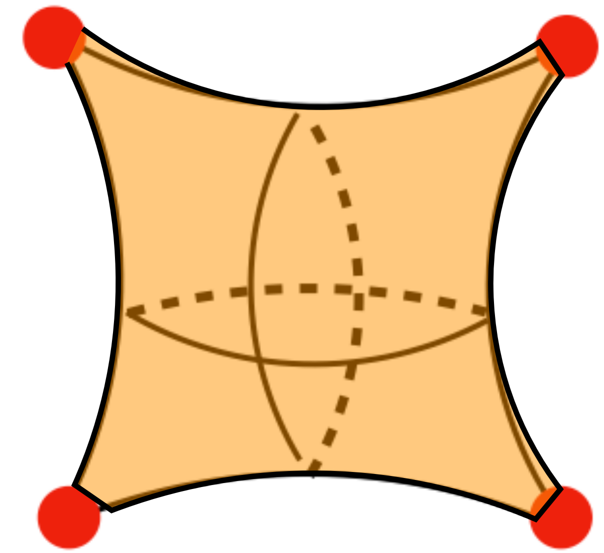

Index theorem on blow-up manifold

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Index theorem on blow-up manifold of T^2/Z_N orbifold

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$n_\pm$: chiral zero modes number , M' : Total flux of S^2 ,

$\frac{N-1}{2N}$: embedded area of S^2 , ξ^F : localized flux at fixed point

Only contribution of flux!

5. Conclusion

Reinterpretation of index formula on T^2/Z_N orbifold

$$n_+ - n_- = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number χ_+ at fixed points

Winding number has contributions of localized flux and curvature.

$$\xi^F = \frac{\xi^R}{2} - m + lN \quad (z_j^f = 0) \quad \Rightarrow \quad \xi^F = \frac{\xi^R}{2} - \chi_+ + lN \quad (z_j^f \neq 0)$$

$$\sum_{z_j^f} \frac{\xi^F}{N} = - \sum_{z_j^f} \frac{\chi_+}{N} + \sum_{z_j^f} \frac{\xi^R}{2N} + l = - \frac{V_+}{N} + 1 + l$$

5. Conclusion

Reinterpretation of index theorem on T^2/Z_N orbifold

$$n_+ - n_- = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number χ_+ at fixed points

Winding number has contributions of localized flux and curvature.

$+1$ removes the contribution of localized curvature in V_+.

$$\sum_{z_j^f} \frac{\xi^F}{N} = -\sum_{z_j^f} \frac{\chi_+}{N} + \sum_{z_j^f} \frac{\xi^R}{2N} + l = -\frac{V_+}{N} + 1 + l$$

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Reinterpretation of index theorem on T^2/Z_N orbifold

$$n_+ - n_- = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number χ_+ at fixed points

Winding number has contributions of localized flux and curvature.

Index theorem implies new zero modes of l = localized mode

$$\sum_{z_j^f} \frac{\xi^F}{N} = -\sum_{z_j^f} \frac{\chi_+}{N} + \sum_{z_j^f} \frac{\xi^R}{2N} + l = -\frac{V_+}{N} + 1 + l$$

5. Outlook

- Further analysis of localized mode
- Extension of the method of blow-up to higher dimensions (T^4/Z_N , T^6/Z_N)

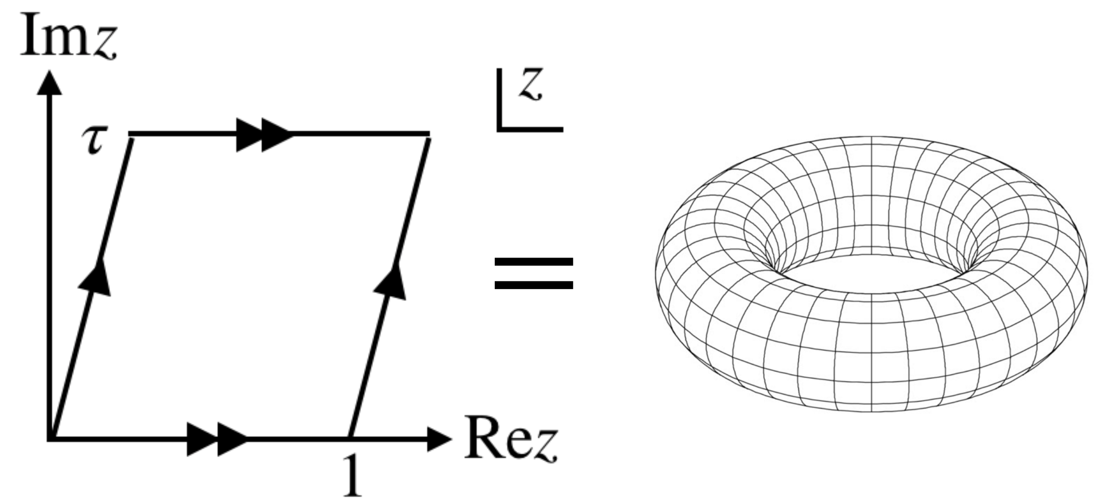
Thank you!

Back up

3. T^2 model

- $\mathscr{M}^4 \times T^2$ **in flux background**

$T^2 : z \sim z + 1 \sim z + \tau$



6d Dirac action

$$S = \int d^4x \int d^2z \bar{\Psi}(x, z) i\Gamma^M D_M \Psi(x, z)$$

$$D_M = \partial_M - iqA_M$$

6d chiral fermion

Mode expansion

2d chiral fermion

$$\Psi(x, z) = \sum_n \psi^{(4)}_{R,n}(x) \otimes \psi^{(2)}_{T^2+,n}(z) + \psi^{(4)}_{L,n}(x) \otimes \psi^{(2)}_{T^2-,n}(z)$$

4d chiral fermion

2d chiral fermion

$$\psi^{(2)}_{T^2+,n}(z) = \begin{pmatrix} \psi_{T^2+,n} \\ 0 \end{pmatrix}, \quad \psi^{(2)}_{T^2-,n}(z) = \begin{pmatrix} 0 \\ \psi_{T^2-,n} \end{pmatrix}$$

3. T^2 model

- $\mathscr{M}^4 \times T^2$ **in flux background**

T^2 sift corresponds to
gauge transformation.

Boundary condition

$$\psi_{T^2\pm,n,j}(z+1) = U_1(z)\,\psi_{T^2\pm,n,j}(z)$$

$$\psi_{T^2\pm,n,j}(z+\tau) = U_2(z)\,\psi_{T^2\pm,n,j}(z)$$

$$U_1(z) = e^{iq\Lambda_1(z)}, U_2(z) = e^{iq\Lambda_2(z)}$$

Flux quantization

$$\frac{qf}{2\pi} \equiv M \in \mathbb{Z}$$

M: flux quantization number f : flux

3. T^2 model

- $\mathscr{M}^4 \times T^2$ **in flux background**

Eigenvalue equation

$$\left.\begin{aligned} 2D_{\bar{z}}\psi_{T^2+,n,j} &= 2\left(\partial_{\bar{z}} - iqA_{\bar{z}}\right)\psi_{T^2+,n,j} = m_n\psi_{T^2-,n,j} \\ -2D_z\psi_{T^2-,n,j} &= -2\left(\partial_z - iqA_z\right)\psi_{T^2-,n,j} = m_n\psi_{T^2+,n,j} \end{aligned}\right\} \quad \begin{aligned} -4D_zD_{\bar{z}}\psi_{T^2+,n,j} &= m_n^2\psi_{T^2+,n,j} \\ -4D_{\bar{z}}D_z\psi_{T^2-,n,j} &= m_n^2\psi_{T^2-,n,j} \end{aligned}$$

$\Rightarrow$ **Zero mode has the eigenvalue** $m_n = 0$.

AS index theorem on T^2

$$\boxed{\underbrace{n_+ - n_-}_{\text{Chiral zero modes \#}} = \int_{T^2}\frac{F}{2\pi} = \underbrace{M}_{\text{flux quantization\#}}}$$

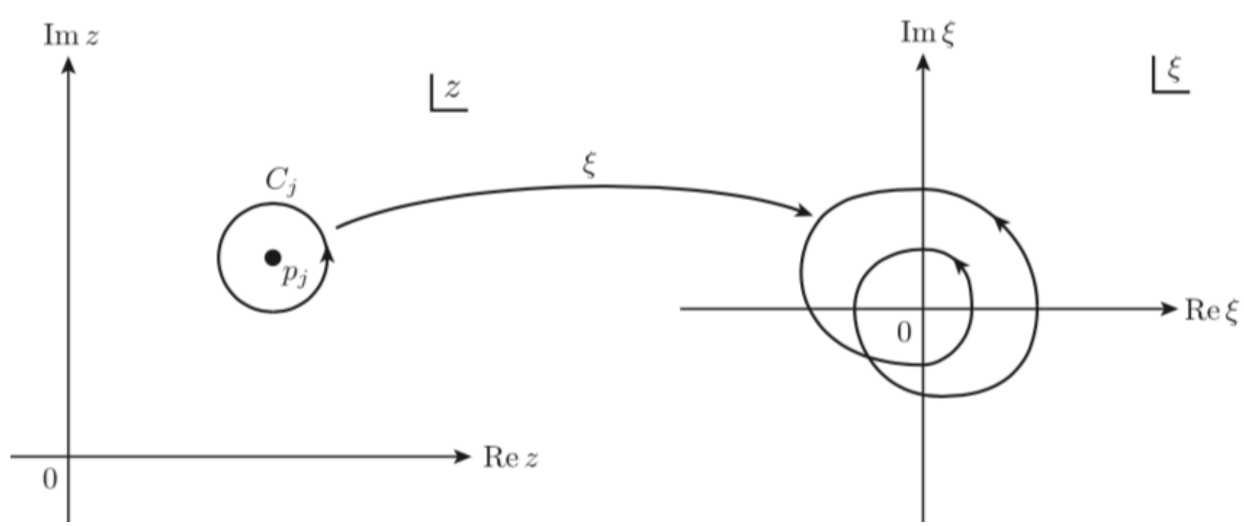
4. T^2/Z_N orbifold model

Winding number

Define winding number of Z_N eigen function $\psi^m_{T^2/Z_N+,n,j}(z)$ at fixed point z_j^f

$$\boxed{\psi^m_{T^2/Z_N+,n,j}(\rho z + z_j^f) = \rho^{\chi_+}\psi^m_{T^2/Z_N+,n,j}(z + z_j^f) \quad \Rightarrow \textbf{winding \# :} \ \chi_+}$$

ex)winding # = 2

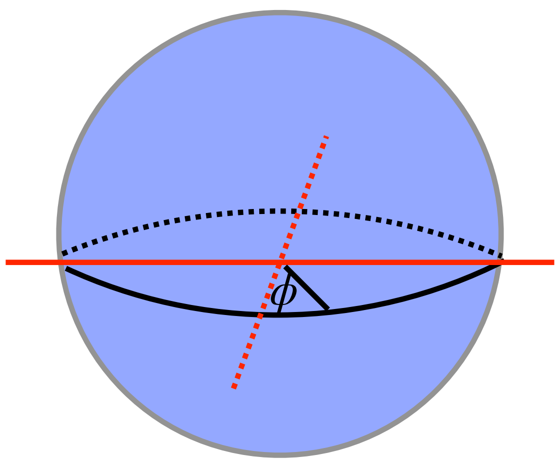


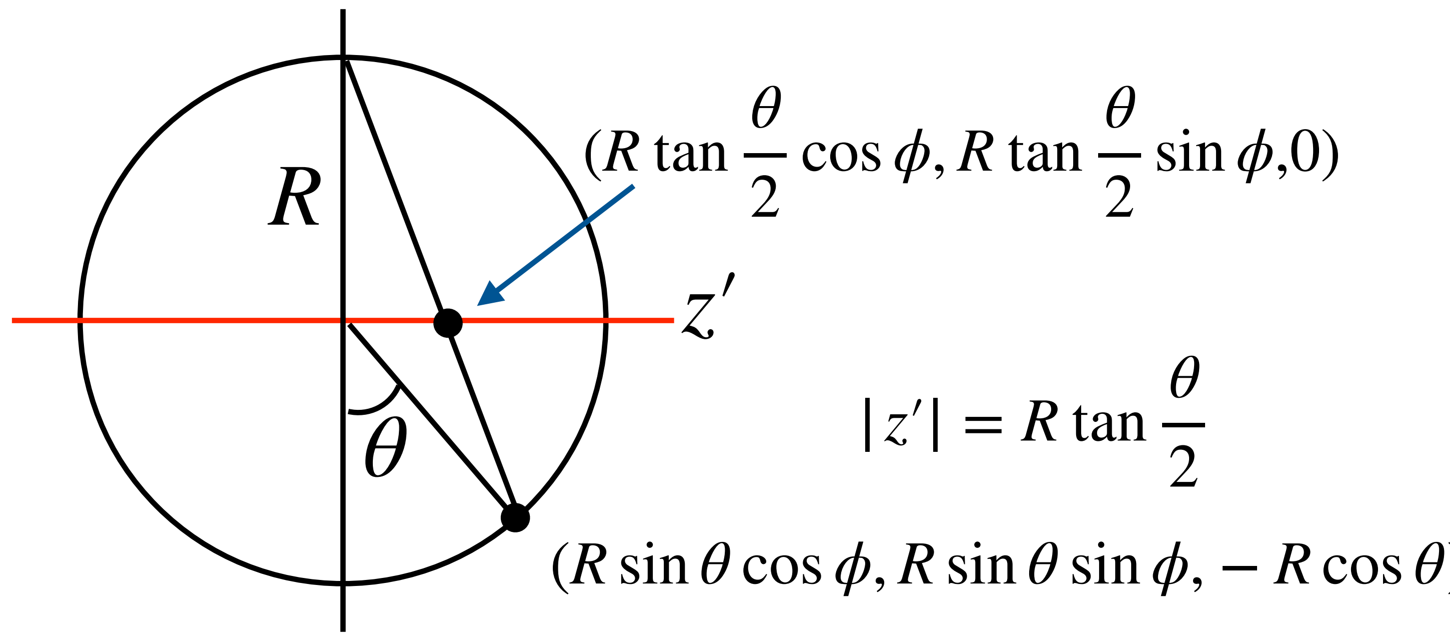
Winding number depends on eigen function & fixed point

5. Magnetized S^2

Magnetized S^2

How to take the coordinate z' of S^2





AS index theorem on S^2

$$n_+ - n_- = \int_{S^2}\frac{F}{2\pi} = M'$$

Chiral zero modes # flux quantization#

5. Magnetized S^2

Magnetized S^2

Dirac equation

$$\frac{R^2+|z'|^2}{R} i\left(\partial_{\bar{z}'} + i\frac{1}{2}\omega_{\bar{z}'} - iA_{\bar{z}'}\right)\psi^n_{s^2,+}(z') = m_n \psi^n_{s^2,-}(z')$$

$$\frac{R^2+|z'|^2}{R} i\left(\partial_{z'} - i\frac{1}{2}\omega_{z'} - iA_{z'}\right)\psi^n_{s^2,-}(z') = m_n \psi^n_{s^2,+}(z')$$

$$\omega_{\bar{z}'} = \frac{i}{2}\frac{2}{R^2+|z'|^2}z',\ \omega_{z'} = -\frac{i}{2}\frac{2}{R^2+|z'|^2}\bar{z}',\ A_{\bar{z}'} = \frac{i}{2}\frac{M'}{R^2+|z'|^2}z',\ A_{z'} = -\frac{i}{2}\frac{M'}{R^2+|z'|^2}\bar{z}'$$

$\Rightarrow$ **Zero mode has the eigenvalue $m_n = 0$.**

Zero mode

$$\psi^0_{s^2,+}(z') = \frac{f_+(z')}{(R^2+|z'|^2)^{\frac{M'-1}{2}}}$$

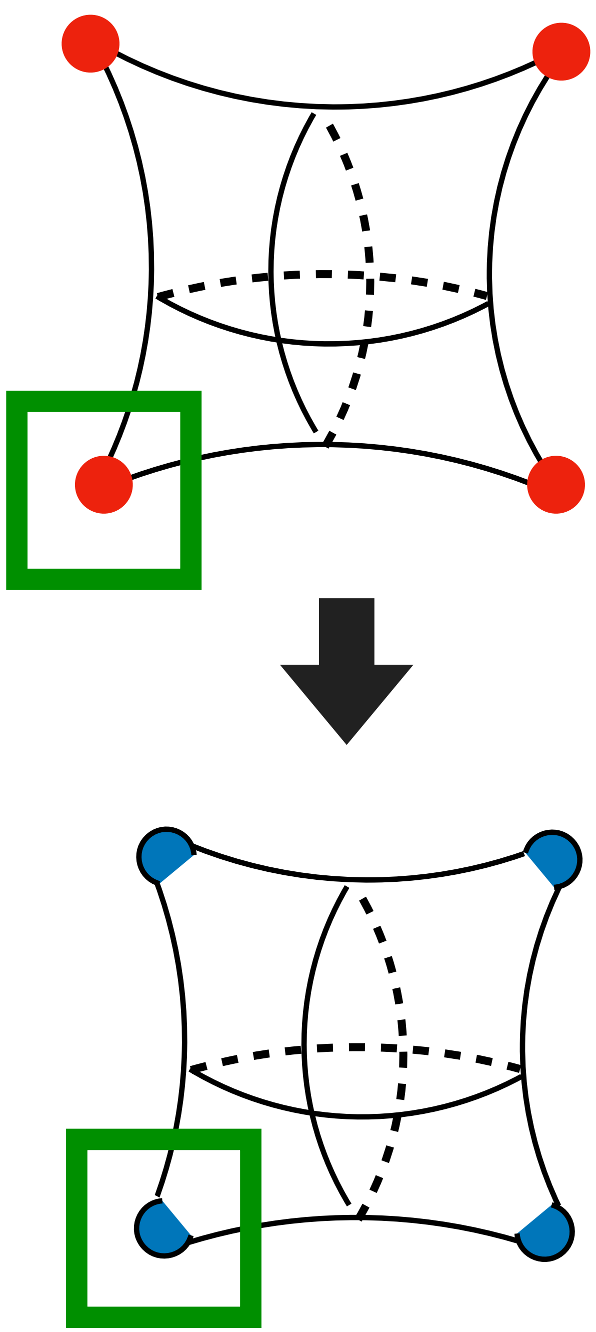
Lowest mode with - chirality

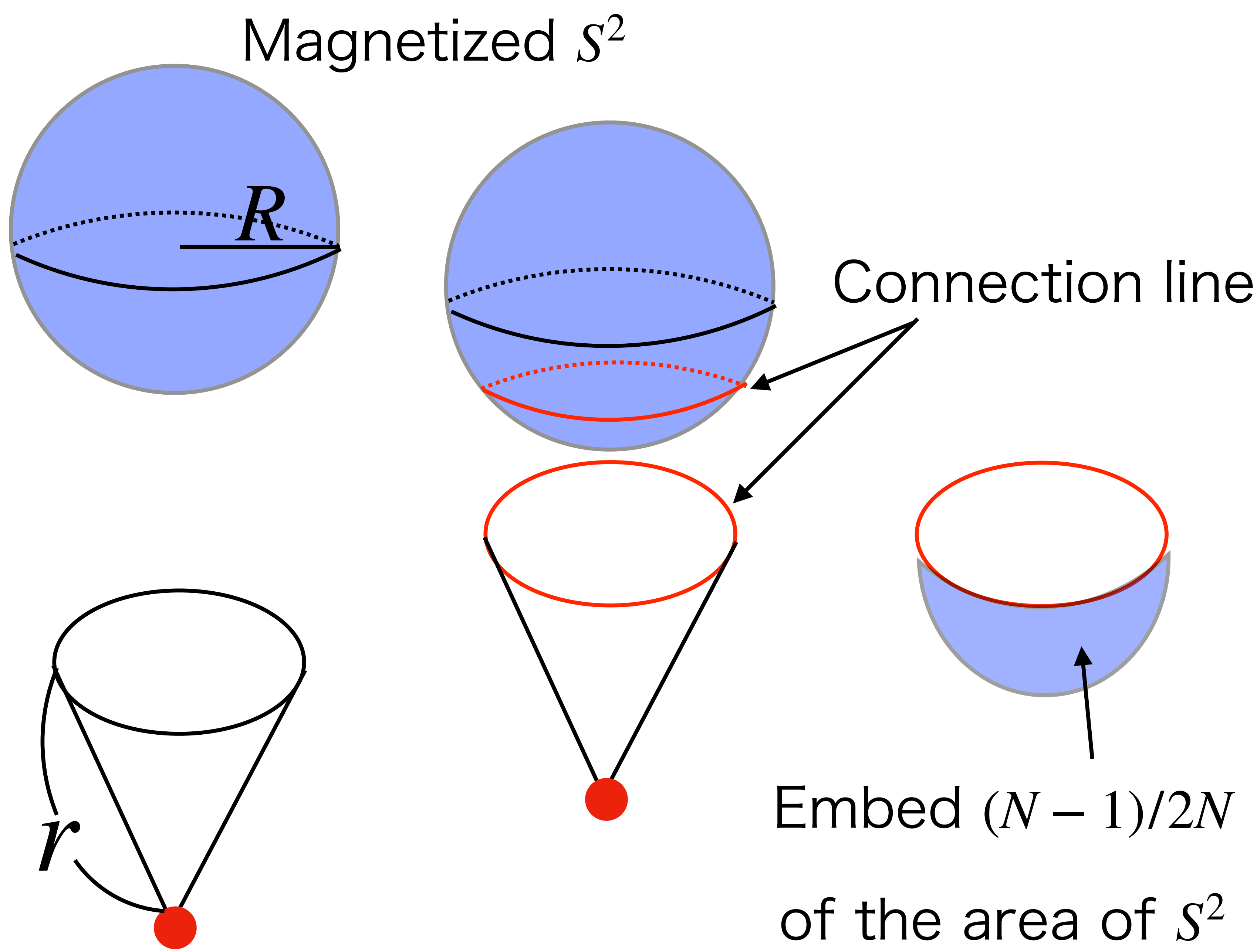
$$\psi^1_{s^2,-}(z') = \frac{f_-(z')}{(R^2+|z'|^2)^{\frac{M'+1}{2}}}$$

$f_+(z'), f_-(z')$: holomorphic function

4. Blow-up of T^2/Z_N orbifold

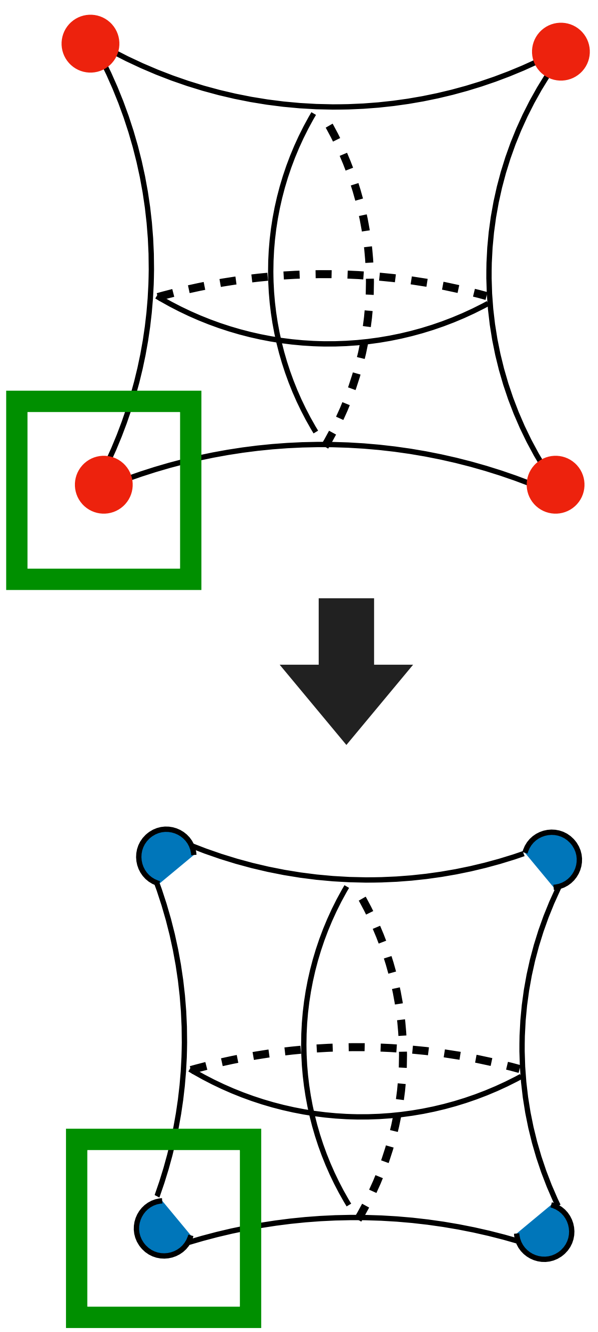
To apply AS index theorem, we introduce blow-up manifold

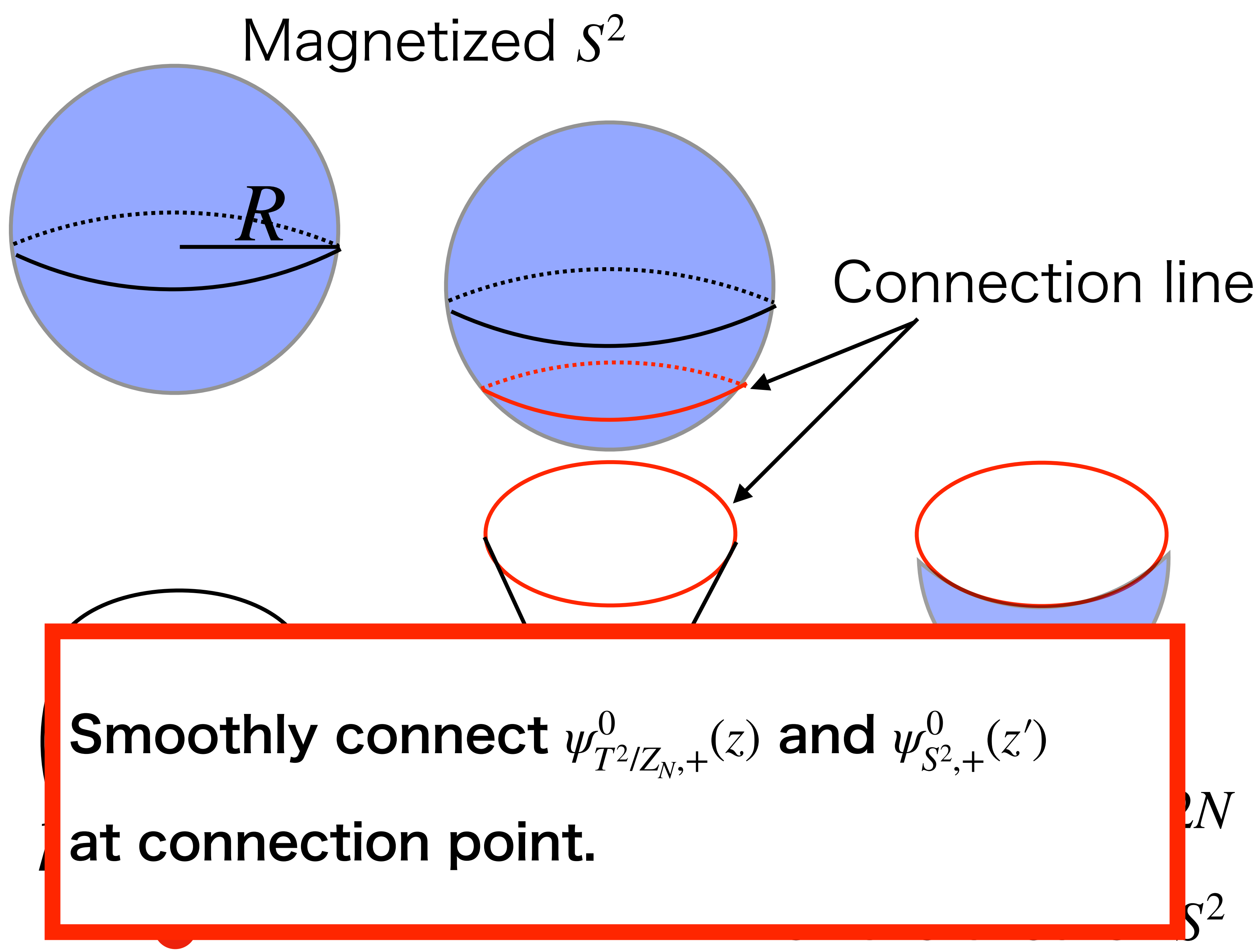




4. Blow-up of T^2/Z_N orbifold

To apply AS index theorem, we introduce blow-up manifold





Smoothly connect $\psi^0_{T^2/Z_N,+}(z)$ **and** $\psi^0_{S^2,+}(z')$ **at connection point.**

6. Blow-up manifold of T^2/Z_N orbifold

``Singular" gauge transformation

$$\tilde{\psi}_{T^2/Z_N,\pm}(z) = U_{\xi^F} U_{\xi^R} \psi_{T^2/Z_N,\pm}(z) \qquad U_{\xi^F} \propto \left(\frac{z}{\bar{z}}\right)^{\frac{\xi^F}{2}},\ U_{\xi^R} \propto \left(\frac{z}{\bar{z}}\right)^{\frac{\xi^R}{4}}$$

$\psi_{T^2/Z_N,+}(z)$

Field strength F

$$\int_{T^2/Z_N} \frac{F}{2\pi} = \frac{M}{N}$$

Gauge field A

+

Winding number χ_+

$\tilde{\psi}_{T^2/Z_N,+}(z)$

Field strength $F+\delta F$

$$\int_{T^2/Z_N} \frac{F+\delta F}{2\pi} = \frac{M}{N} + \frac{\xi^F}{N}$$

ξ^F : localized flux at fixed point

Gauge field $A+\delta A$ $\qquad \delta A = iU_{\xi^F} dU_{\xi^F}^{-1}$

Localized curvature ξ^R

$$\int_{T^2/Z_N\ fixedpoint} \frac{\delta R}{2\pi} = \frac{\xi^R}{N}$$

+ No winding number

4. Blow-up of T^2/Z_N orbifold

“Singular” gauge transformation

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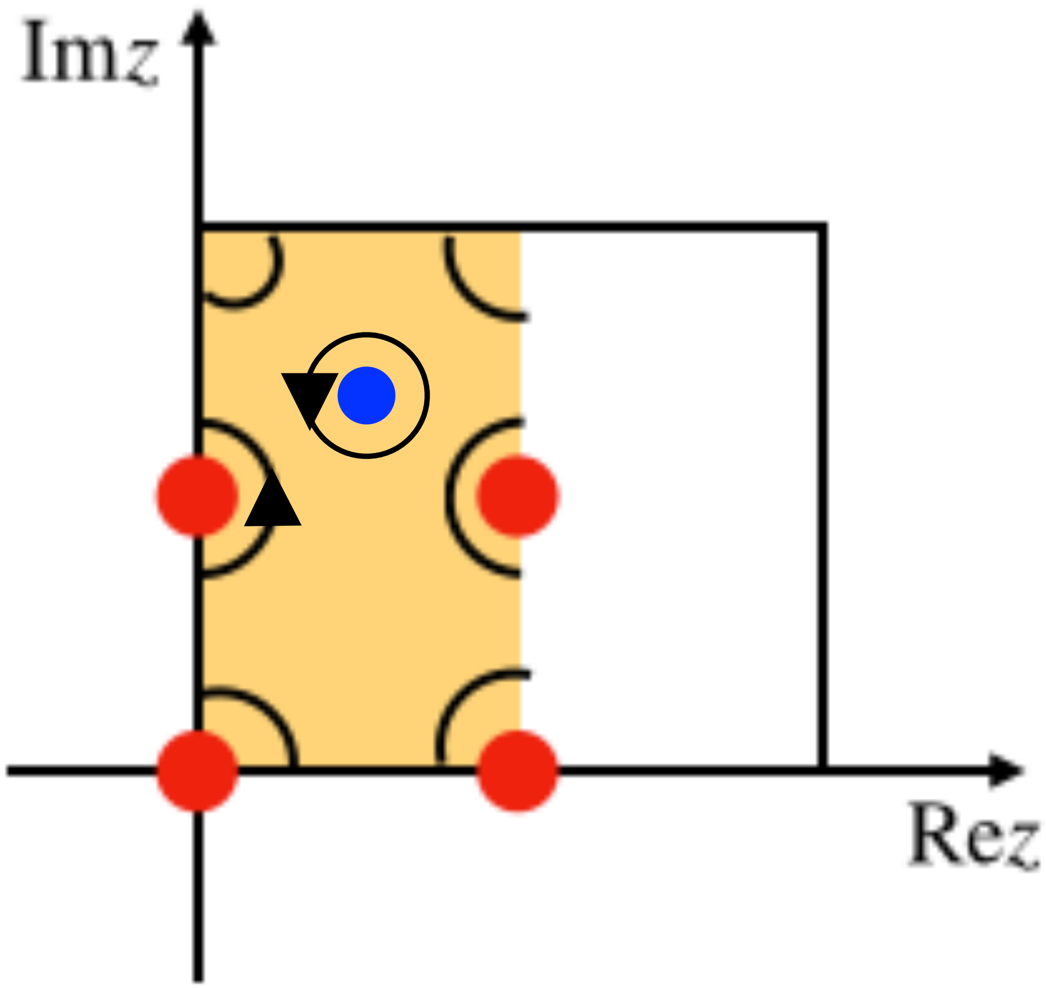
6. Blow-up of T^2/Z_N orbifold

Localized curvature at fixed point ξ^R



At fixed points
→Winding is closed at 180°

At the other points
→Winding is closed at 360°

At fixed points, there is deflection angle $2\pi - \frac{2\pi}{N} = 2\pi \frac{N-1}{N}$.

6. Blow-up of T^2/Z_N orbifold

Localized curvature at fixed point ξ^R

Gauss-Bonnet theorem

$$\int_{\mathscr{M}} \frac{R}{2\pi} = \chi(\mathscr{M})$$

R : Curvature χ : Euler characteristic

$+$

$$(deflection\ angle) = 2\pi\chi$$

→ Localized curvature at fixed point $\frac{\xi^R}{N} = \frac{N-1}{N}$

5. Conclusion

Reinterpretation of index formula on T^2/Z_N orbifold

$$(\text{generation } \#) = \frac{M}{N} - \frac{V_+}{N} + 1$$

M : flux quanta、V_+ : sum of winding number χ_+ at fixed points

Winding number has contributions of localized flux and curvature.

Index theorem implies the existence of l new zero modes = localized modes

$$\sum_{z_j^f} \frac{\xi^F}{N} = - \sum_{z_j^f} \frac{\chi_+}{N} + \sum_{z_j^f} \frac{\xi^R}{2N} + l = - \frac{V_+}{N} + 1 + l$$