Topological equivalence and invariants of Calabi-Yau threefolds

New invariants, and identification of topological data.

UNIVERSITY OF OXFORD

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based on (upcoming) work with Andre Lukas, Thomas Harvey, Aditi Chandra, and Andrei Constantin

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- We are able to construct **many** Calabi-Yau threefolds (CY3s) - various methods [Candelas et al, '88] [Kreuzer, Skarke, '02]. Only a **single** topological type of each of CY1s and CY2s.

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  - How many pairs of Hodge numbers in Kreuzer-Skarke? $30{,}108$. At **least this many distinct manifolds.**
  - Very loose upper bounds on KS - 1.65×10^{428} manifolds. Mostly one polytope!

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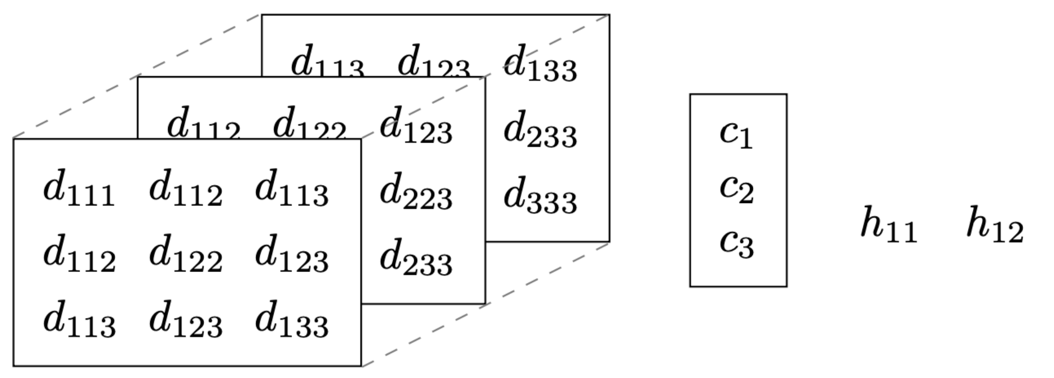

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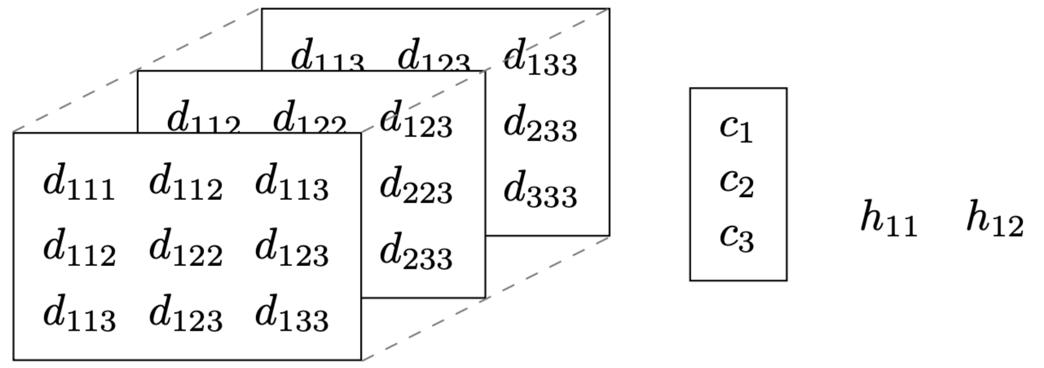

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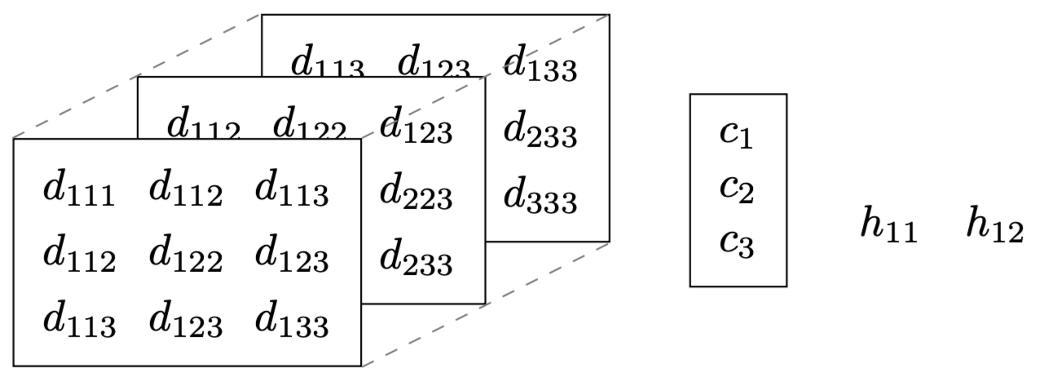

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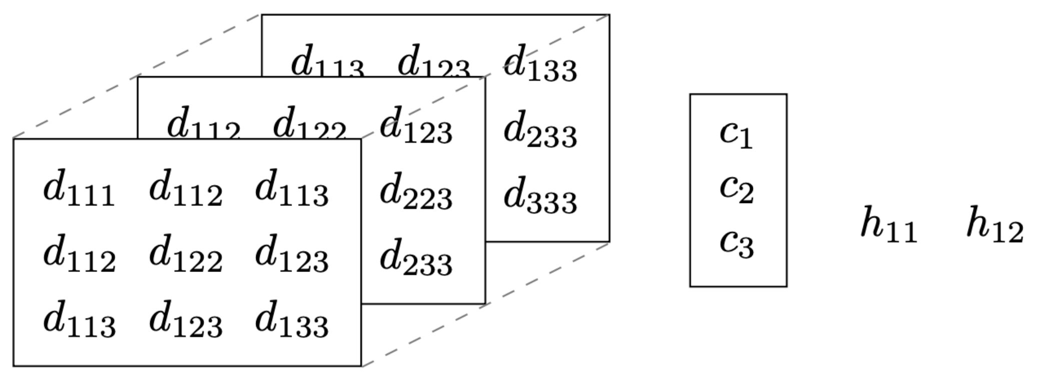

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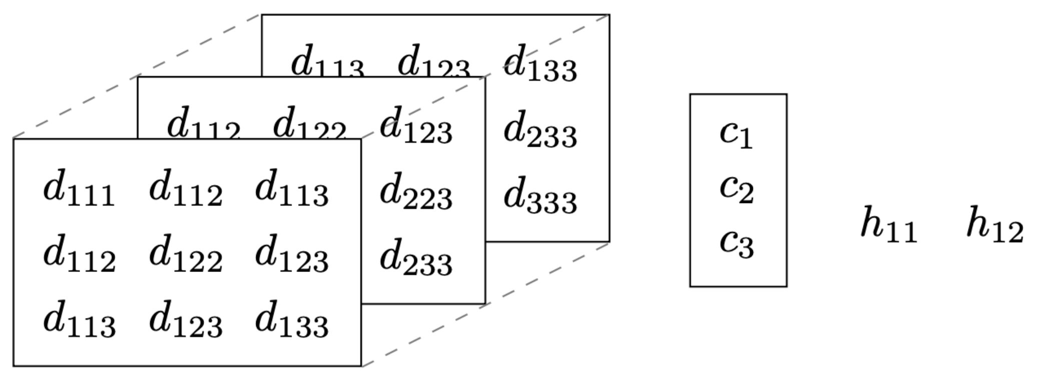

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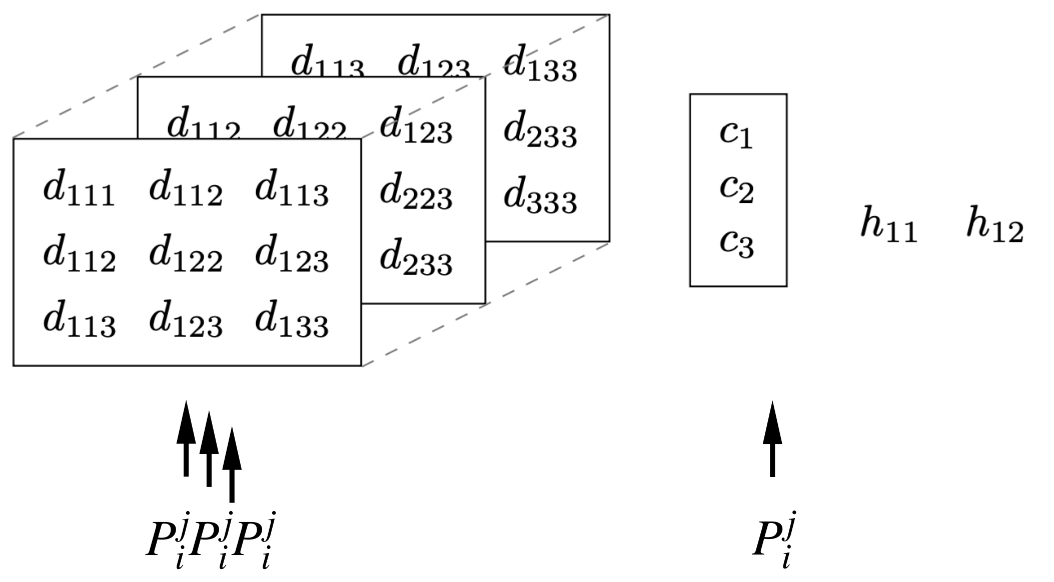

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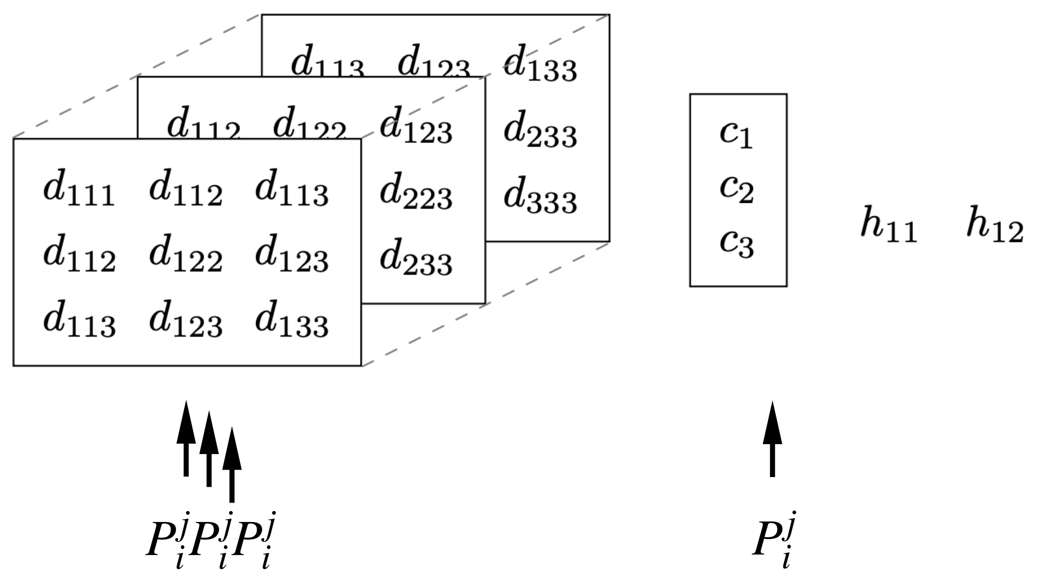

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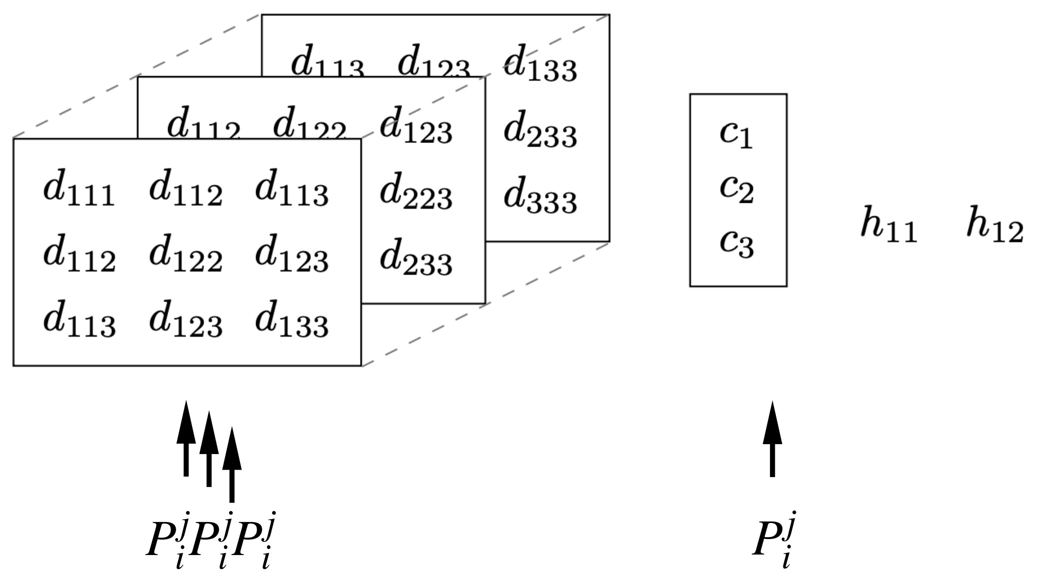

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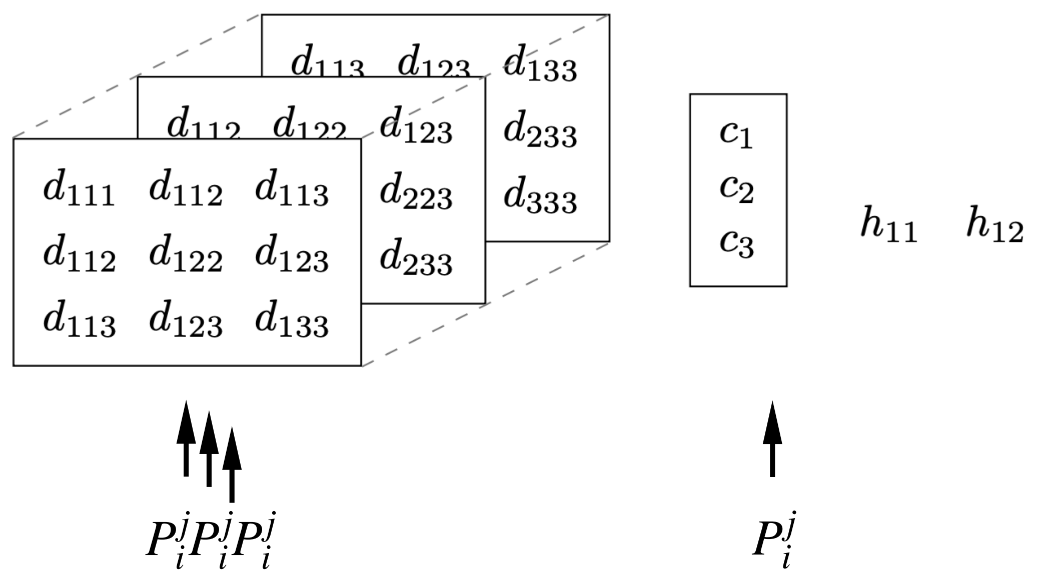

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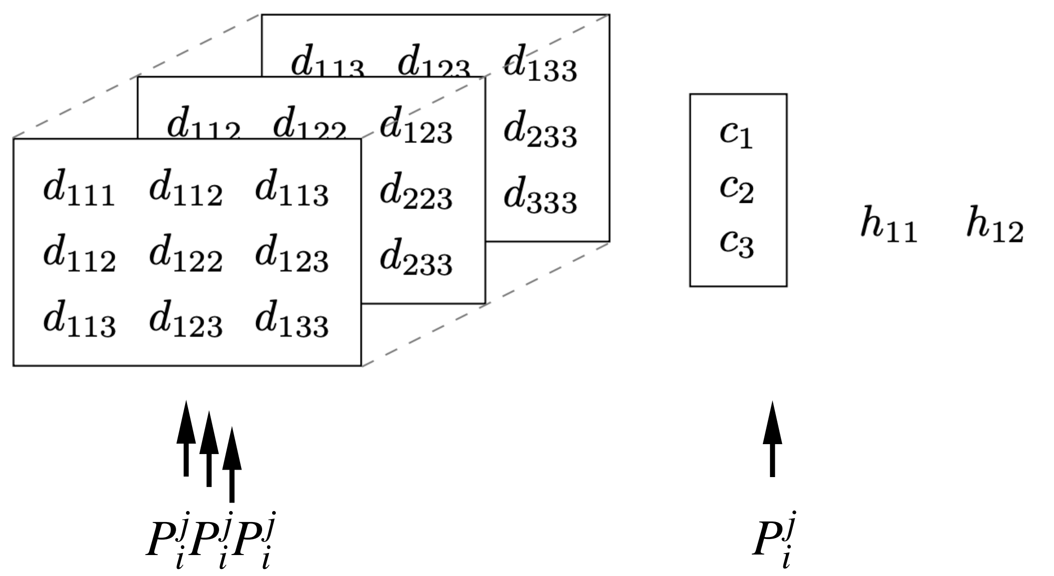

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$$SL(N,\mathbb{Z}) \subset SL(N,\mathbb{R}) \subset SL(N,\mathbb{C})$$

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Quadratic and cubic forms are **much more complicated.**

Clearly $\mathrm{gcd}(\{d_{rst}\})$ and $\mathrm{gcd}(\{c_r\})$ are preserved [Hubsch, '92].

- Some more complicated GCD invariants exist, related to (e.g.) the GCD of the diagonal elements $\{d_{rrr}\}$.
- Other invariants related to limiting mixed Hodge structures in infinite distance limits exist [Grimm, Ruehle, van de Heisteeg, '19]. For the cases in this talk, these are less powerful than those discussed below.

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Linears: $\mathrm{Sym}^1(\mathbf{4}) = \mathbf{4}$

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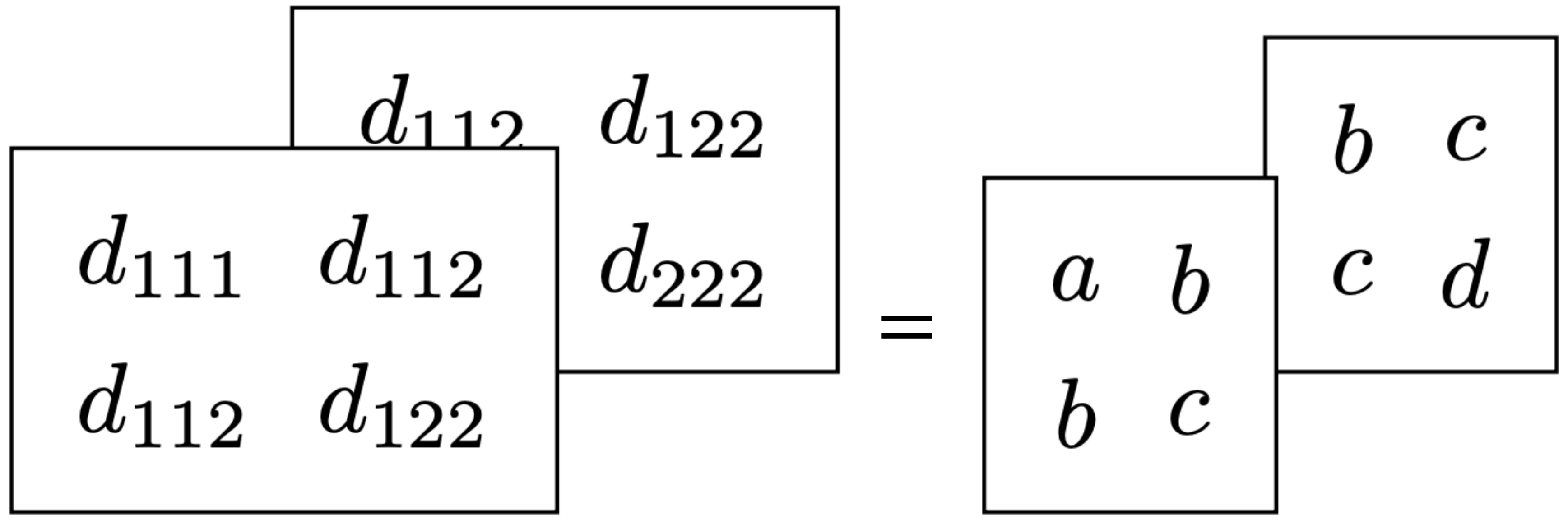

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$$\begin{pmatrix} d_{111} & d_{112} \\ d_{112} & d_{122} \end{pmatrix}, \begin{pmatrix} d_{112} & d_{122} \\ & d_{222} \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} b & c \\ c & d \end{pmatrix}$$

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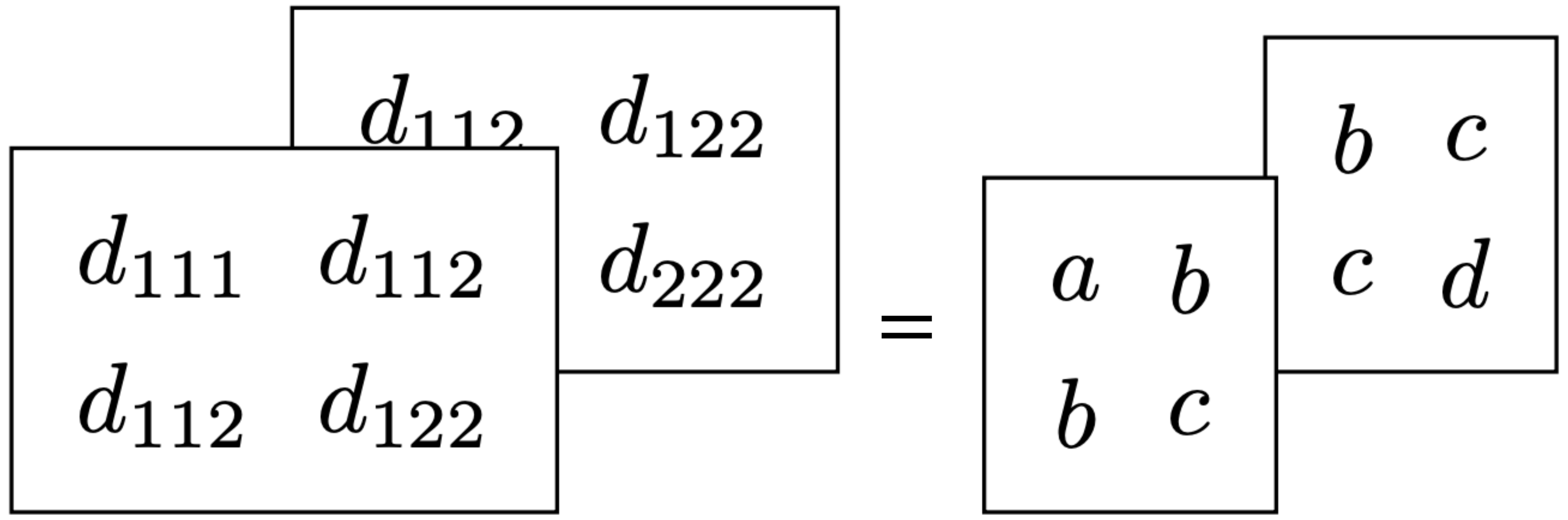

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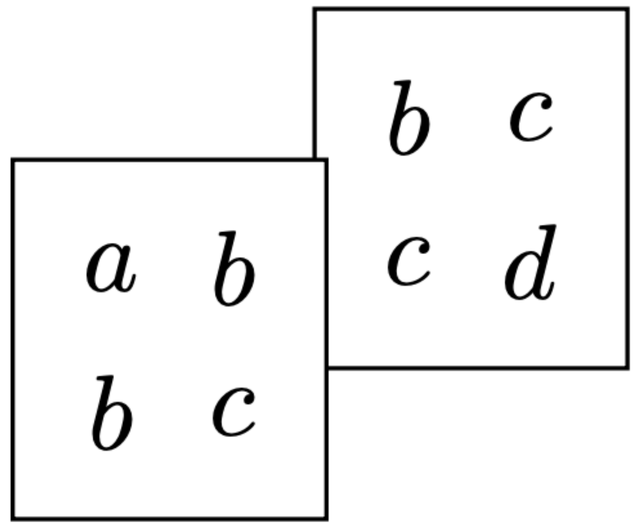

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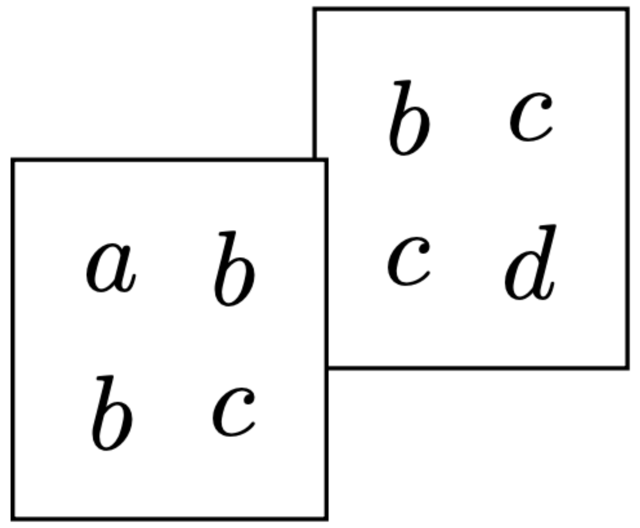

- General problem: identify the **singlets** of $\mathrm{Sym}^k(\mathrm{Sym}^3(\mathbf{N}))$ for all k and $N = h^{11}$. Gives us the singlets of $GL(N, \mathbb{Z})$ (up to sign).

* *also have* $\mathrm{Sym}^k(\mathrm{Sym}^3(\mathbf{N}) \otimes \mathbf{N})$ *if combined with the data from* c_2

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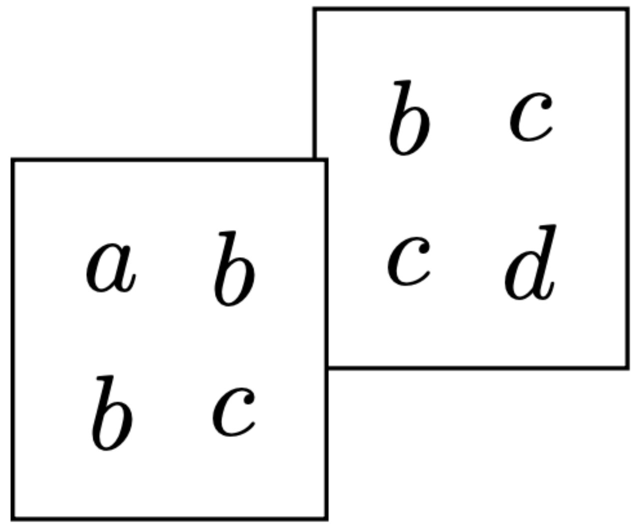

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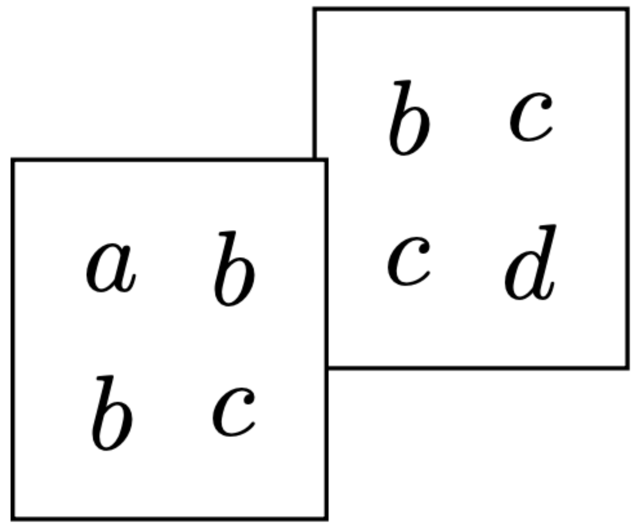

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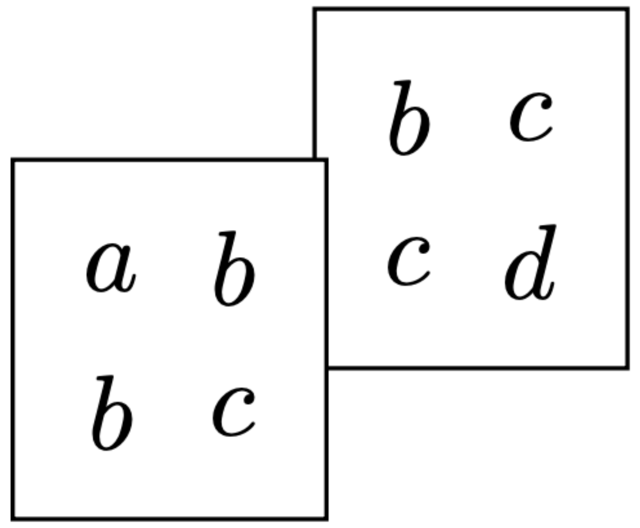

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$h^{1,1}$	1	2	3	4	5	6	7
Degrees	1	4	4,6	8,16*	10	10*,12*	14*
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- Known lowest degrees of singlets and total number of (algebraically independent) singlets expected. Starred have not been determined explicitly

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The results, for $h^{11} \leq 5$:

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- E.g. - the degree-10 $h^{11} = 5$ invariant has 7000 independent coefficients, each multiplying an S_5 orbit of a particular monomial. **It is large.**

10D_Invariant = 14592 a07 a08 a09 a11 a12 a14 a21 a22 a24
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a08 a09 a11^2 a12 a14 a21 a22^2 a24 +
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a21^2 a30^2 + a08^2 a09^2 a11^2 a22^2 a30^2 +
a07^2 a08^2 a14^2 a22^2 a30^2 + a07^2 a09^2 a11^2
a24^2 a30^2 + a07^2 a08^2 a12^2 a24^2 a30^2) +
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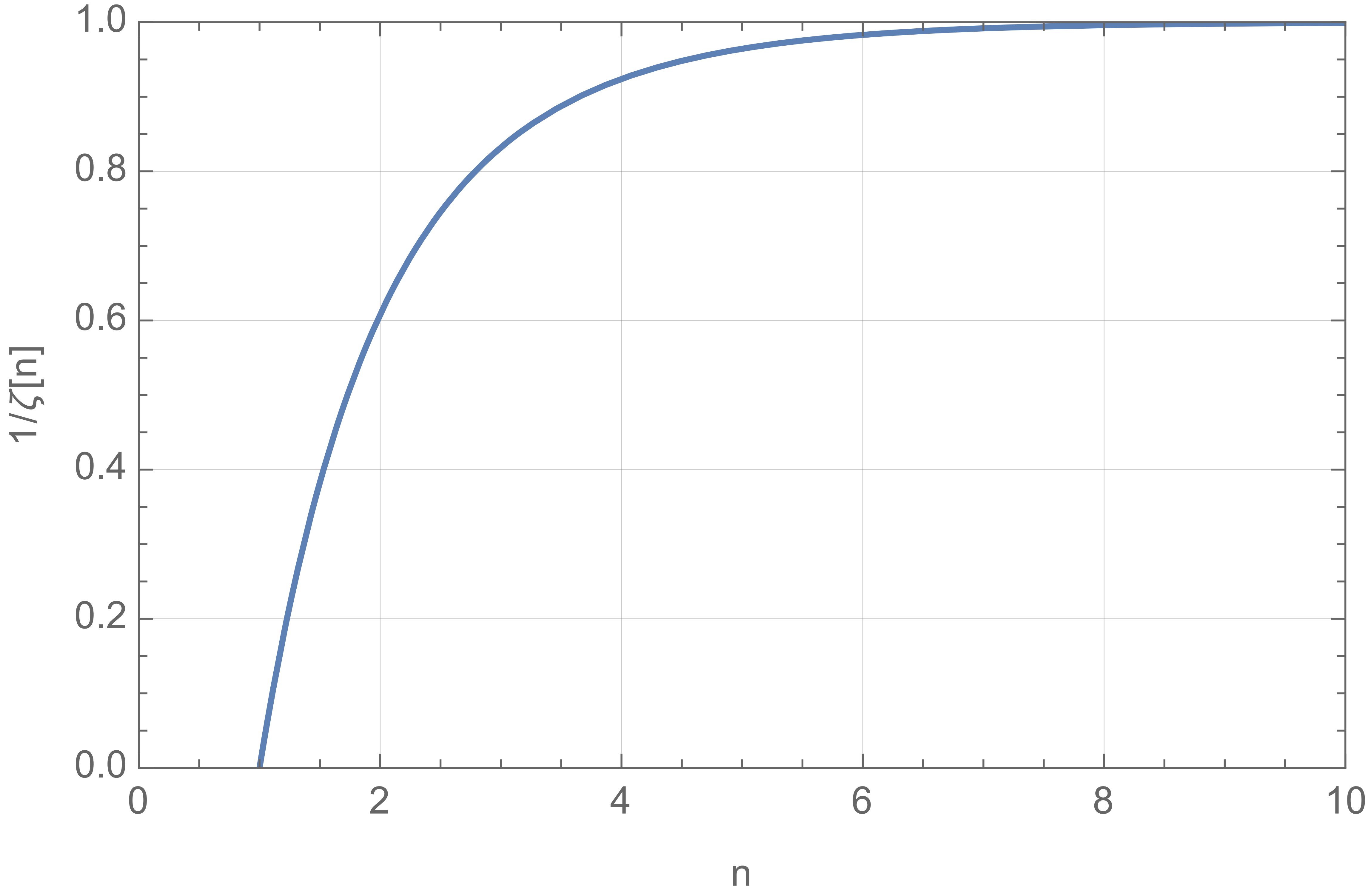

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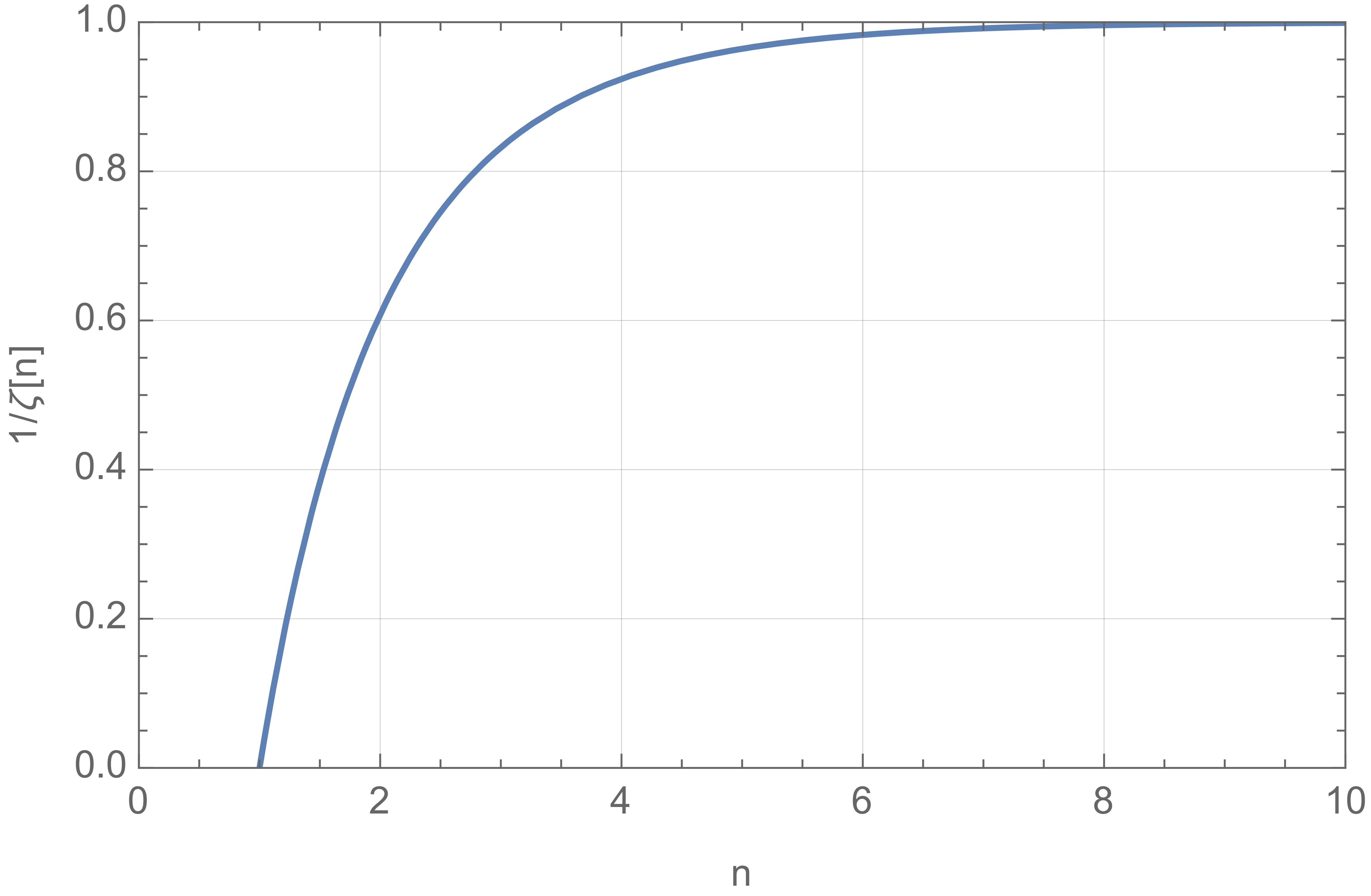

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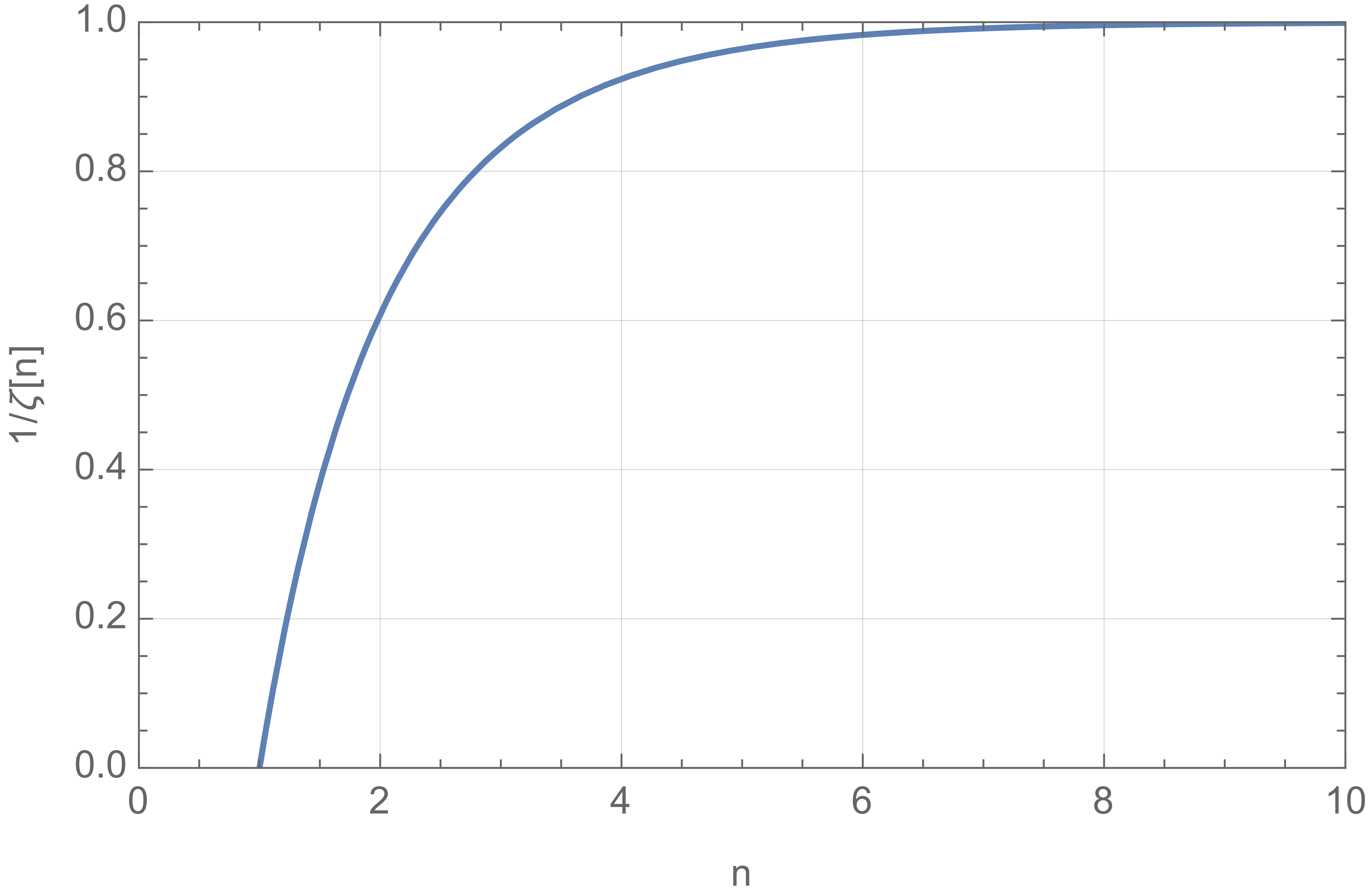

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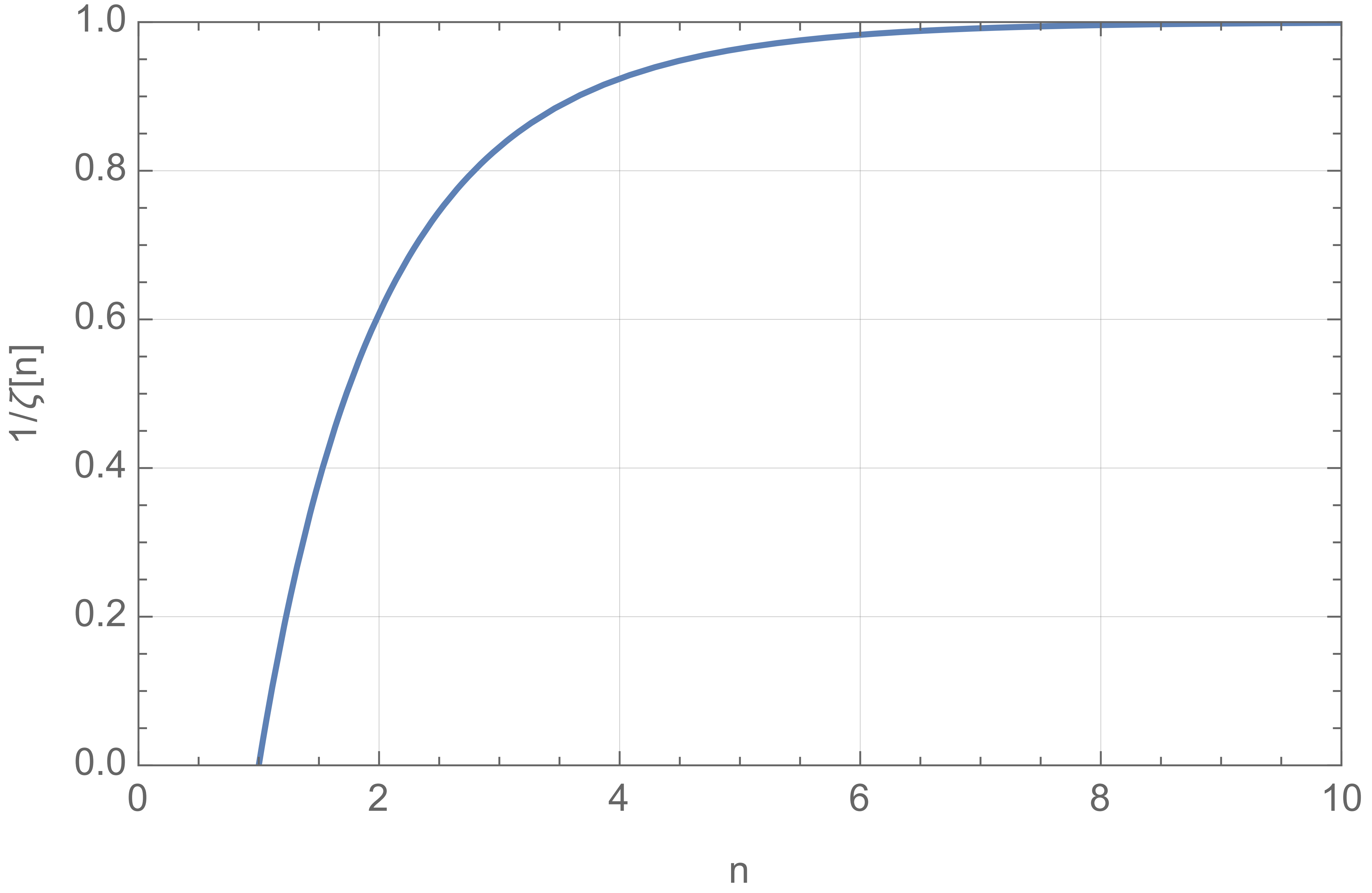

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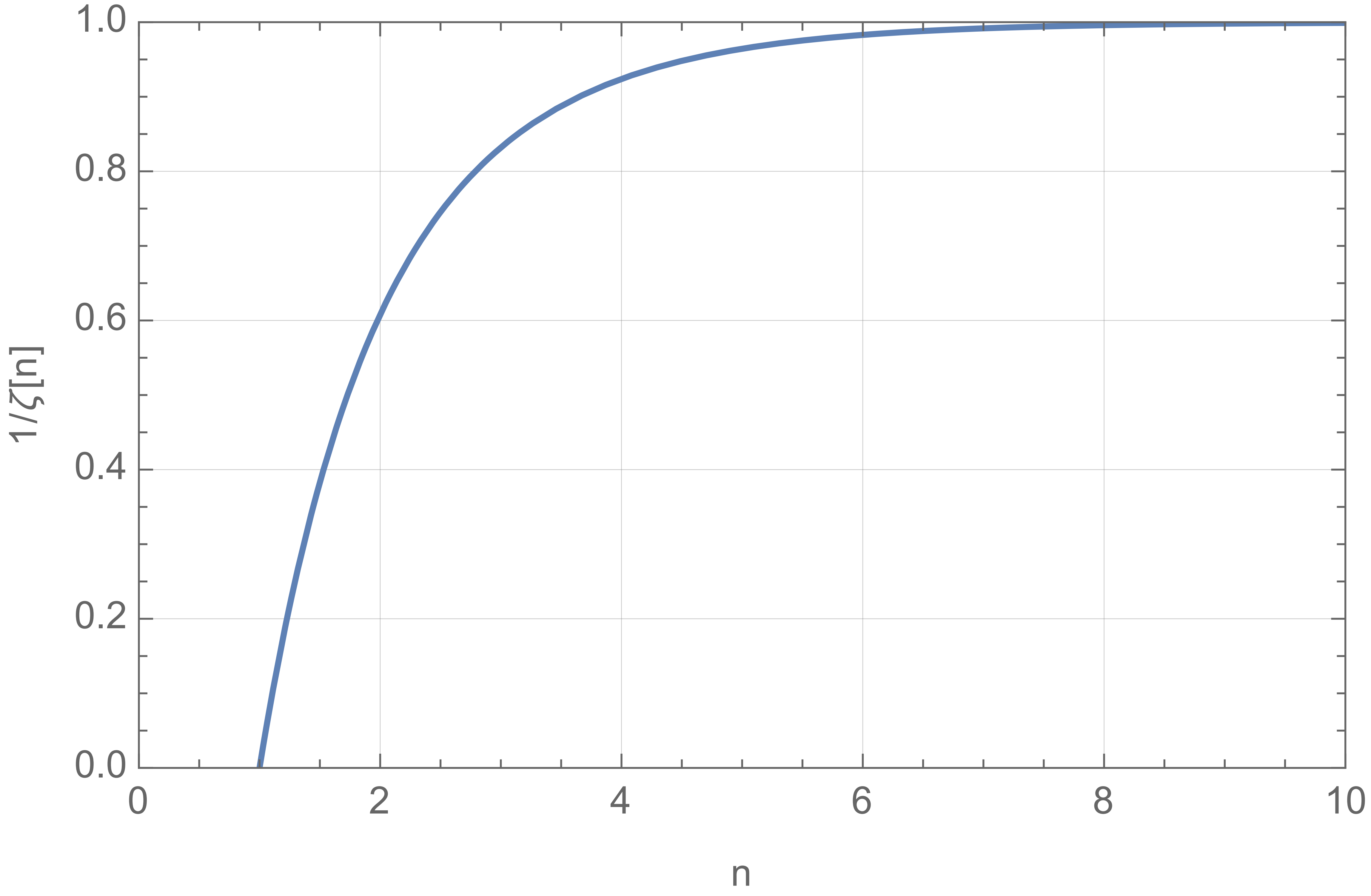

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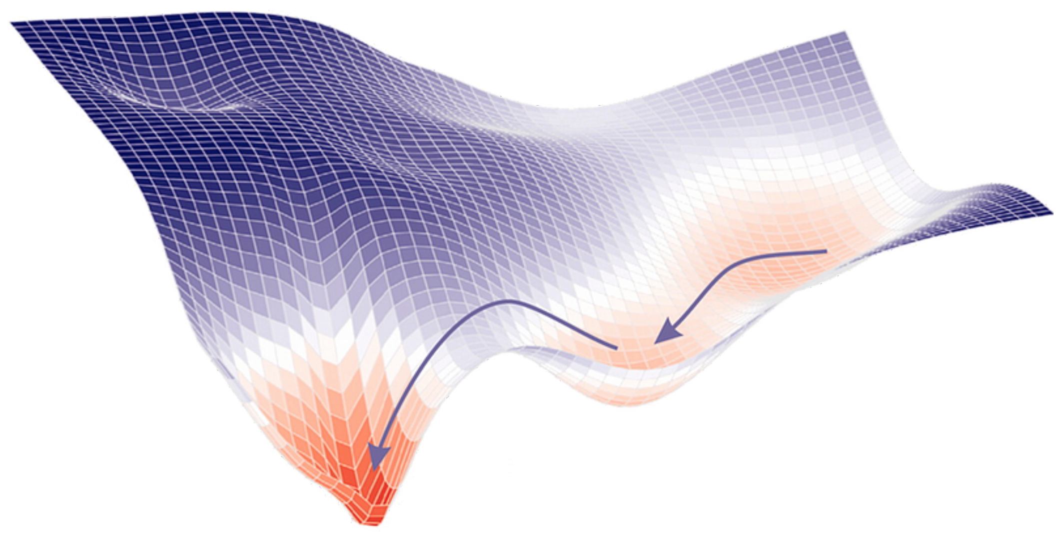

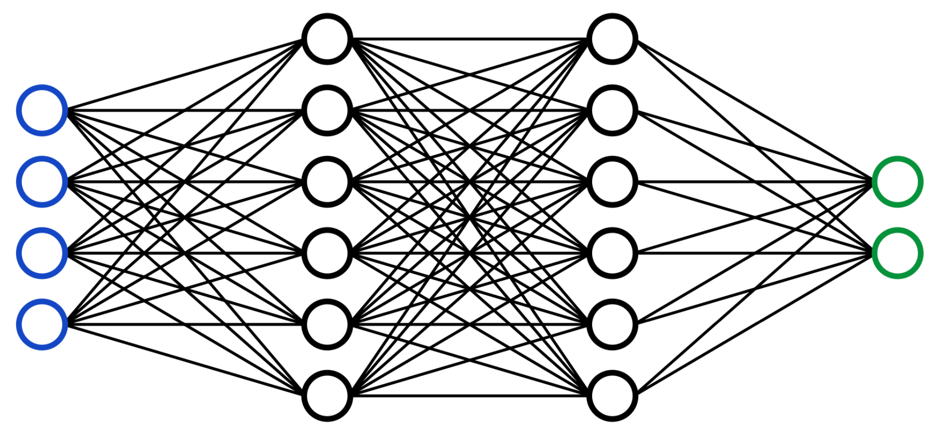

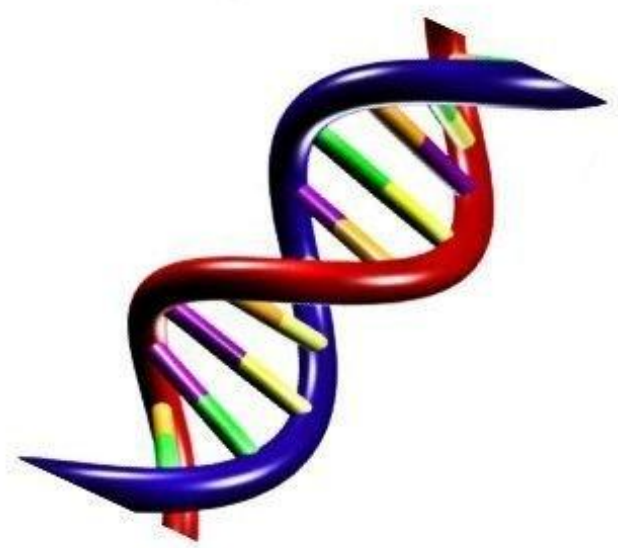

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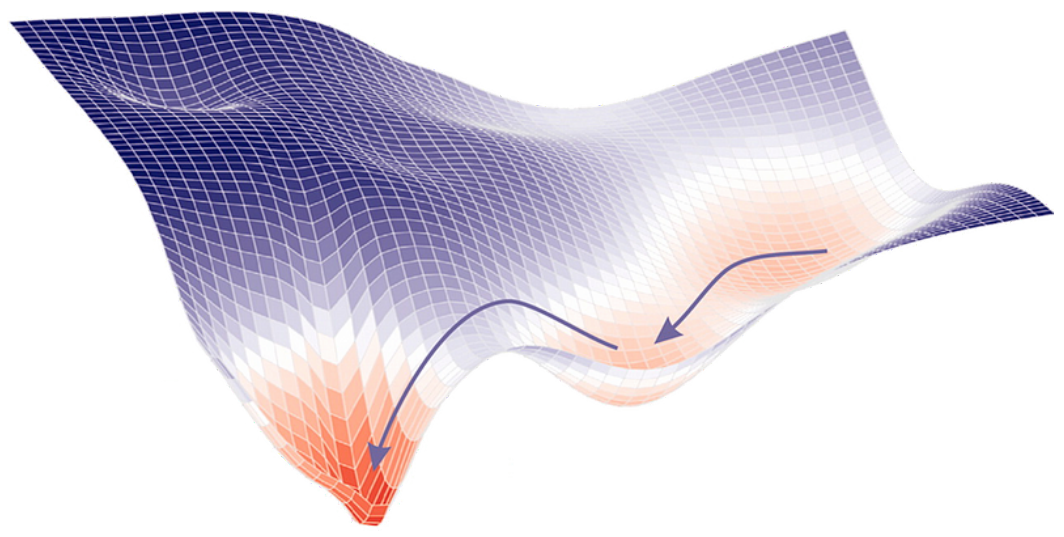

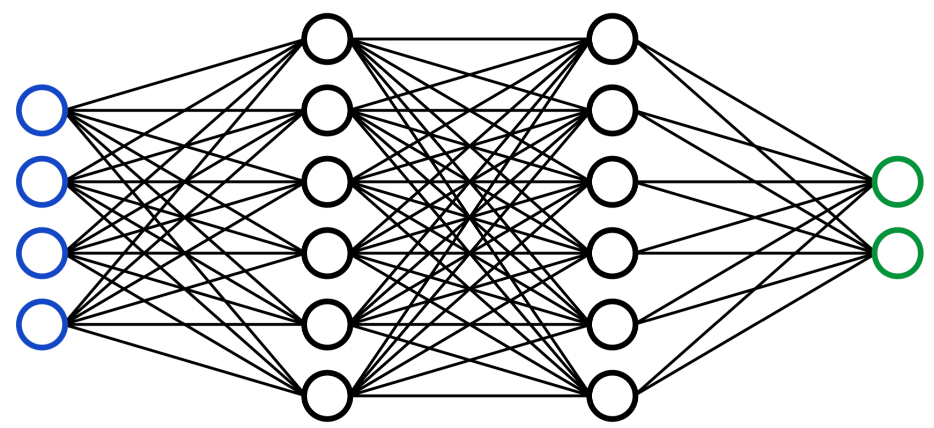

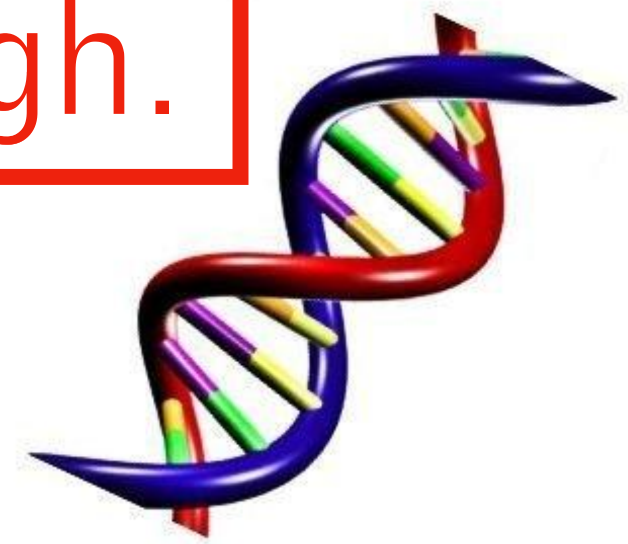

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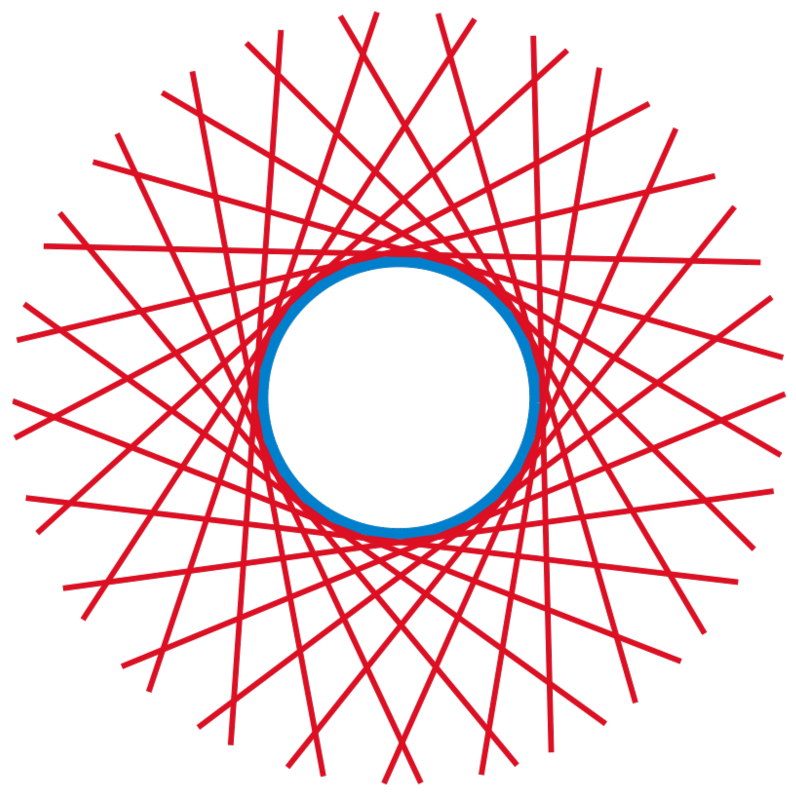

Use line bundle cohomology to distinguish between X and X'. The quantity that is sensitive only to $c_2(X)$ and $d_{rst}(\mathrm{X})$ is the **Euler characteristic.**

Full line bundle cohomology contains too much info (e.g. about the complex structure)/too slow to generate.

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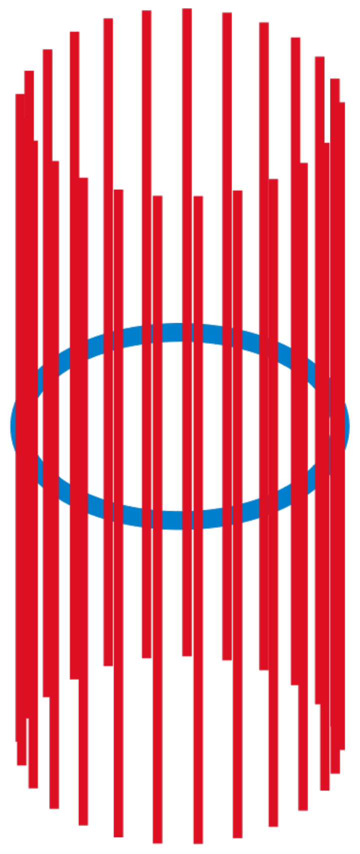

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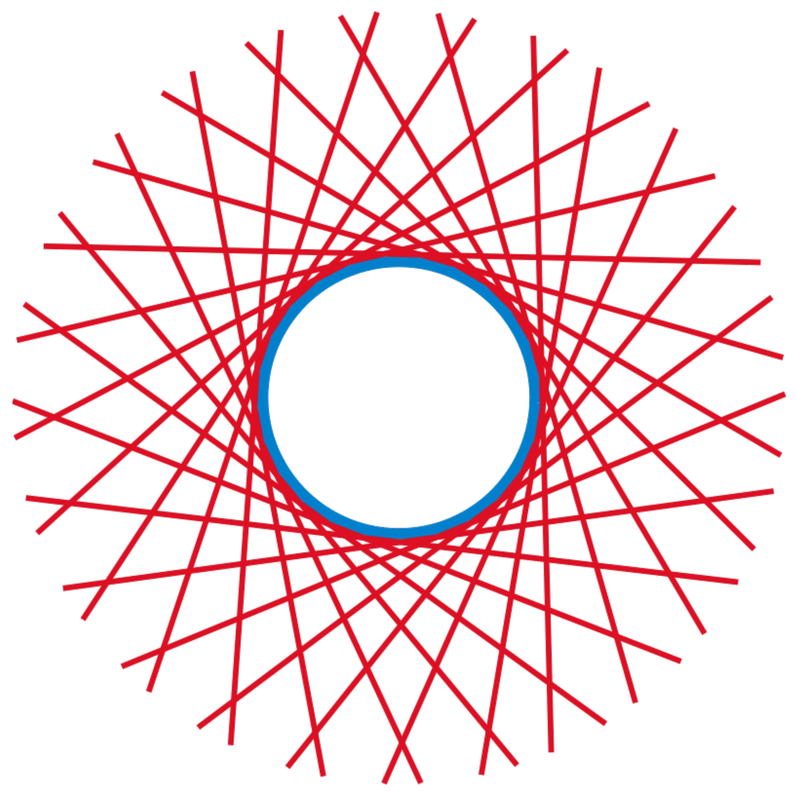

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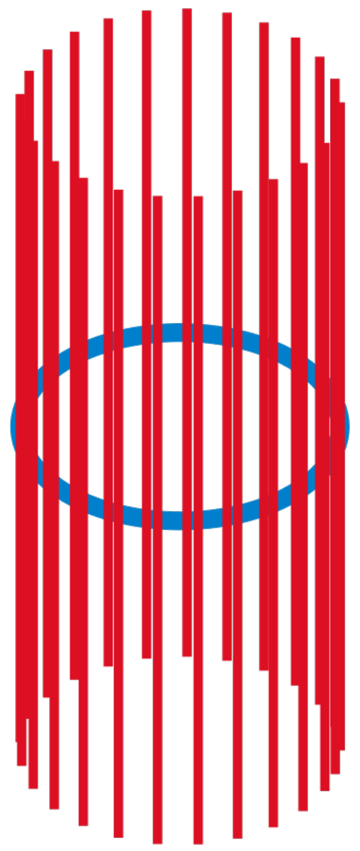

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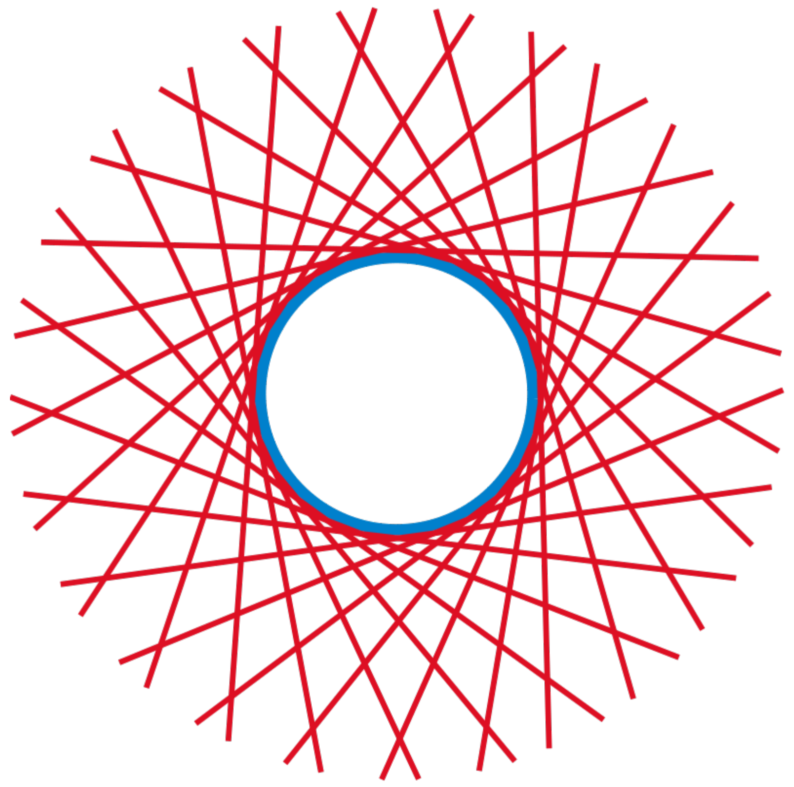

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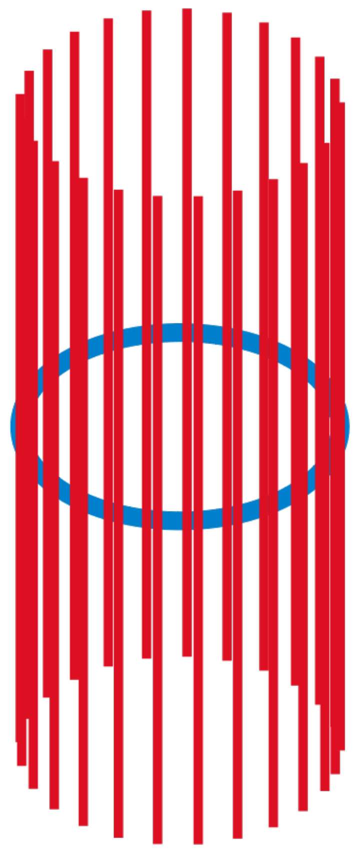
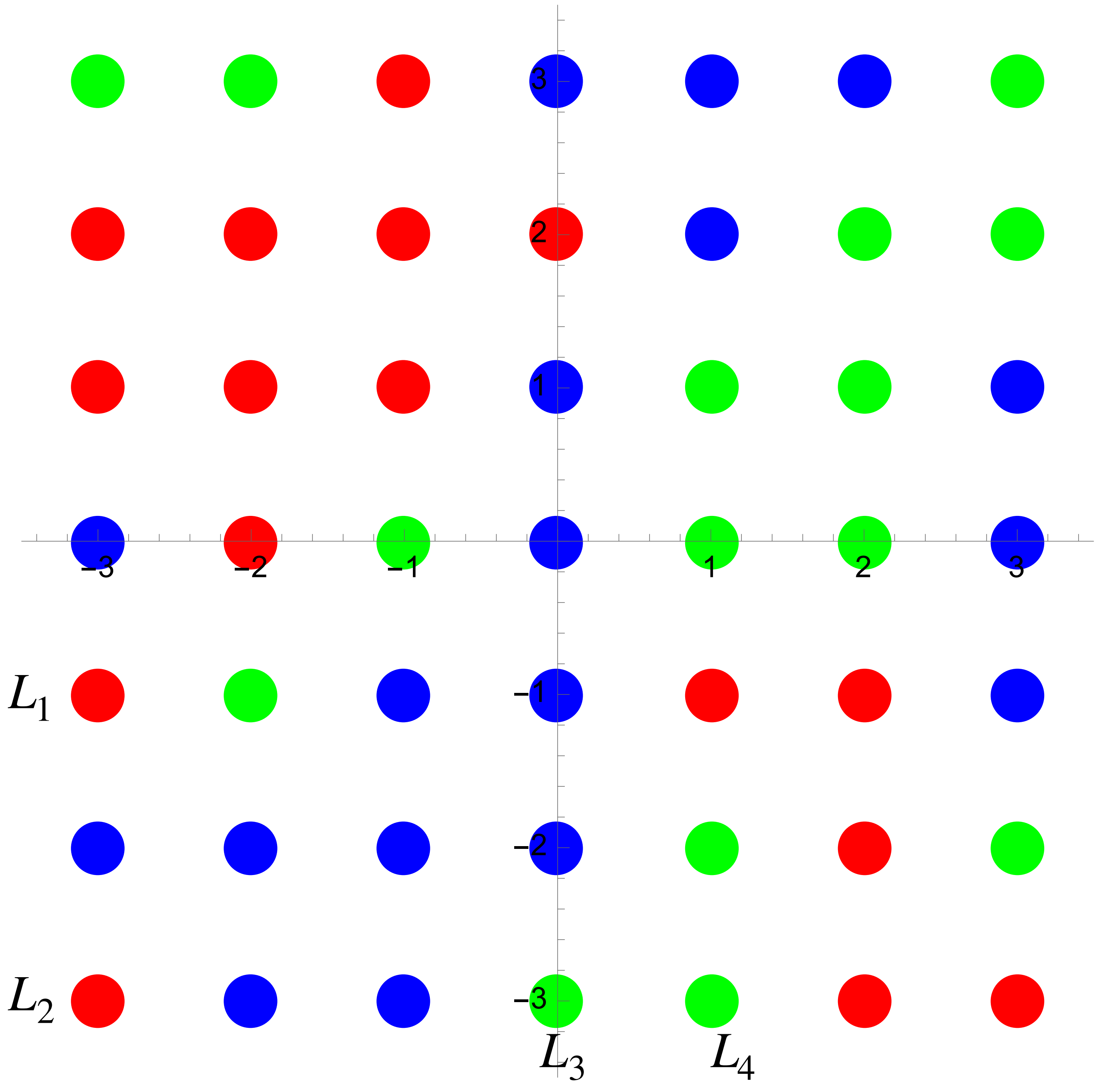


Line bundle $c_1^r(L)$ is an **integer** vector k^r, and so we can **reframe the problem** to that of finding which L (or k^r) on X are mapped to which L' (or k'^r) on X'. **Do this in a box.**

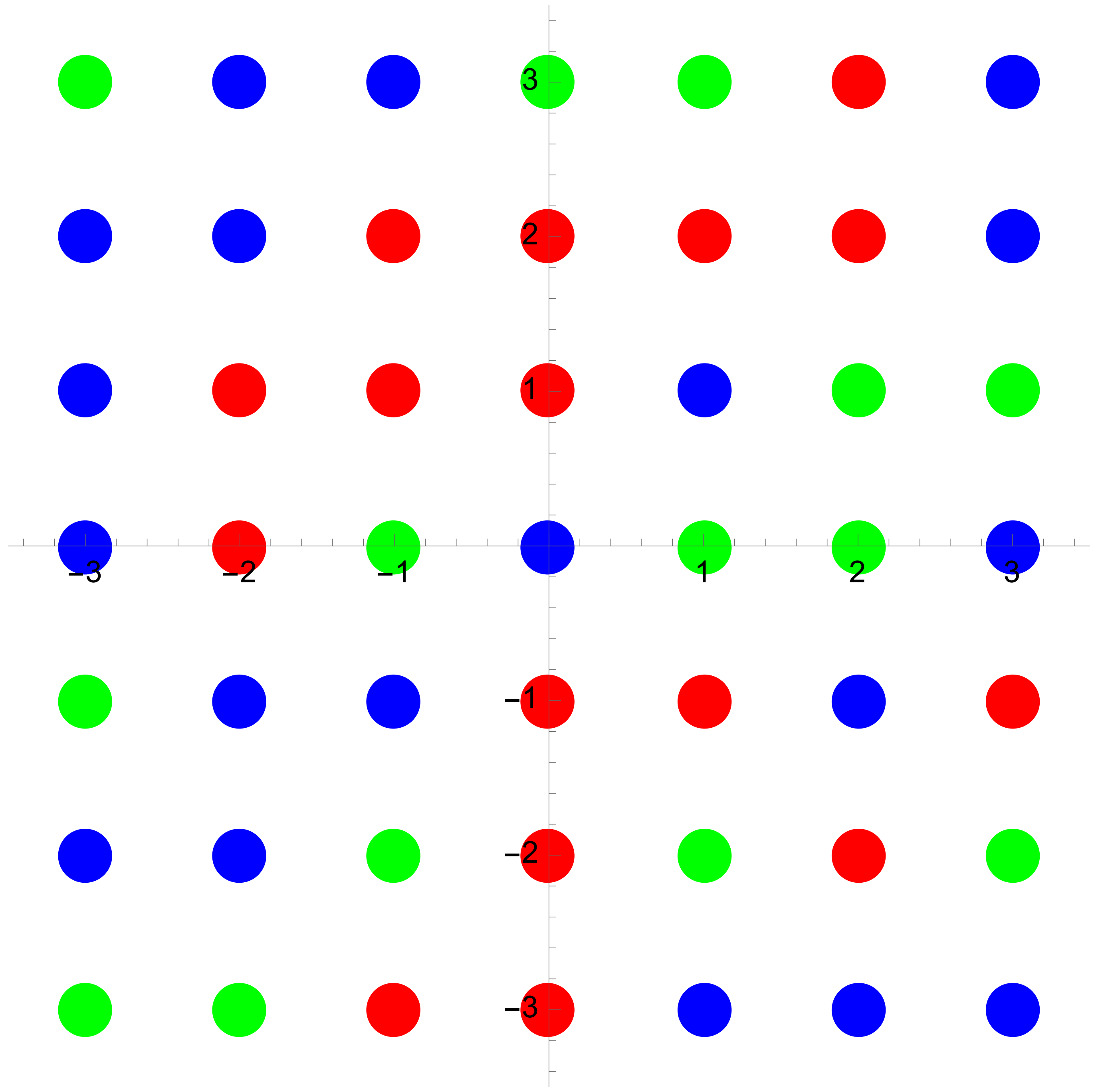
*In this case we can split the Euler characteristic into ‘linear’ and ‘cubic’ parts

Find the map

X data



X' data



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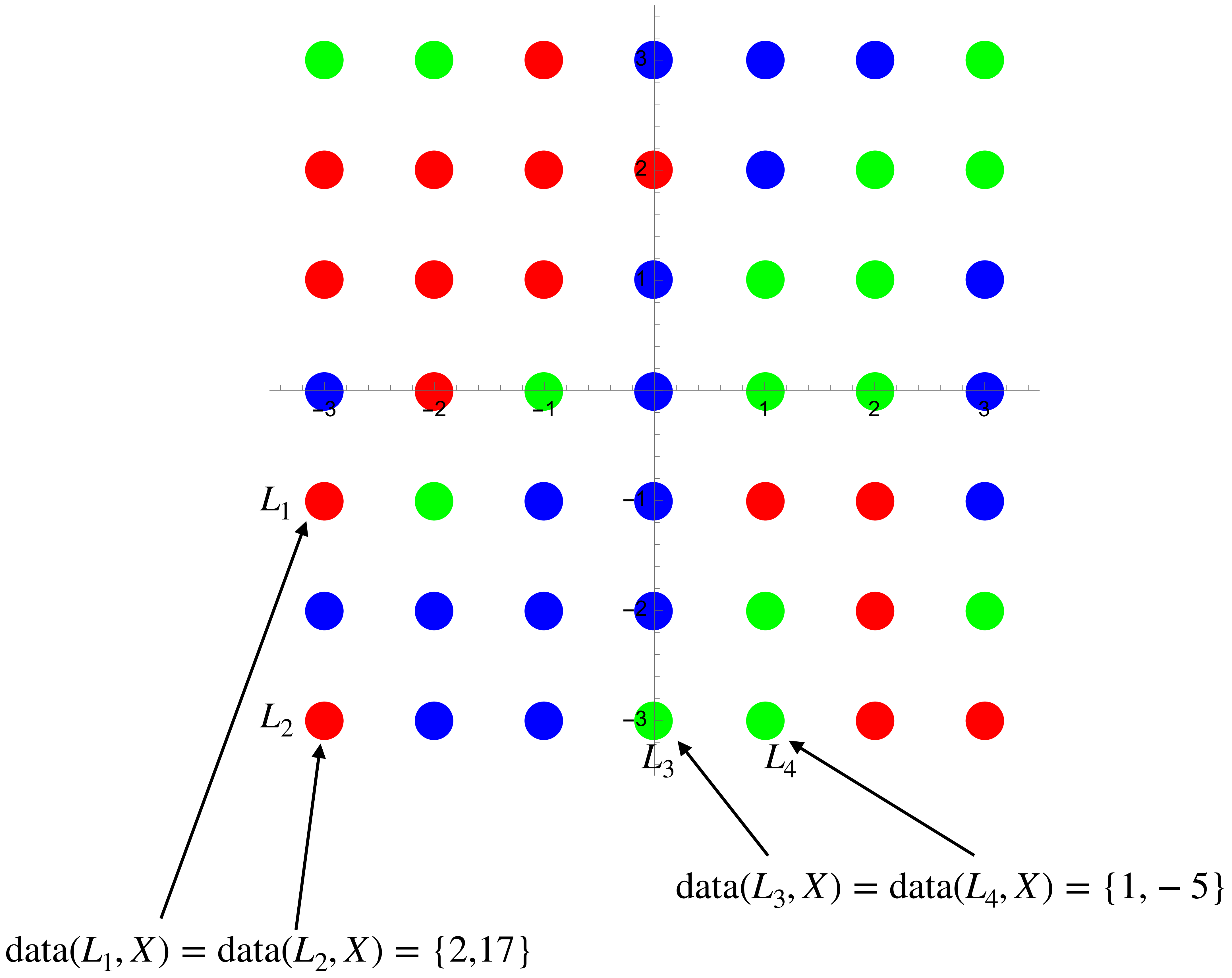

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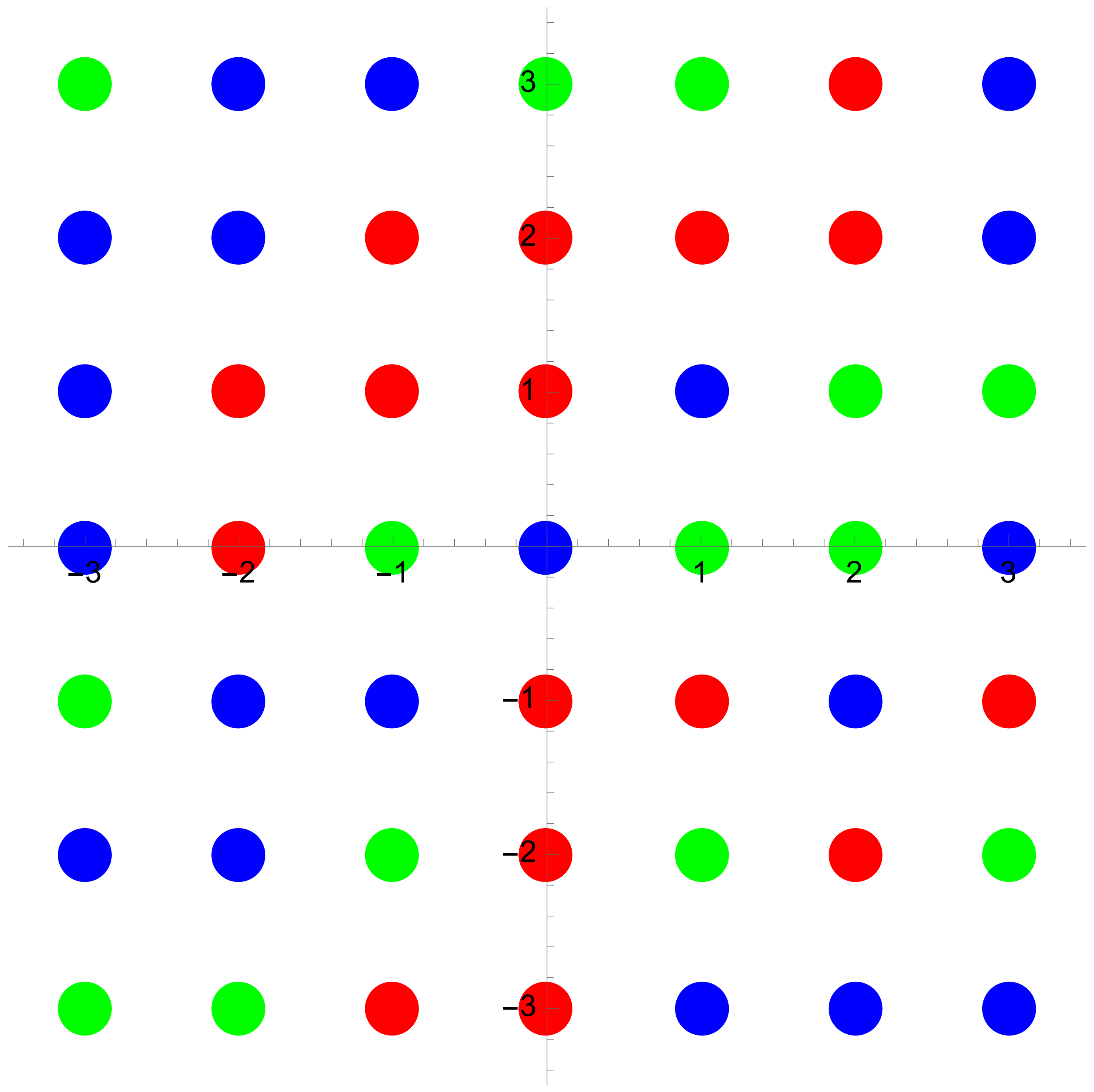

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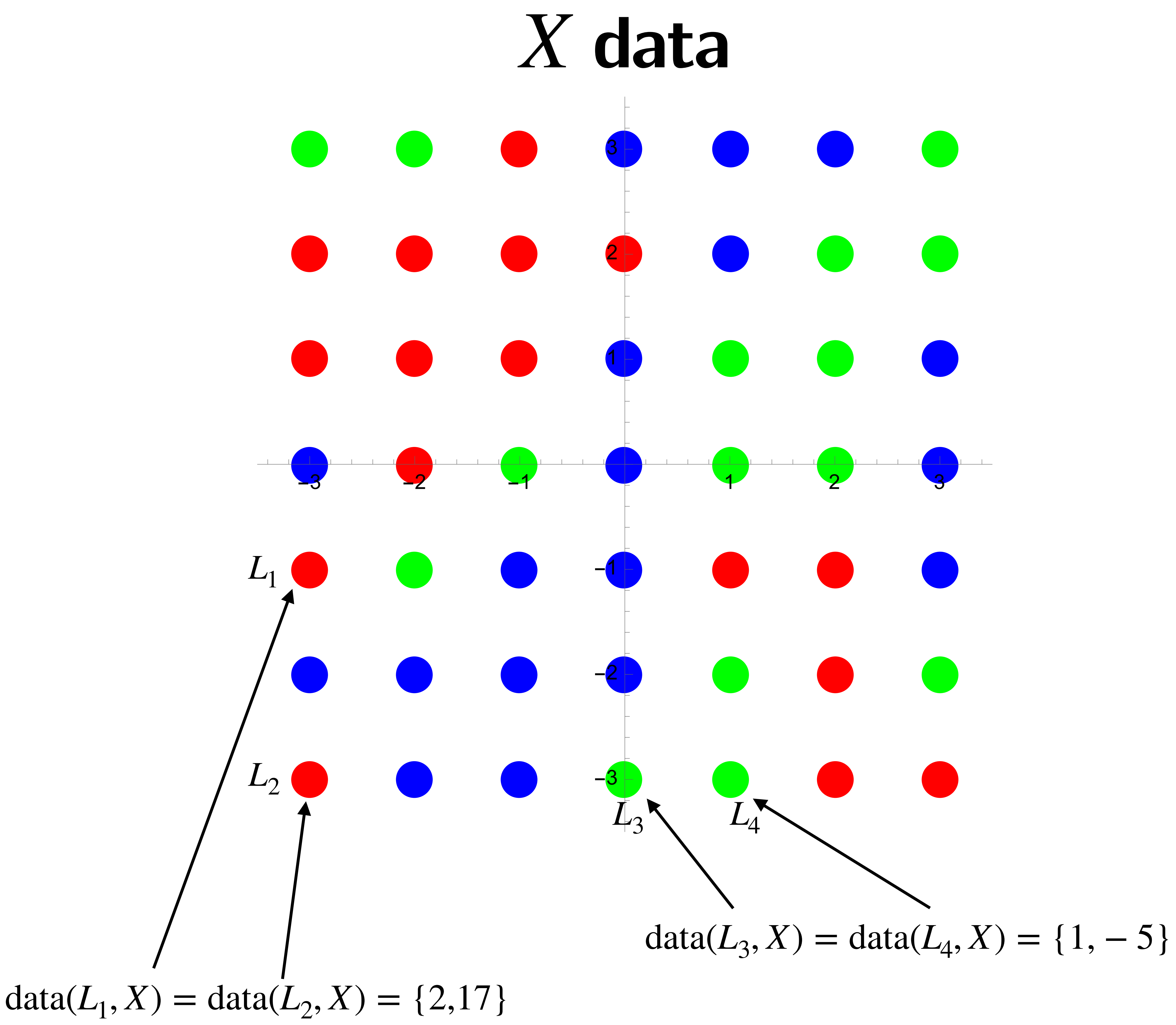
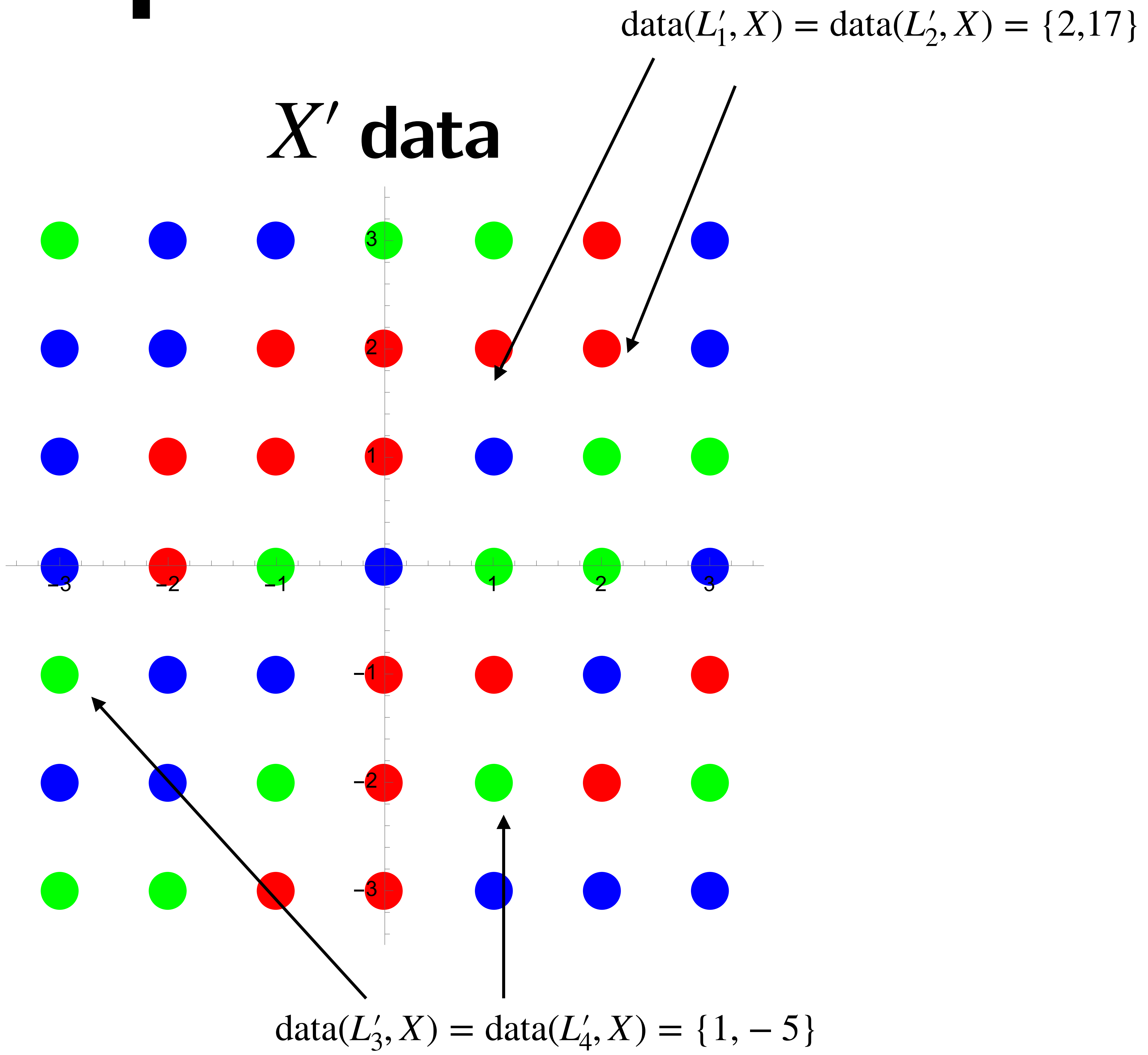


X data
L_1
L_2
L_3
L_4
data(L_3, X) = data(L_4, X) = {1, −5}
data(L_1, X) = data(L_2, X) = {2,17}



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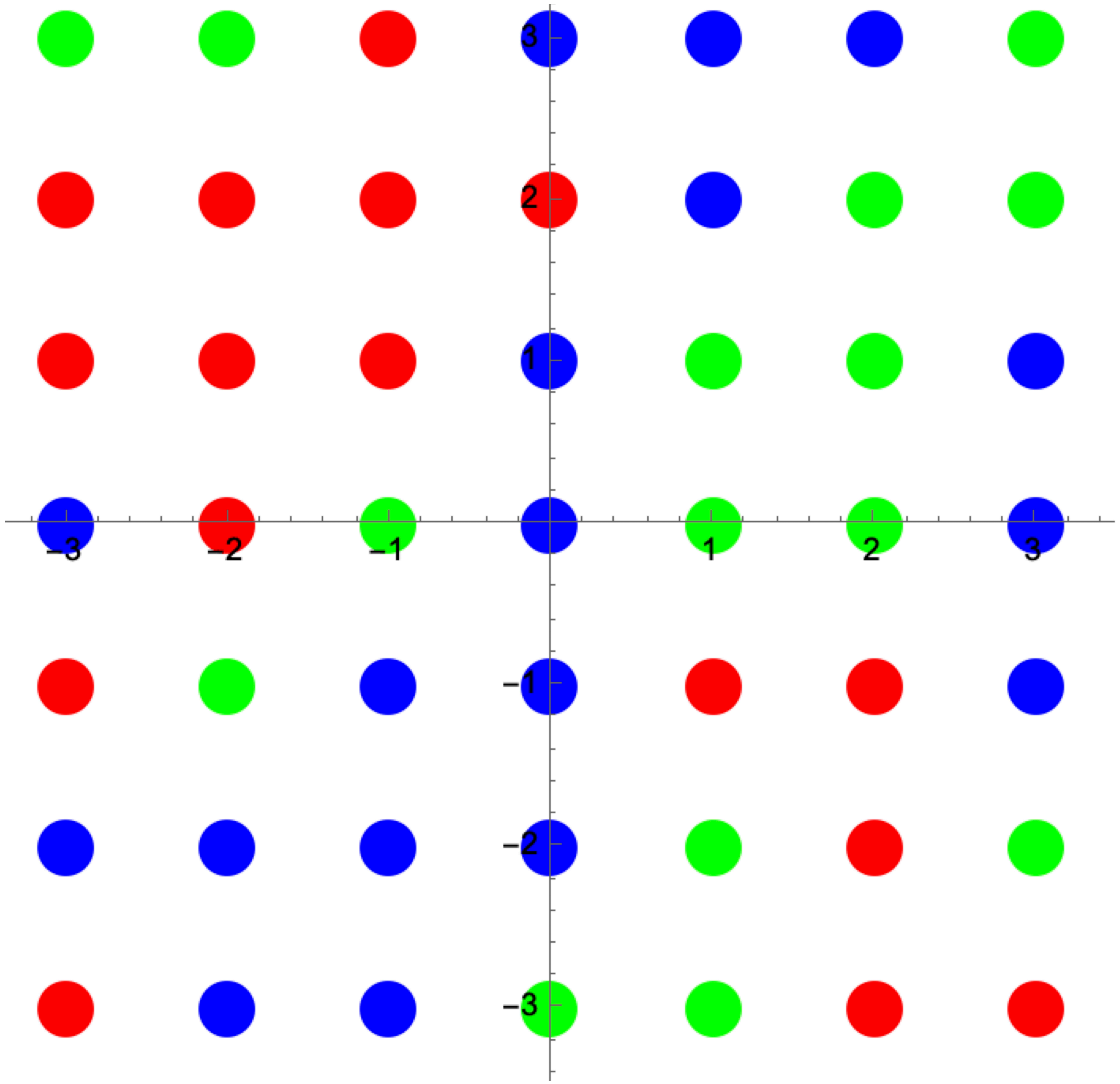
A $GL(N, \mathbb{Z})$ transformation



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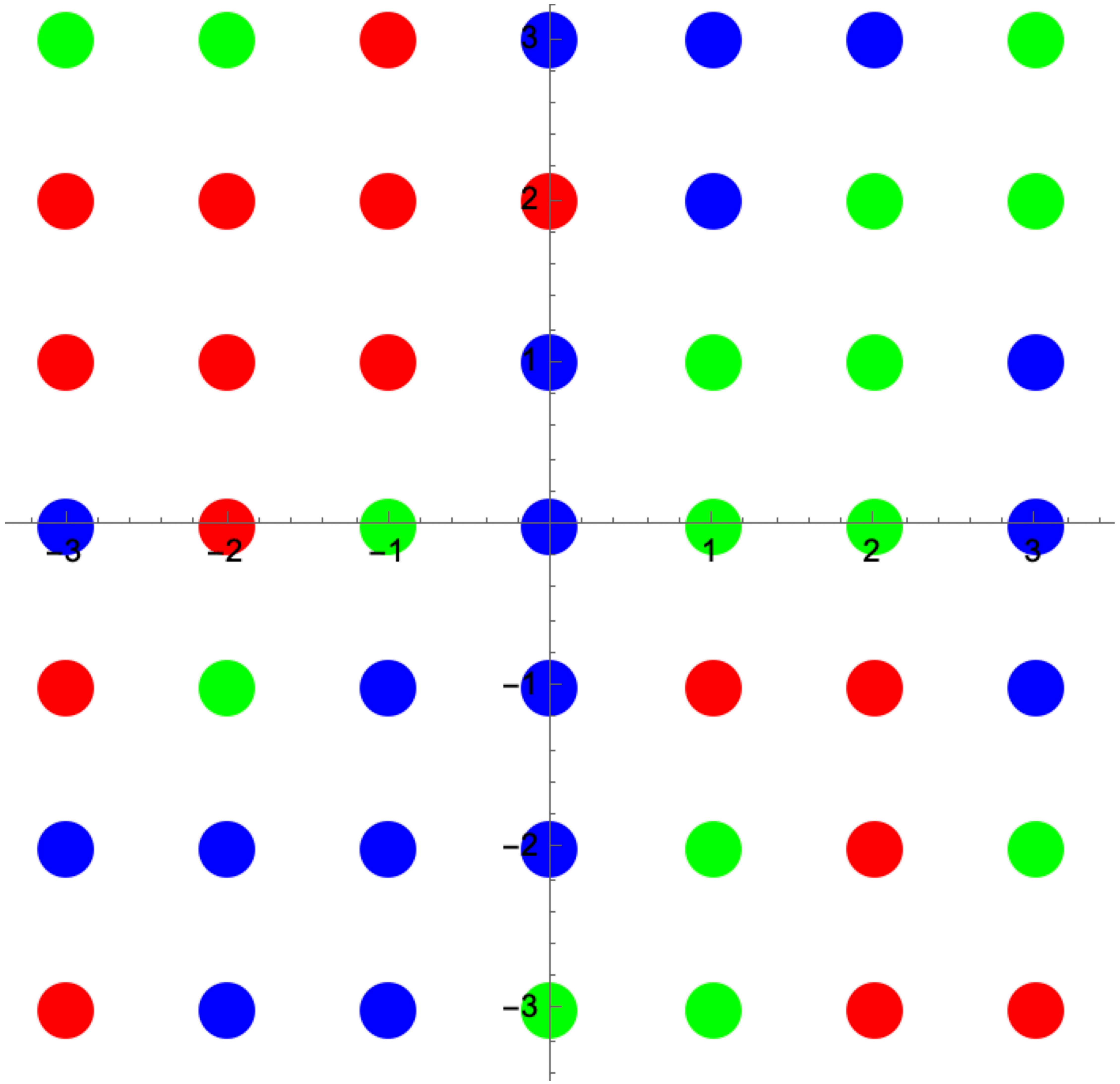

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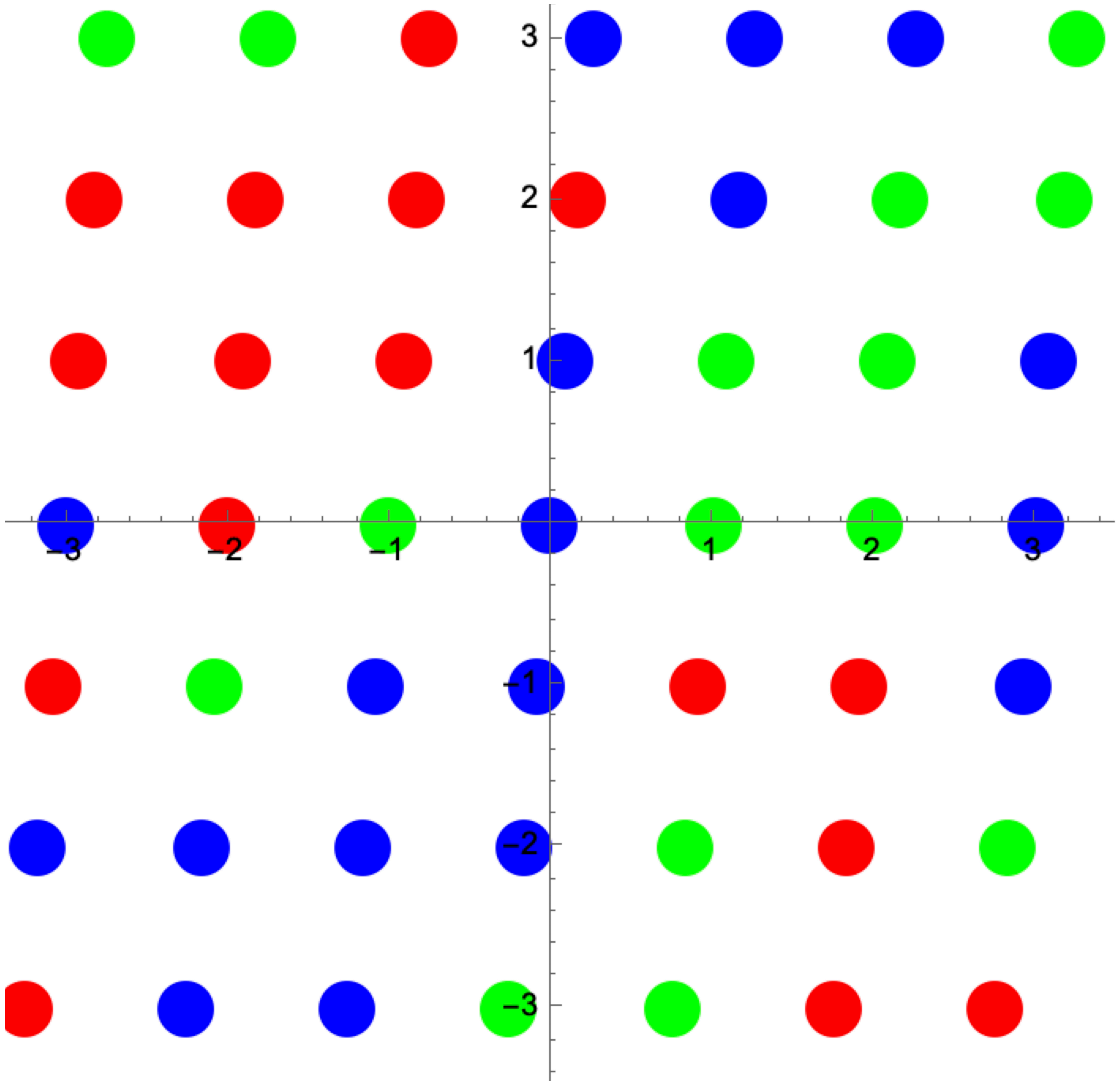

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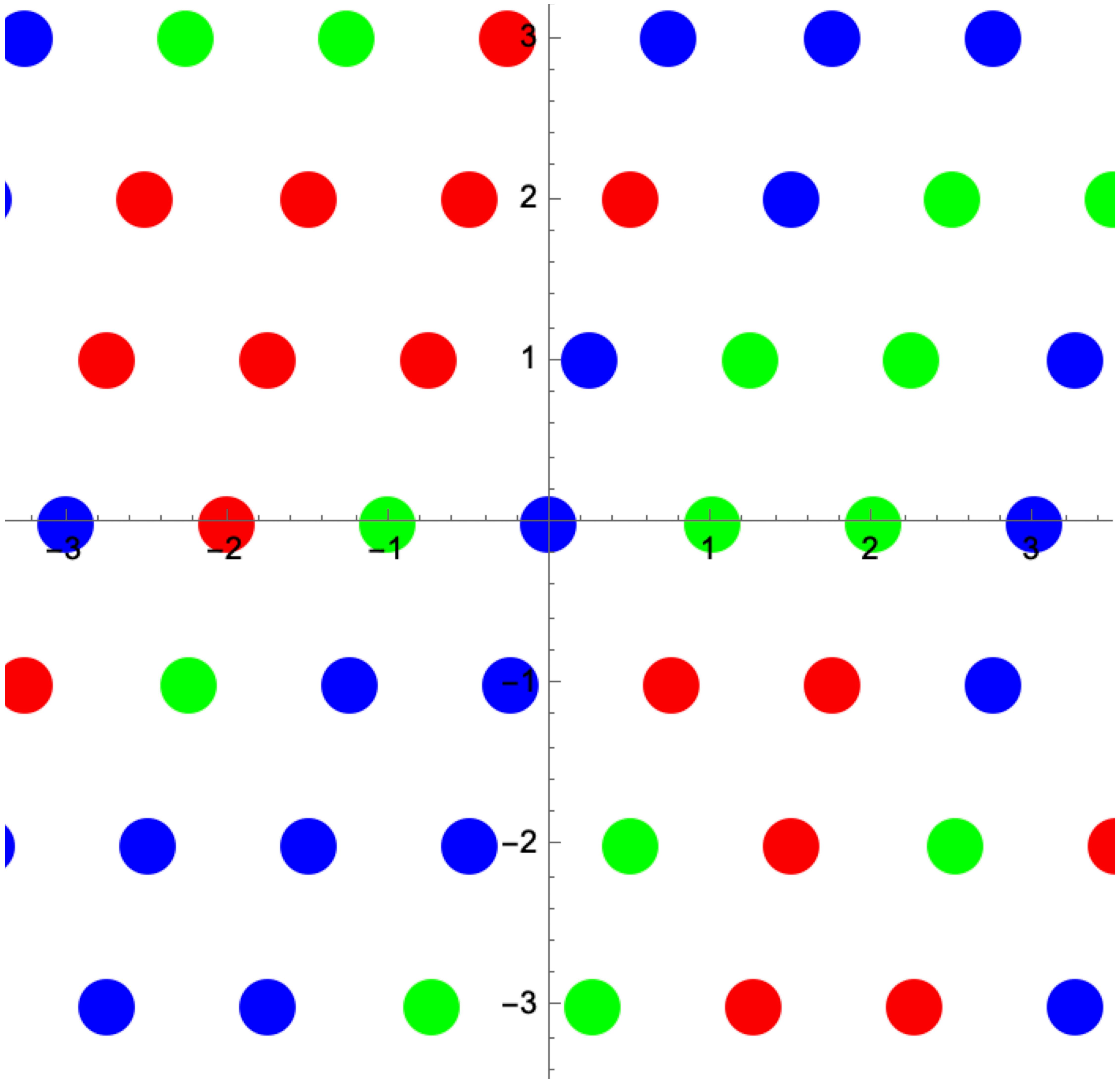

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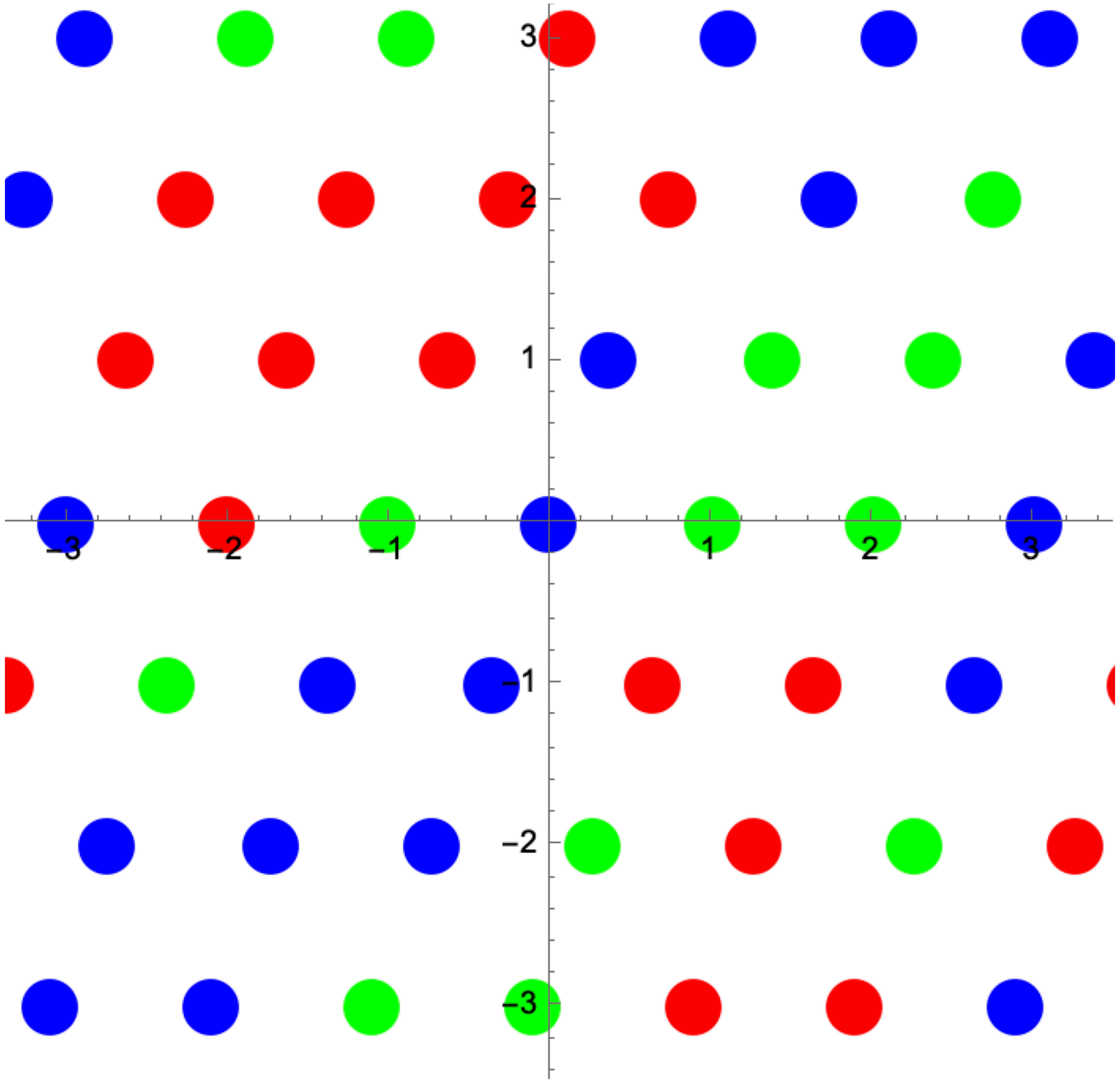

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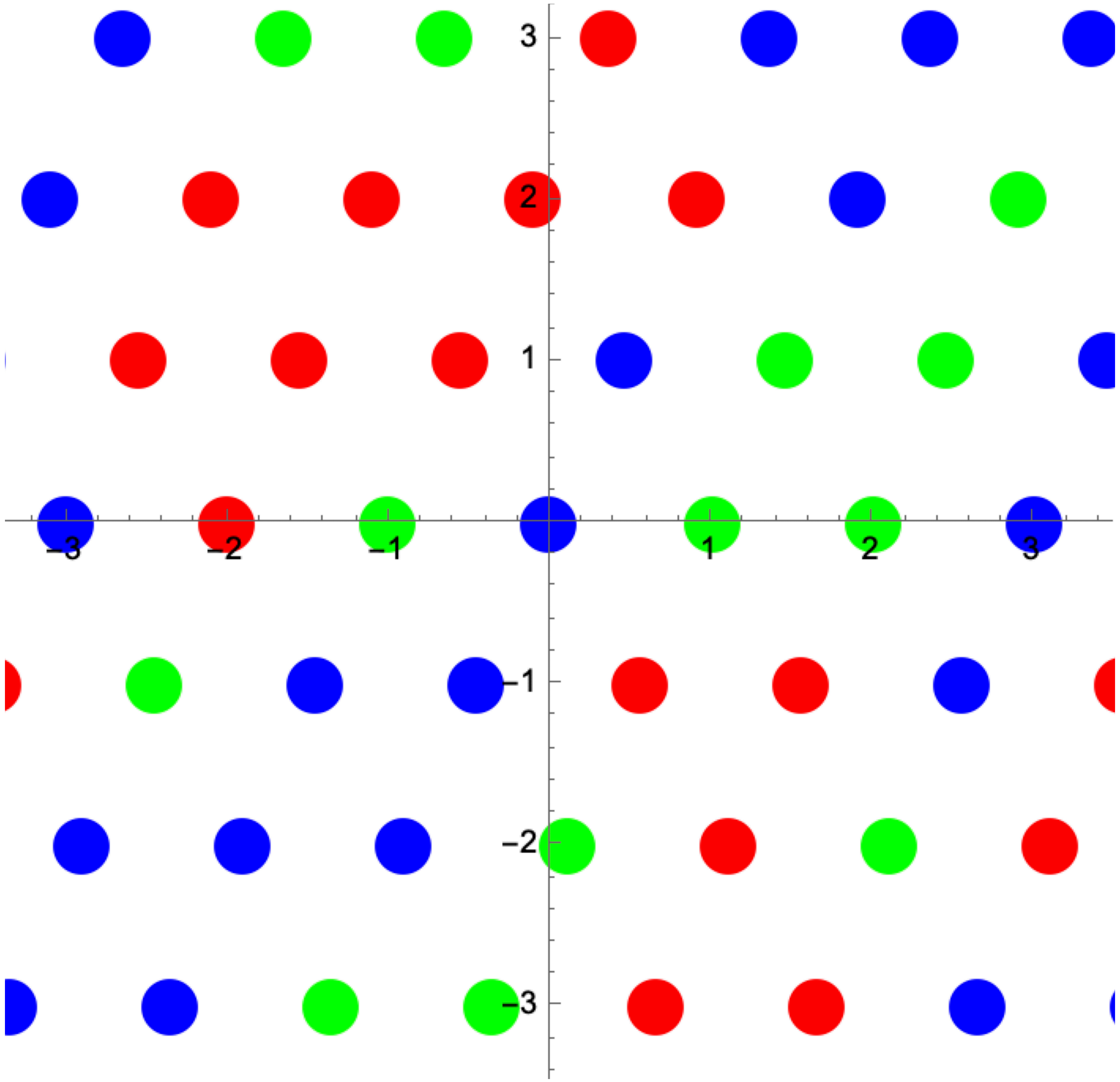

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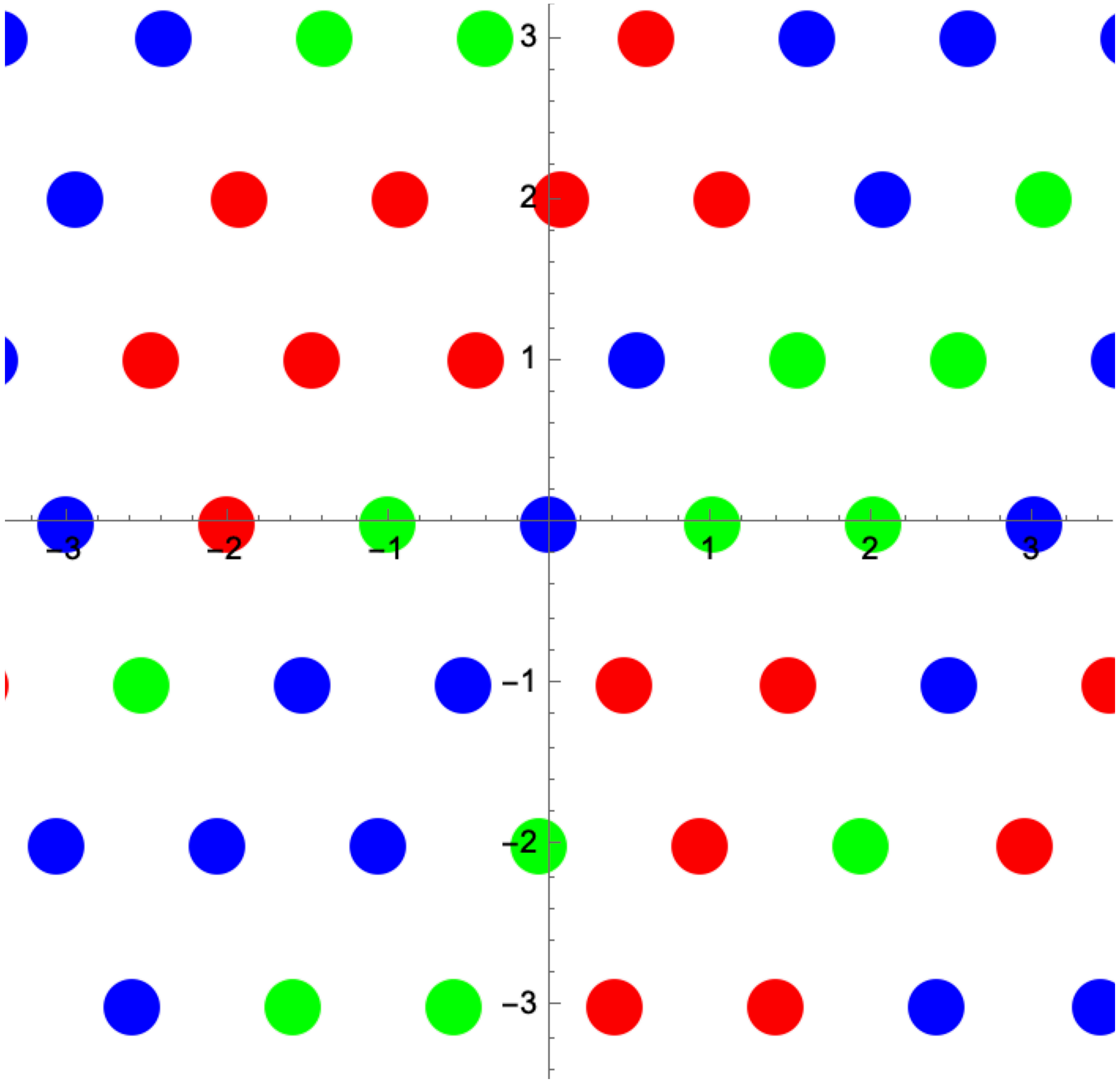

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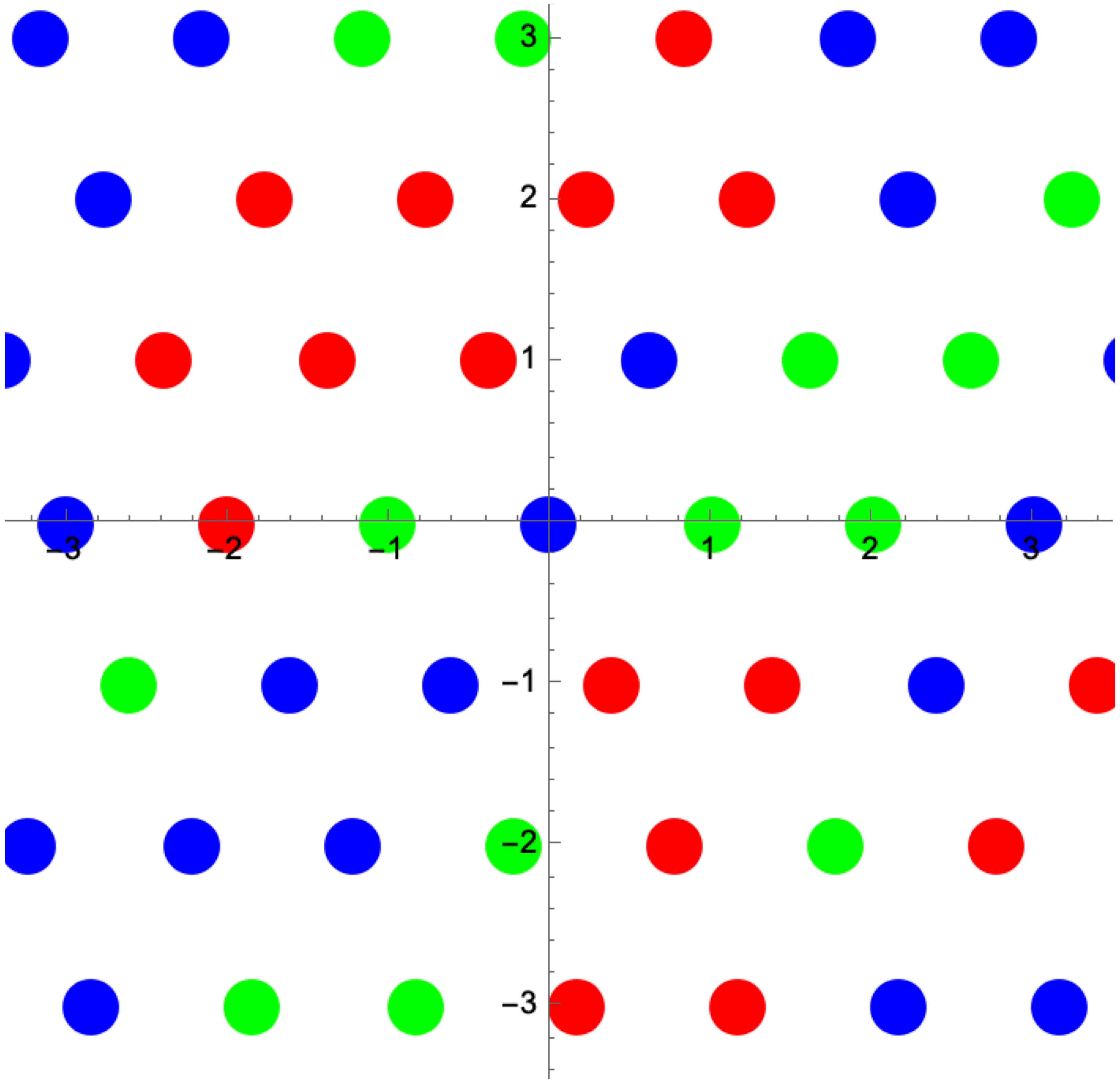

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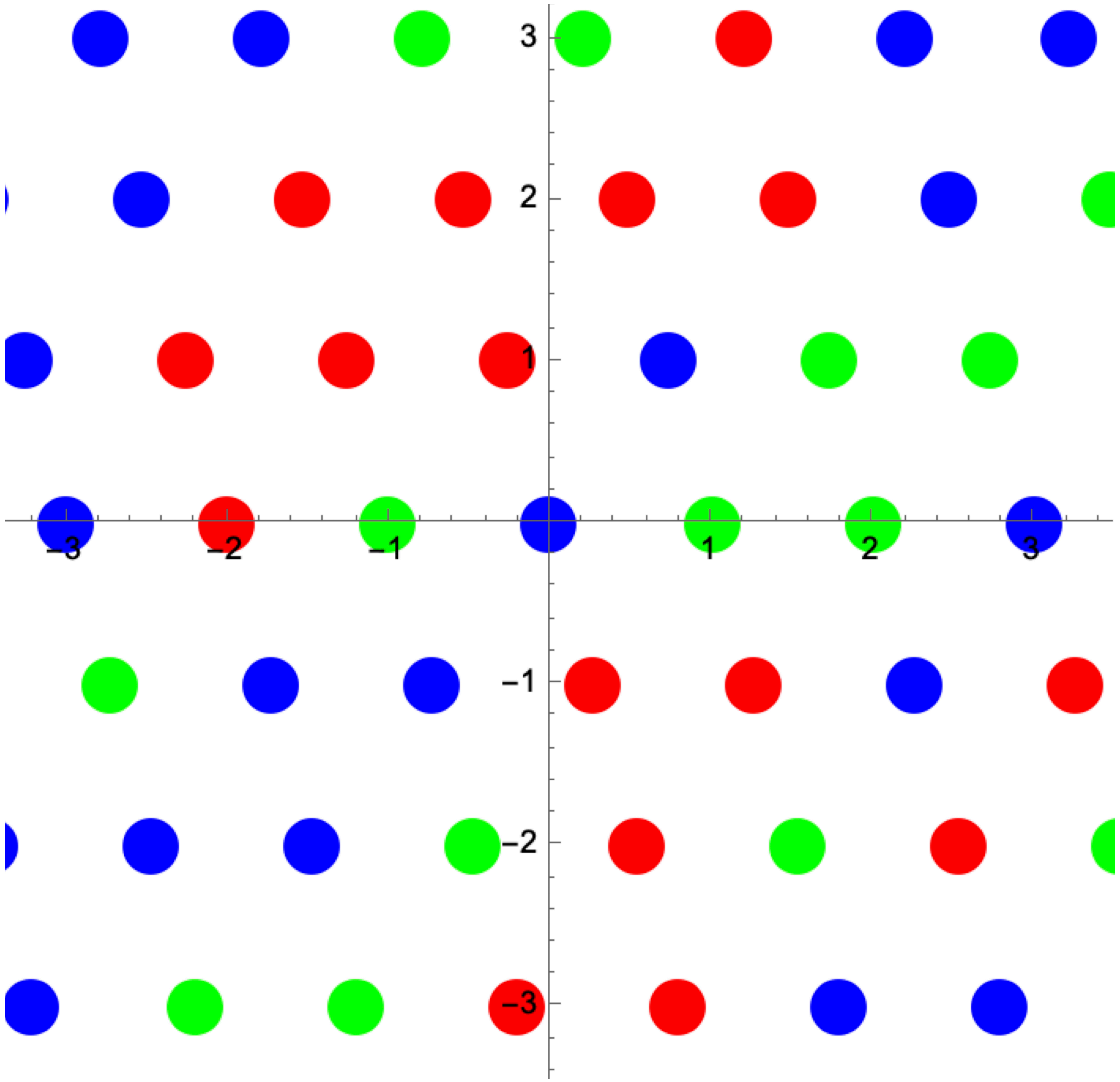

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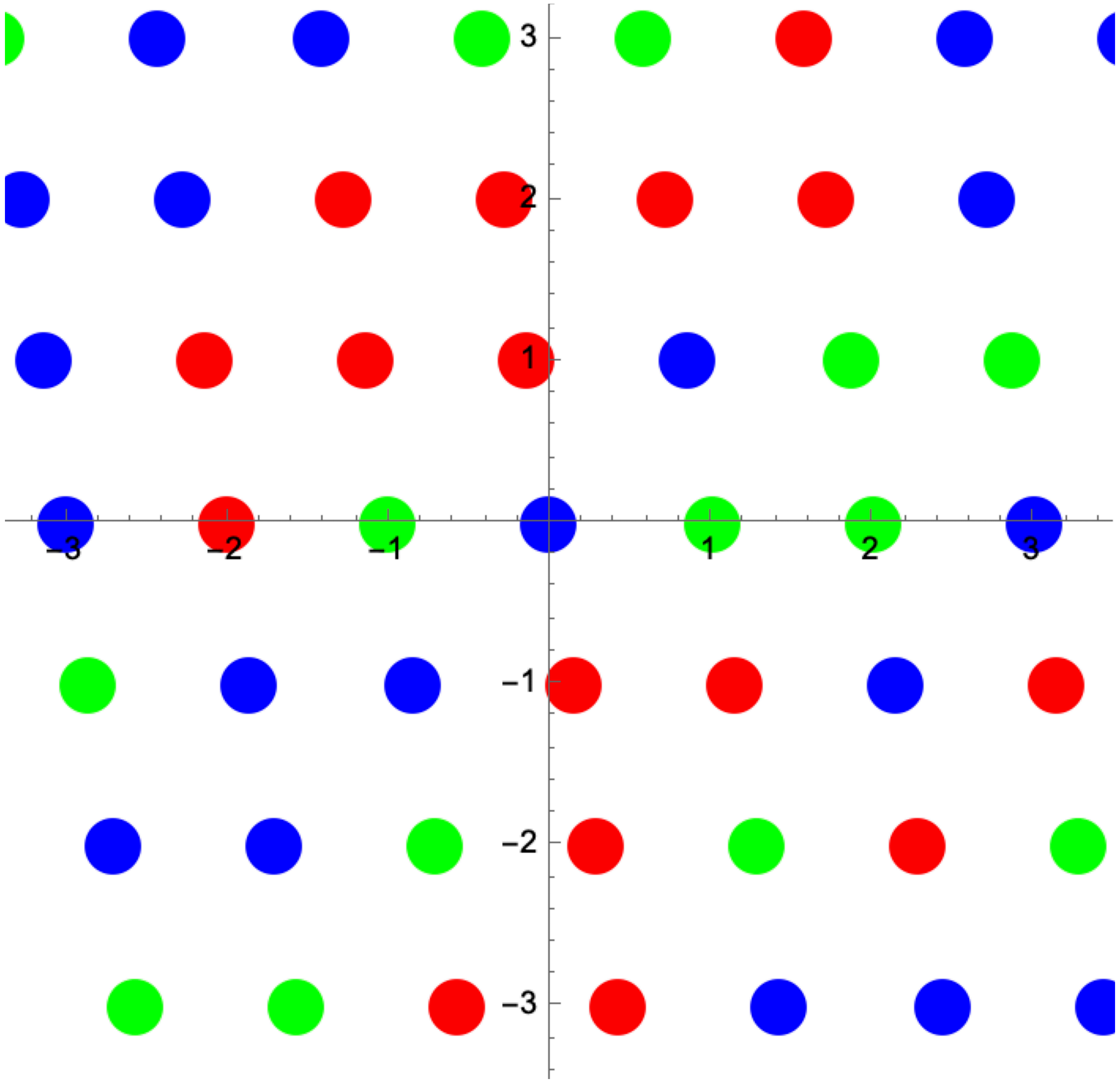

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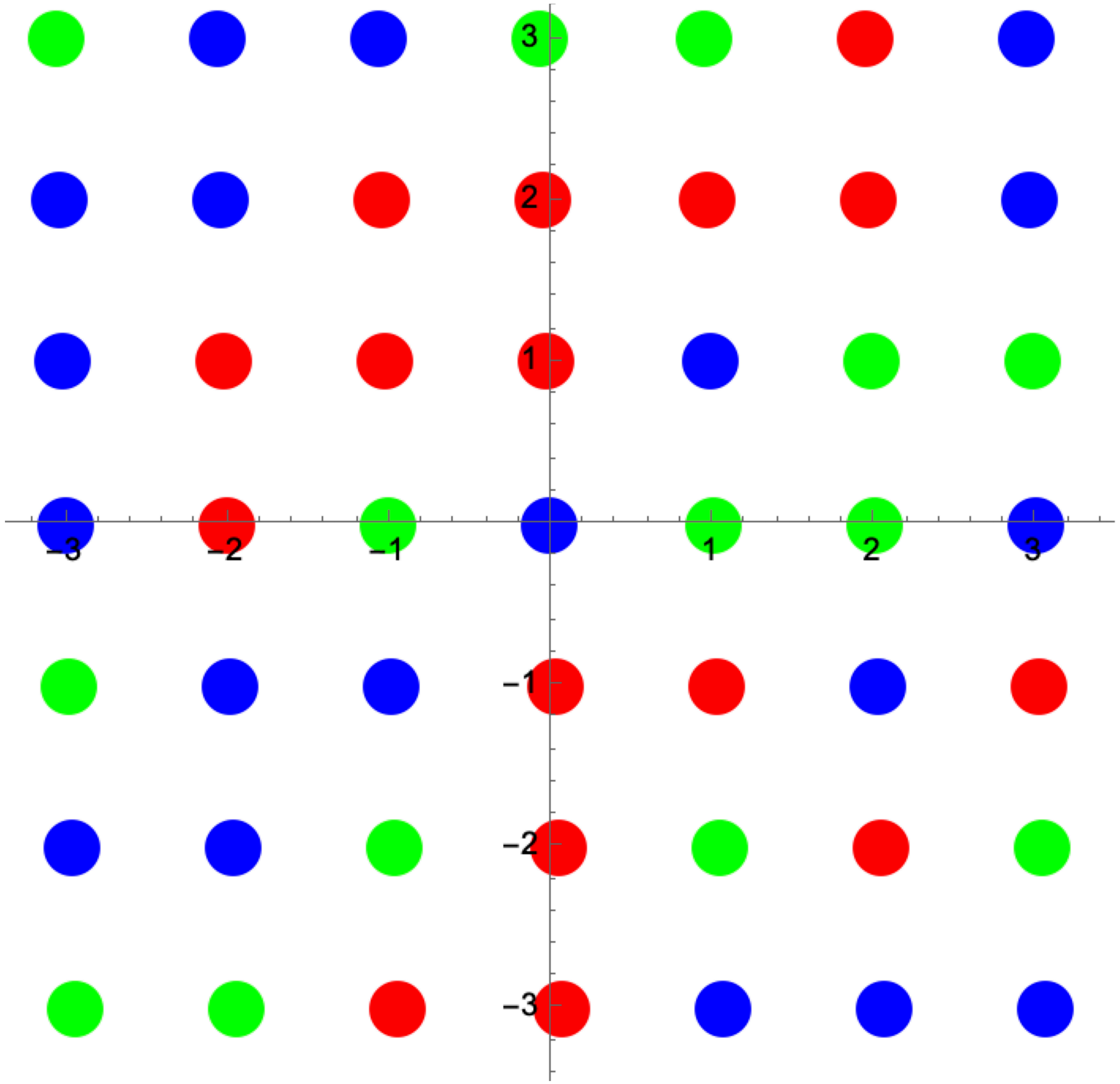

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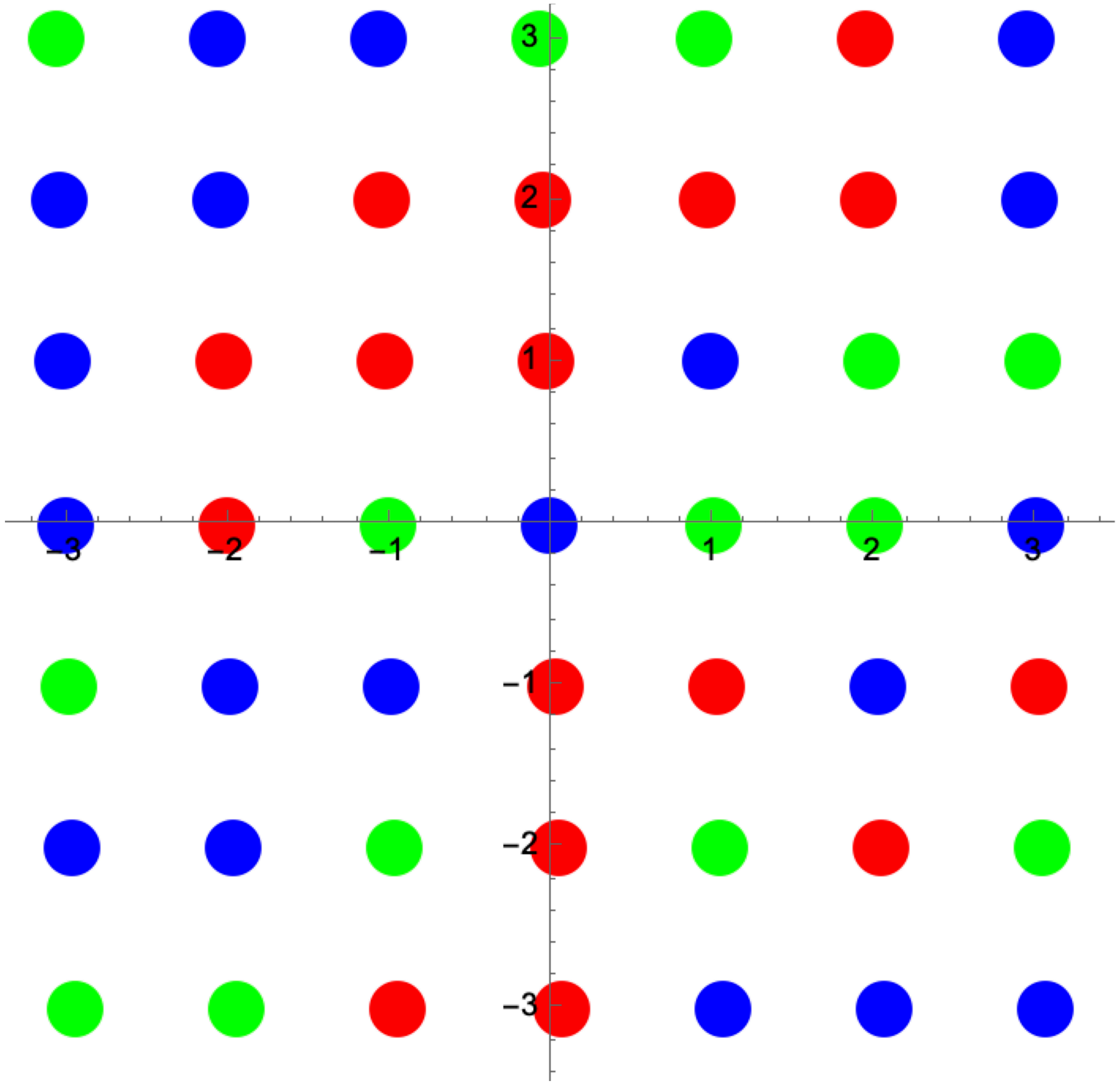

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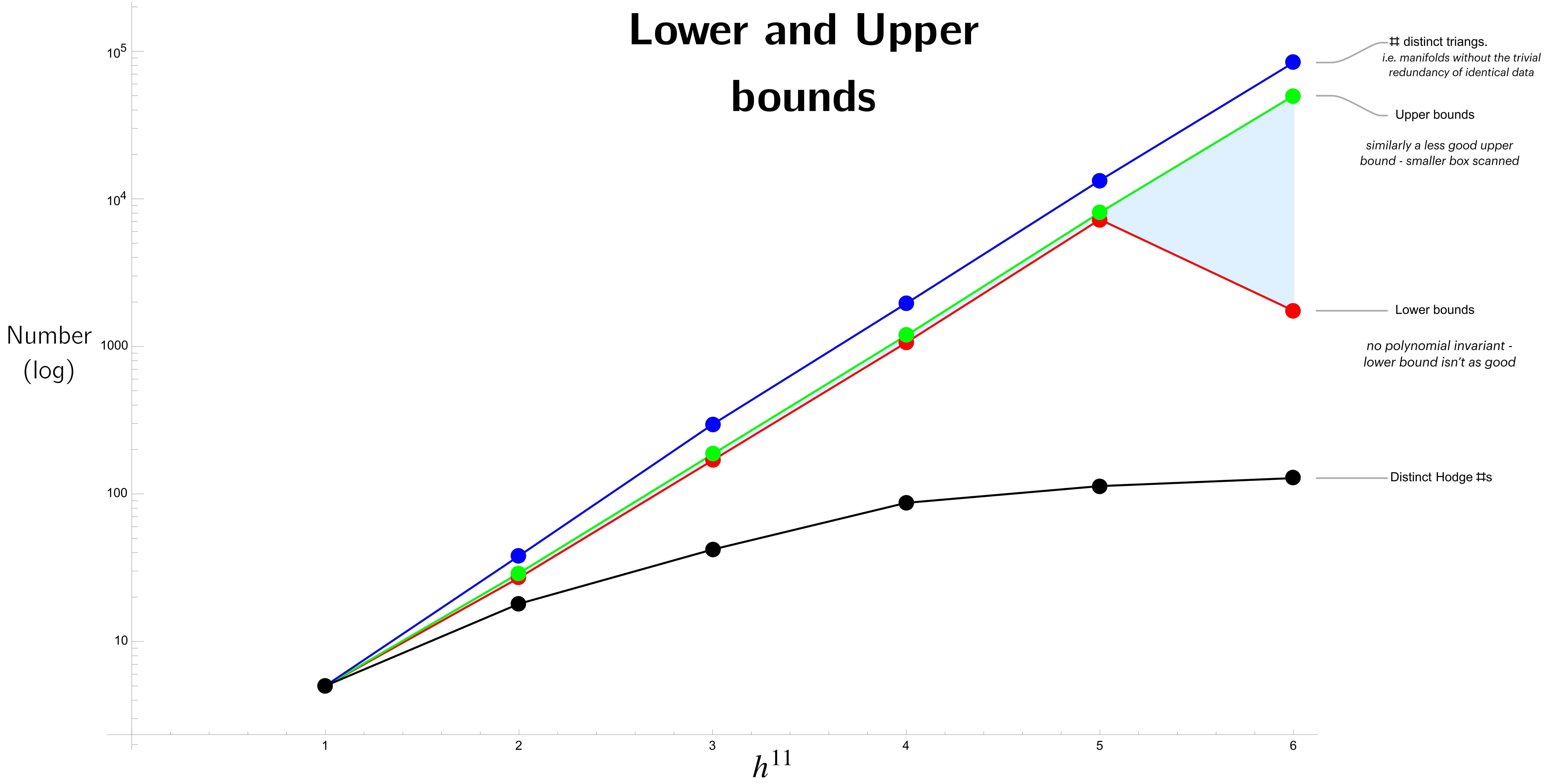
$h^{1,1}$	# Polytopes	# Triangs.	# Distinct Triangs.	Hodge #s	Lower Bound	Upper Bound
1	5	5	5	5	5	5
2	36	48	38	18	27	29
3	243	525	296	42	169	186
4	1185	5,330	1,954	87	1,061	1186
5	4896	56,714	13,330	113	7,244	8078
6	16607	584,281	83,906	128	1,744	TBC

Lower and Upper bounds



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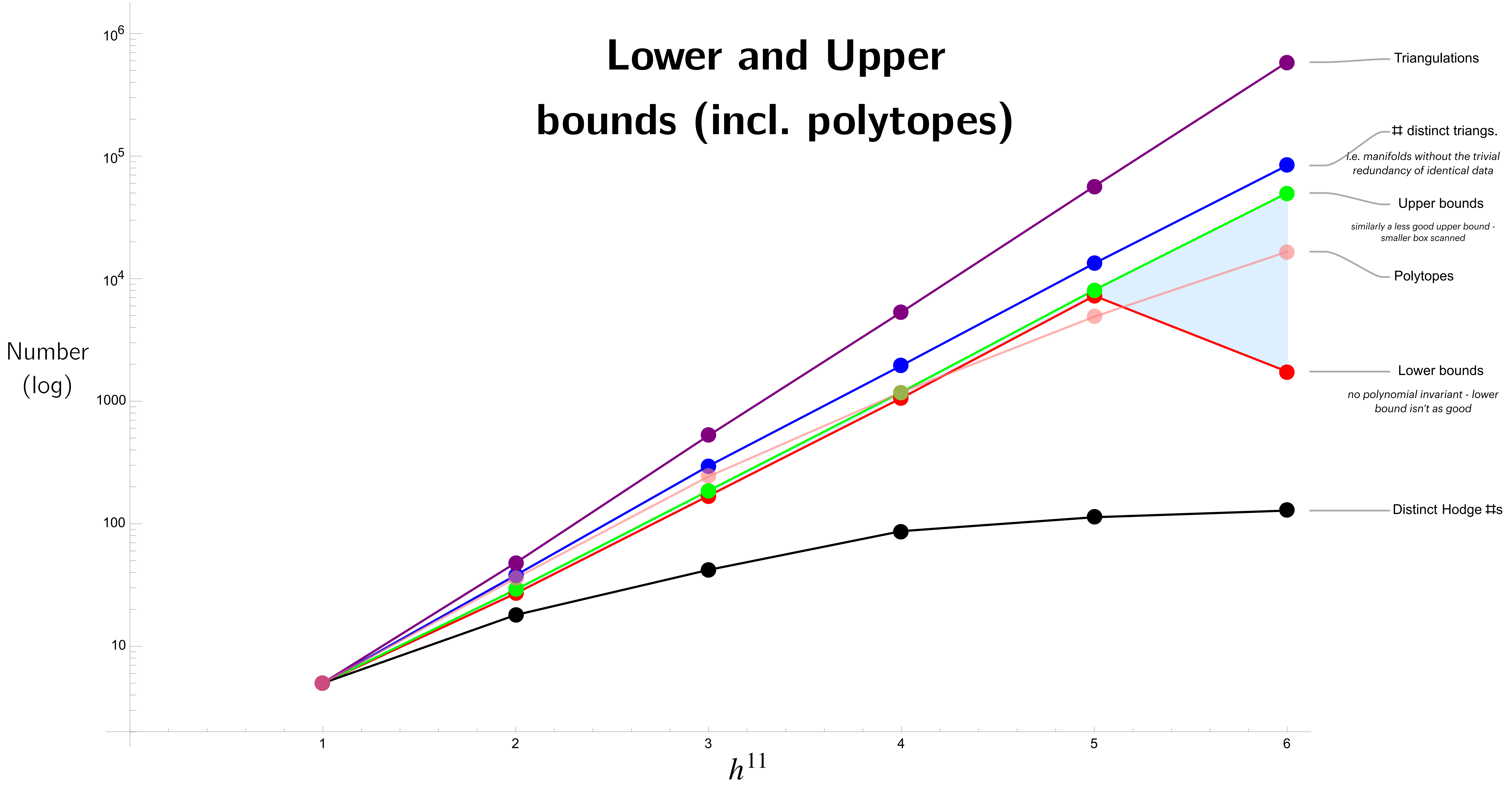
Lower and Upper bounds (incl. polytopes)



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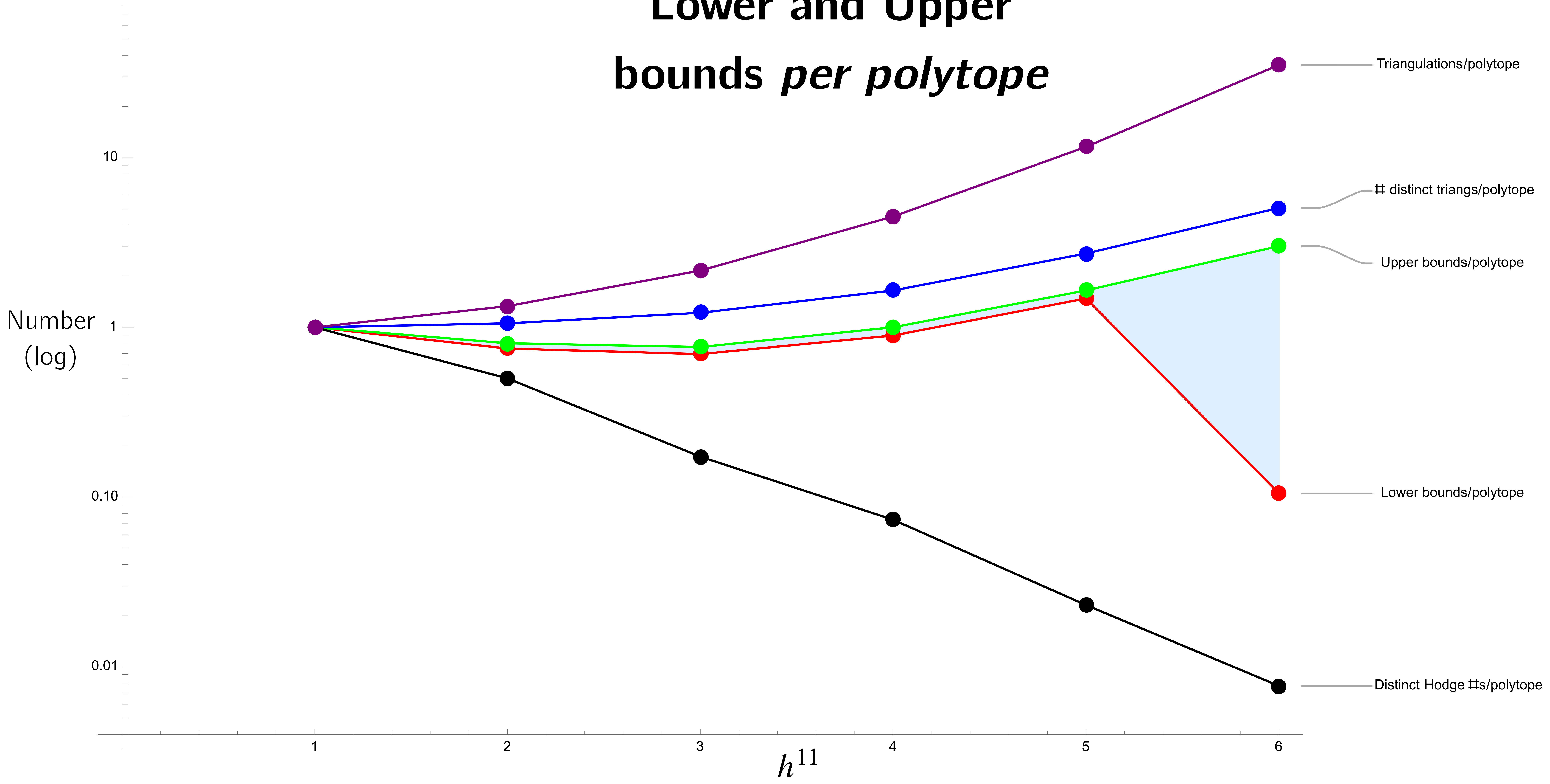
Lower and Upper bounds *per polytope*



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Outlook

- Is there **topological meaning** to a GCD invariant? What about the polynomial singlets? Do they represent interesting properties of the CYs?
- If we want to **extend to higher Picard number** - can we generate them more efficiently? Linear algebra may be too slow.
    - A **partial recurrence relation** exists for the degree-$2h^{11}$ invariant. Can it be completed to a full recurrence relation?
    - Can these invariants be used to **bound** the number of Calabi-Yaus at a particular h^{11}?
    - Add the limiting mixed Hodge structure invariants mentioned above.