Line Bundle Cohomology

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Outline:

1. The need for ultra-fast methods for cohomology computations
   in heterotic model building
2. Some results on line bundle cohomology for CY threefolds
   with Picard numbers 4 and 5

First order objective of StringPheno

Explain the core structures in Particle Physics: interactions, spectrum and flavour parameters

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Why three families of quarks and leptons?

Why a hierarchy of masses?

$$m_{\text{top}} = 172{,}440 \text{ MeV}, \; m_e = 0.511 \text{ MeV}, \; m_\nu < 0.1 \cdot 10^{-6} \text{ MeV}$$

Problem 1

topology

+

geometry

$X, V, \tilde{V}$

complicated map →

particle spectrum

+

couplings

cohomology formulae: provide great simplifications to this map in heterotic string compactifications

Spectrum and couplings

- correct spectrum: all SM multiplets (plus, possibly RH neutrinos) and no exotics;

  Required topological data: $h^{\bullet}(X, V^{\wedge n})$

- correct couplings: all SM couplings and no dangerous operators, such as those inducing fast proton decay or rapid flavour-changing processes

Useful: additional symmetries inherited from the compactification, (e.g. extra U(1)s or discrete symmetries) to forbid (or at least suppress) dangerous operators, and dictate certain forms for the couplings, so as to explain the observed patterns for fermion masses and mixing angles

Example: SU(5) models from the $E_8 \times E_8$ heterotic string on smooth CY3 with split bundles

$$V = \bigoplus_{a=1}^{5} L_a$$

SU(5) multiplets with S(U(1)[5]) charges, $(q_1, q_2, q_3, q_4, q_5) \sim (q_1, q_2, q_3, q_4, q_5) + (q, q, q, q, q)$

pattern of charges: $\mathbf{10}_{\mathbf{e}_a},\ \overline{\mathbf{5}}_{\mathbf{e}_a+\mathbf{e}_b},\ \mathbf{5}_{-\mathbf{e}_a-\mathbf{e}_b},\ \mathbf{1}_{\mathbf{e}_a-\mathbf{e}_b},\ H^u_{-\mathbf{e}_a-\mathbf{e}_b},\ H^d_{\mathbf{e}_a+\mathbf{e}_b}$

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Bottom-up example:

Spectrum: $\mathbf{10}_1, \ \mathbf{10}_2, \ \mathbf{10}_3, \ 3\,\overline{\mathbf{5}}_{1,2}, \ H^u_{-3,-4}, \ H^d_{3,4}, \ \mathbf{1}_{4,-3}, \ \mathbf{1}_{3,-2}, \ \mathbf{1}_{3,-1}, \ \mathbf{1}_{5,-4}$

$\mu-$term: forbidden with any number of singlet insertions;

Proton decay: dim-4 op. $\overline{\mathbf{5}}\overline{\mathbf{5}}\mathbf{10}$ forbidden; dim-5 op. suppressed: $\mathbf{1}_{5,-4}\mathbf{1}_{4,-3}\mathbf{1}_{4,-3}\overline{\mathbf{5}}_{1,2}\mathbf{10}_3\mathbf{10}_3\mathbf{10}_3 + O(\mathbf{1}^4)$

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$H^u_{-3,-4}$

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$$H^u_{-3,-4}\left(\begin{array}{c|ccc} & \mathbf{10}_1 & \mathbf{10}_2 & \mathbf{10}_3 \\ \hline \mathbf{10}_1 & \mathbf{1}_{4,-3}\mathbf{1}_{3,-1}\mathbf{1}_{3,-1} & \mathbf{1}_{4,-3}\mathbf{1}_{3,-1}\mathbf{1}_{3,-2} & \mathbf{1}_{4,-3}\mathbf{1}_{3,-1} \\ \mathbf{10}_2 & \mathbf{1}_{4,-3}\mathbf{1}_{3,-1}\mathbf{1}_{3,-2} & \mathbf{1}_{4,-3}\mathbf{1}_{3,-2}\mathbf{1}_{3,-2} & \mathbf{1}_{4,-3}\mathbf{1}_{3,-2} \\ \mathbf{10}_3 & \mathbf{1}_{4,-3}\mathbf{1}_{3,-1} & \mathbf{1}_{4,-3}\mathbf{1}_{3,-1} & \mathbf{1}_{4,-3} \end{array}\right)$$

Down Yukawa matrix, $H^d\overline{\mathbf{5}}^{(i)}\,\mathbf{10}^{(j)}$:

$$H^d_{3,4}\left(\begin{array}{c|ccc} & \mathbf{10}_1 & \mathbf{10}_2 & \mathbf{10}_3 \\ \hline \overline{\mathbf{5}}_{1,2} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3}\mathbf{1}_{3,-1} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3}\mathbf{1}_{3,-2} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3} \\ \overline{\mathbf{5}}_{1,2} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3}\mathbf{1}_{3,-1} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3}\mathbf{1}_{3,-2} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3} \\ \overline{\mathbf{5}}_{1,2} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3}\mathbf{1}_{3,-1} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3}\mathbf{1}_{3,-2} & \mathbf{1}_{5,-4}\mathbf{1}_{4,-3} \end{array}\right)$$

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WIP with Lucas Leung, Thomas Harvey, Andre Lukas:

identify viable patterns of charge assignments and their string realisations; see also Dudas&Palti, 0912.0853

Problem 2

topology
+
geometry

complicated
map

particle spectrum
+
couplings

huge space of possibilities for $X, V, \tilde{V}$

Solution: heuristic search methods (RL, GAs, QA…)

Line bundle cohomology formulae

based on 2306.03147 with Steve Abel, Thomas Harvey, Andre Lukas and Luca Nutricati
also based on earlier work with Callum Brodie, James Gray, Andre Lukas, Fabian Ruehle
(see 2112.12107 for a review)

Computational cost of line bundle cohomology (using spectral sequences):

$$\sim O\left(\left(\rho(X)^{\dim(X)}\deg(L)^{\dim(X)}\right)^3\right)$$

Example: for a line bundle of (multi)-degree 10 on a Calabi-Yau threefold with $h^{1,1}(X) = \rho(X) = 4$ Kähler parameters, the estimate is

$\sim 10^{14}$ elementary operations

which reaches the limits of a standard machine

A Picard number 4 example: the tetraquadric manifold

$$X^{(4,68)}_{7862} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left[\begin{matrix} 2 \\ 2 \\ 2 \\ 2 \end{matrix} \right]$$

$\{J_1, J_2, J_3, J_4\}$: generators of the Kähler cone inherited from $(\mathbb{P}^1)^{\times 4}$

$L \to X$ line bundle with $c_1(L) = k^i J_i$

Euler characteristic:

$$\chi(X, L) = \int_X \mathrm{ch}(L) \cdot \mathrm{td}(X) = 2 \sum_{i=1}^{4} \left(k_i + \prod_{j \neq i} k_j \right)$$

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Formula: $$h^0(X,L) = \begin{cases} 1, \text{ for } k^i = 0 \\ \chi(X,L), \text{ for } L \in \mathscr{K}(X) \\ \chi(X,L') \text{ , if } \exists c_1(L') = M_{i_1}M_{i_2}\ldots M_{i_n}c_1(L) \in \overline{\mathscr{K}(X)}\setminus \mathscr{O}_X \\ (1+k^A)(1+k^B) \text{ , if } k^A, k^B \geq 0 \text{ and the other two integers vanish} \\ 0, \text{ otherwise} \end{cases}$$

$$M_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix},$$

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Effective cone of X: infinite number of simplicial cones, corresponding to the Kähler cones of isomorphic CY-threefolds obtained from X through a sequence of flops, concretely by the action of an infinite group generated by $\{M_i\}$.

For flops of generic CICYs, see 2112.12106 C.Brodie, AC, A.Lukas and F.Ruehle.

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Higher cohomologies: $$h^1(X,L) - h^2(X,L) = h^0(X,L) - h^3(X,L) - \chi(X,L)$$

$$h^3(X,L) = h^0(X,L^*)$$

On X almost all line bundles satisfy either $h^1(X,L) = 0$ or $h^2(X,L) = 0$.

Exception: two zero entries, one entry $k^A < -1$and one entry $k^B > 1$. In this case, $h^1(X,L) + h^2(X,L) = -2(1 + k_A k_B)$

A Picard number 5 example: the split tetraquadric manifold

$$X^{(5,45)}_{7447} = \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{matrix} \left[\begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix} \right]$$

Very similar story: infinite number of flops through each of the five walls of the Kähler cone.

Simple formula for the zeroth cohomology in terms of the Euler characteristic and five matrices.

Formula for higher cohomologies follows the same pattern.

Cohomology formulae for smooth quotients of these manifolds by discrete group actions?
Partial results: yes - equivariant cohomology dimensions split as evenly as possible between the various reps.

Summary and Outlook

- Heterotic line bundle models offer a rich phenomenology.

- Fast line bundle cohomology computations: an essential tool for model building, especially when coupled with heuristic methods of search.

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- Line bundle cohomology dimensions capture a great deal of geometric information about the base manifold, such as threefold flops, rigid divisors, certain GW invariants.

- A better method of encoding the patterns arising in line bundle cohomology: generating functions.
  Examples include surfaces, threefolds and higher dimensional manifolds of (almost) Fano, Calabi-Yau and general type.