Black hole extremality in nonlinear electrodynamics

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@ String phenomenology 2023

2023/07/04

Based on collaboration w/

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Ref. arXiv:2305.17062 [hep-th]

Weak gravity conjecture (WGC)

[Arkani-Hamed-Motl-Nicolis-Vafa ’06]

- Roughly speaking, WGC claims that gravity is the weakest force
- WGC requires existence of a charged state which satisfies

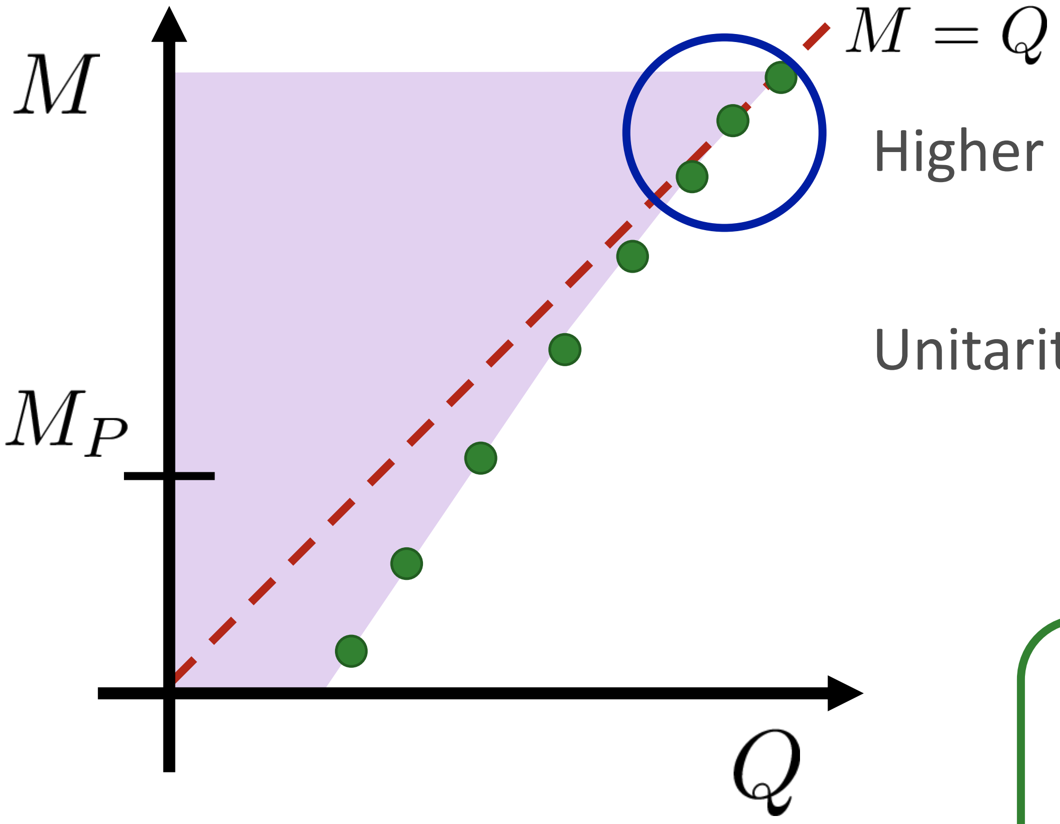
$$\boxed{e^2 q^2 \geq \frac{m^2}{2M_P^2}} \quad \Rightarrow \quad F_{\text{gauge}} = \frac{(eq)^2}{4\pi r^2} \geq F_{\text{gravity}} = \frac{m^2}{8\pi M_P^2 r^2}$$

- Generalization of “no global symmetry in quantum gravity”
- p-form gauge field ⇒ axion WGC

Charged BH spectrum and monotonicity

- Expected charge state spectrum

Heterotic string compactified on $\mathbb{T}^6$



Higher derivative corrections to the BH solution

[Kats-Motl-Padi ’07]

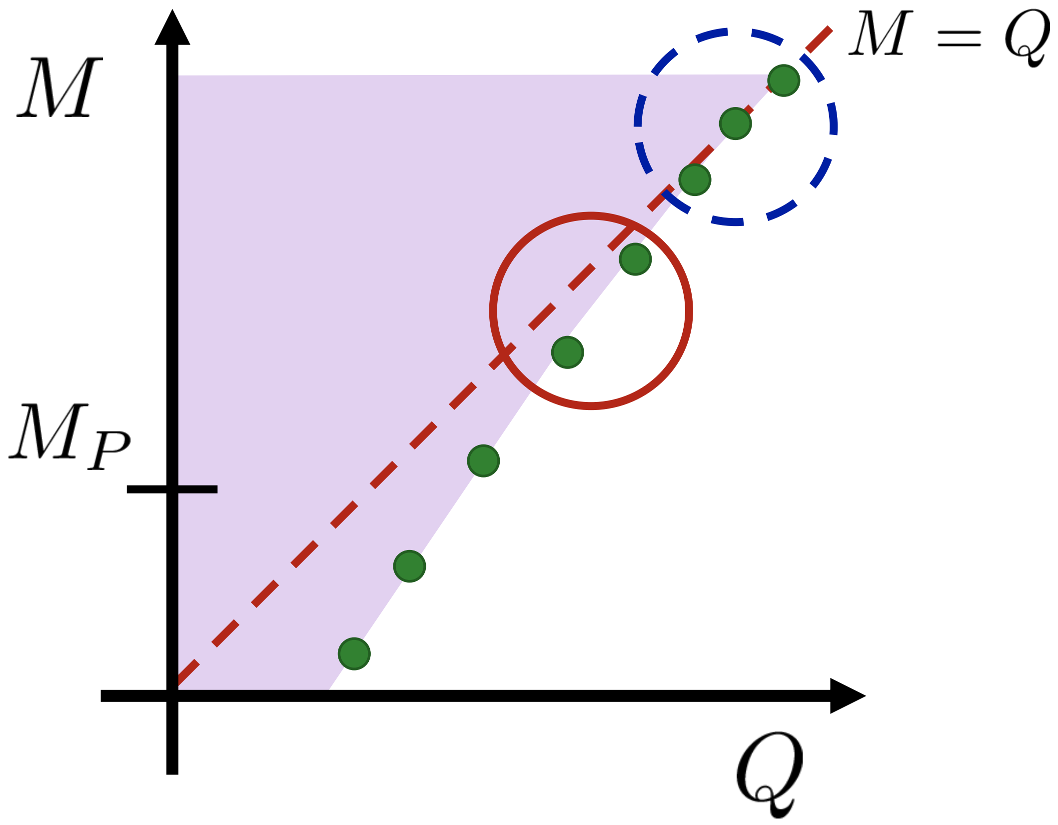
Unitarity and Causality of the scattering amplitude

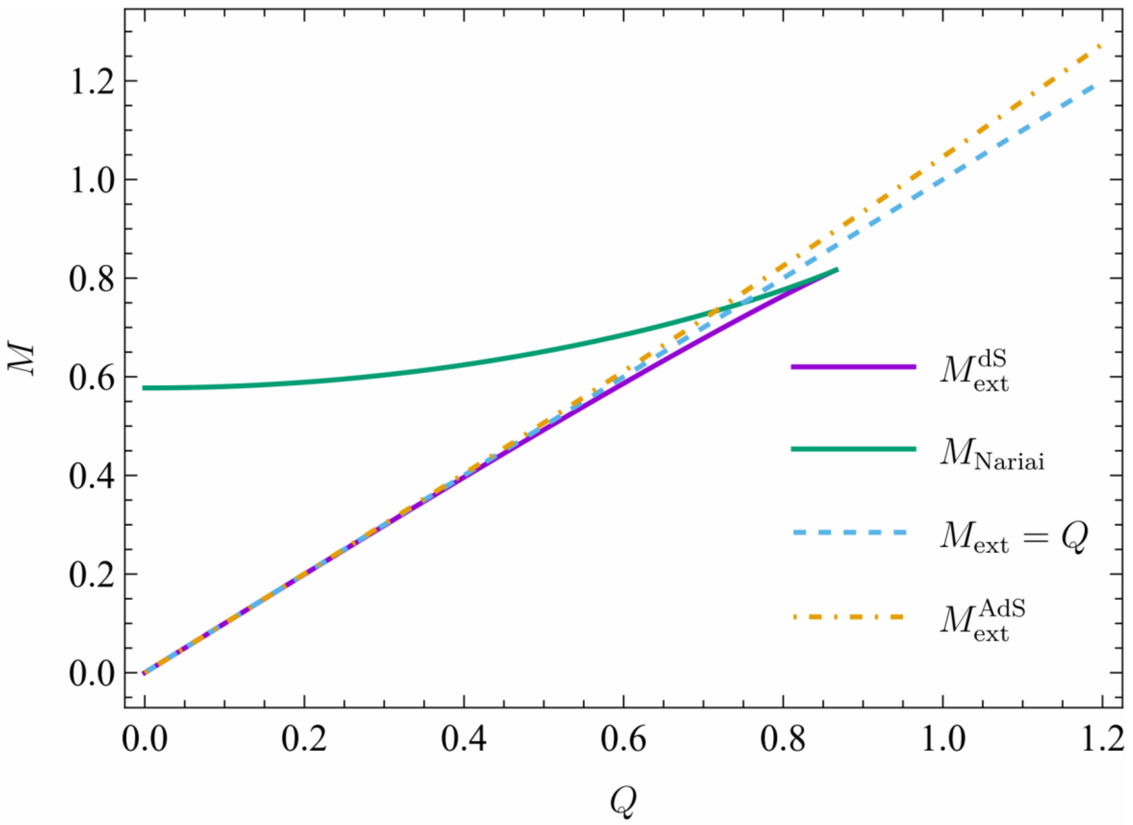
[Hamada-Noumi-Shiu ’19, Bellazzini et al ’19, Charles ’19
Loges-Noumi-Shiu ’19, ’20, Arkani-Hamed et al ’21]

1. The states w/ $M \leq Q$ appear at various scales
2. Monotonicity of the extremal curve
3. At $M \gg M_P$, extremal bound $M \geq Q$

"Monotonicity" and black hole extremality

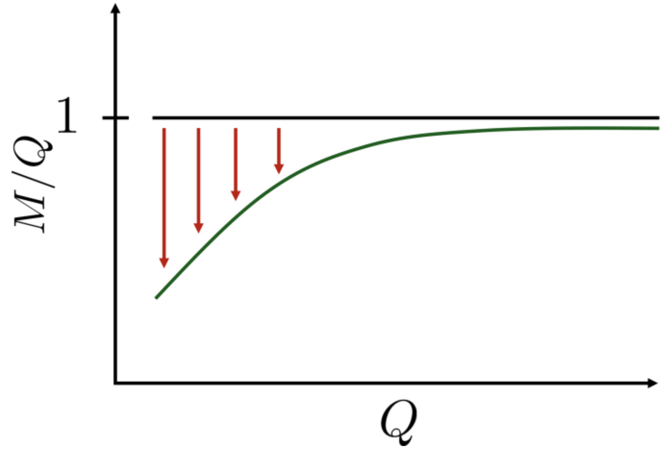
- We study the monotonicity of the BH extremal curve via the BH solution in nonlinear electrodynamics



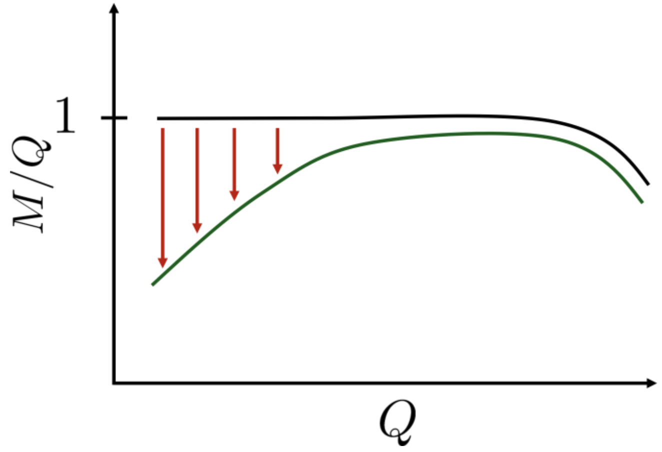


Messages of this talk

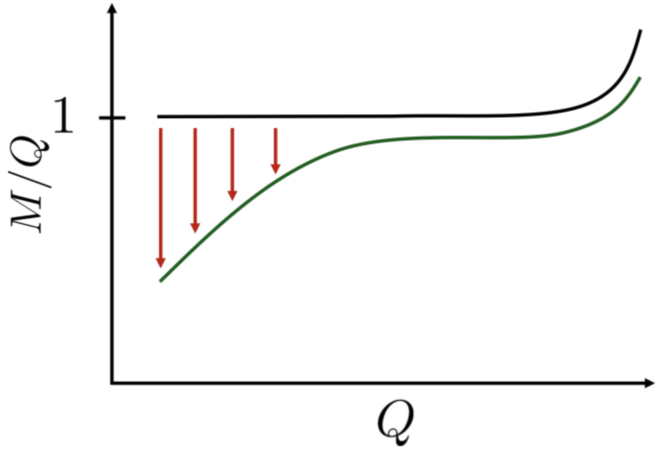
- Black hole mass-to-charge ratio



Asymptotically flat

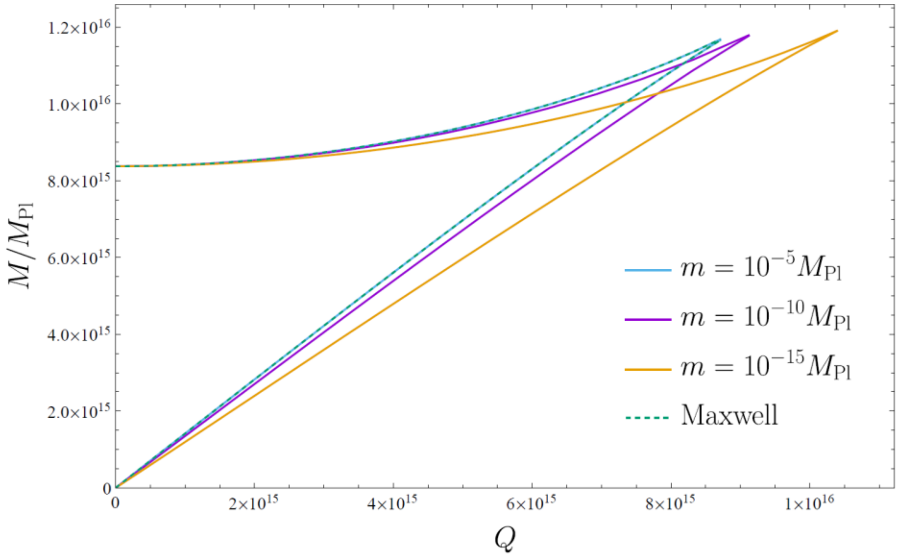


Asymptotically dS



Asymptotically AdS

- De Sitter black hole spectrum



Nonlinear electrodynamics

- Let us consider the following action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi G}(\mathcal{R} - 2\Lambda) + \boldsymbol{L}(\mathcal{F}, \mathcal{G})\right]$$

Maxwell theory

$$\boldsymbol{L}(\mathcal{F}, \mathcal{G}) = -\frac{1}{g_e^2}\mathcal{F}$$

$$\mathcal{F} := \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad \mathcal{G} := \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{8}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

- We assume this Lagrangian is analytic at $\mathcal{F} = \mathcal{G} = 0$ and parity invariant

- Energy-momentum tensor

$$T_{\mu\nu} = -\frac{\partial \boldsymbol{L}}{\partial \mathcal{F}}F_{\mu\rho}F_{\nu}{}^{\rho} + g_{\mu\nu}\left[\boldsymbol{L} - \frac{\partial \boldsymbol{L}}{\partial \mathcal{G}}\mathcal{G}\right]$$

Nonlinear electrodynamics

- Maxwell equation

$$\nabla_\mu \left[\frac{\partial \boldsymbol{L}}{\partial \mathcal{F}} F^{\mu\nu} + \frac{\partial \boldsymbol{L}}{\partial \mathcal{G}} \tilde{F}^{\mu\nu} \right] = 0, \quad \nabla_\mu \tilde{F}^{\mu\nu} = 0$$

- The higher derivative operators modify definition of the electric charge and the Gauss law accordingly
- For an electrically charged BH, we derive a BH solution including this

Dual 2-form [Nomura-Yoshida ’22,…], perturbative iteration [Kats-Motl-Padi ’06,…]

- For a magnetic BH, we don’t need this modification due to the Bianch id.

Magnetic black holes in nonlinear electrodynamics

- Let us consider the following spherically symmetric metric and gauge configuration

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_2^2$$

$$A = -\frac{n}{2}\cos\theta\,\mathrm{d}\varphi, \qquad F = \frac{n}{2}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi$$

- Magnetic charge and magnetic gauge coupling

$$n = \frac{1}{2\pi}\int_{S^2} F$$

$$\frac{1}{4g_e^2}F_{\mu\nu}F^{\mu\nu} = \frac{1}{g_e^2}\mathcal{F} = \frac{g_m^2 n^2}{32\pi^2 r^4} = \frac{Q_m^2}{32\pi^2 r^4} = \frac{k_m n^2}{8\pi r^4}$$

w/

$$Q_m = g_m n, \quad g_m = \frac{2\pi}{g_e}, \quad k_m = \frac{g_m^2}{4\pi}$$

Magnetic black holes in nonlinear electrodynamics

- Einstein equation reduces to

$$(r\partial_r + 1)f(r) = 1 - \Lambda r^2 + 8\pi G r^2 \boldsymbol{L}\left(\frac{n^2}{8r^4}, 0\right)$$

and we obtain

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3} + \frac{8\pi G}{r} \int \mathrm{d}r' r'^2 \boldsymbol{L}\left(\frac{n^2}{8r'^4}, 0\right)$$

- The horizon radius r_H is characterized by $f(r_H) = 0$
- The BH mass parameter is written as the function of r_H, n, Λ

$$M(r_H, n, \Lambda) = \frac{r_H}{2G} - \frac{\Lambda r_H^3}{6G} + 4\pi \int^{r_H} \mathrm{d}r r^2 \boldsymbol{L}\left(\frac{n^2}{8r^4}, 0\right)$$

Extremality of magnetic black holes in nonlinear electrodynamics

- The extremal condition of this charged BH is given by

$$\frac{\partial M(r_H, n, \Lambda)}{\partial r_H} = 0$$

- The extremal BH mass is obtained by substituting this solution to $M(r_H, n)$

<u>E.g. Reissner-Nordström solution</u>

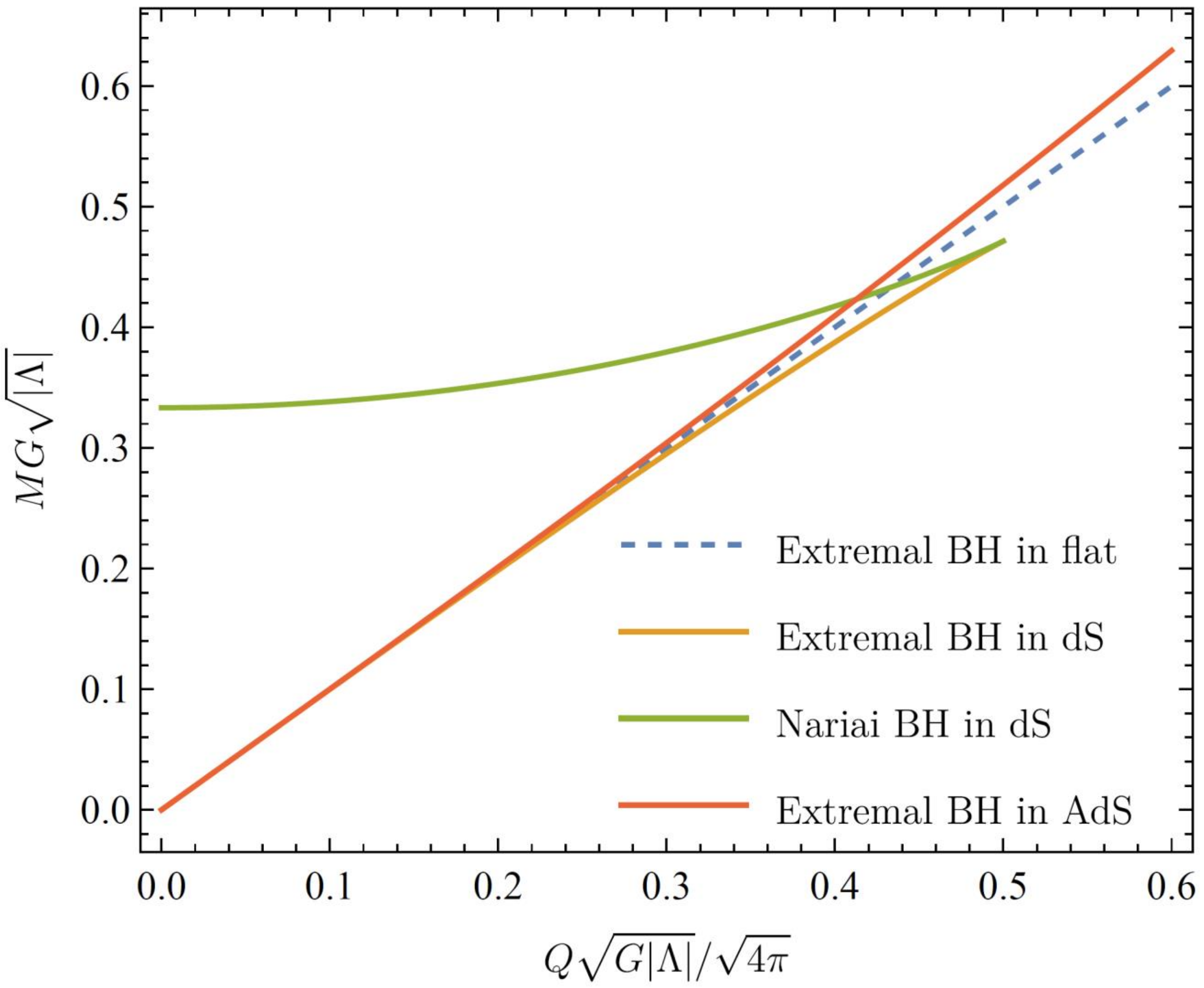
$$\boldsymbol{L} = -\frac{1}{g_e^2}\mathcal{F}, \quad f(r) = 1 - \frac{2GM}{r} + \frac{Gkn^2}{r^2}$$

$$M = \frac{r_H}{2G} + \frac{kn^2}{2r_H}, \quad \frac{\partial M}{\partial r_H} = \frac{1}{2G} - \frac{kn^2}{2r_H^2}$$

$$\Rightarrow r_H = \sqrt{Gkn^2}, \quad GM^2 = kn^2$$

Black hole extremality in Einstein-Maxwell theory

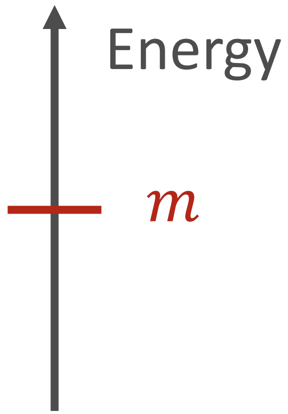
- Charge-Mass relation of BH in Einstein-Maxwell theory



Euler-Heisenberg model

- Euler-Heisenberg (EH) effective action
  → Integrating out the charged matters

[Heisenberg-Euler '36, Schwinger '51]



$$Z = \int \mathcal{D}A_\mu \mathcal{D}\phi \, \exp(\mathrm{i}S[\phi, A_\mu])$$
$$= \int \mathcal{D}A_\mu \, \exp(\mathrm{i}S_{\text{eff}}[A_\mu])$$

- Euler-Heisenberg Lagrangian

$$X := \sqrt{2(\mathcal{F} + \mathrm{i}\mathcal{G})}$$

$$\boldsymbol{L}_{\mathrm{EH}}(\mathcal{F}, \mathcal{G}) = \begin{cases} -\dfrac{\mathcal{F}}{g_e^2} + \dfrac{1}{32\pi^2} \displaystyle\int_0^\infty \dfrac{\mathrm{d}s}{s} \mathrm{e}^{-sm^2} \left[\dfrac{\mathcal{G}}{\mathrm{Im}\cos(sX)} - \dfrac{1}{s^2} + \dfrac{\mathcal{F}}{3} \right] & \text{(scalar)} \\ -\dfrac{\mathcal{F}}{g_e^2} - \dfrac{1}{32\pi^2} \displaystyle\int_0^\infty \dfrac{\mathrm{d}s}{s} \mathrm{e}^{-sm^2} \left[4\dfrac{\mathrm{Re}\cosh sX}{\mathrm{Im}\cosh(s)} - \dfrac{4}{s^2} - \dfrac{8}{3}\mathcal{F} \right] & \text{(fermion)} \end{cases}$$

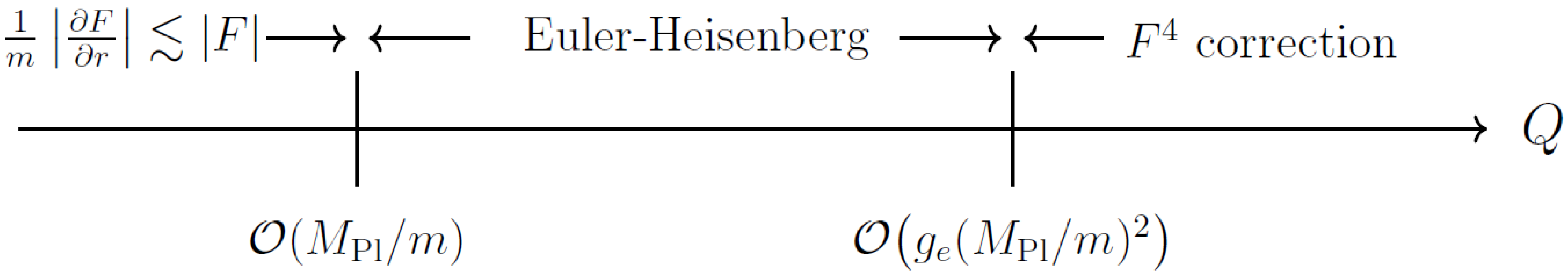
Validity of EH model

- The constant background flux is assumed in the derivation of the EH Lagrangian

→ The change of the flux should smaller than the Compton length of the charged particle

$$\frac{1}{m}\left|\frac{\partial F}{\partial r}\right| \lesssim |F|$$

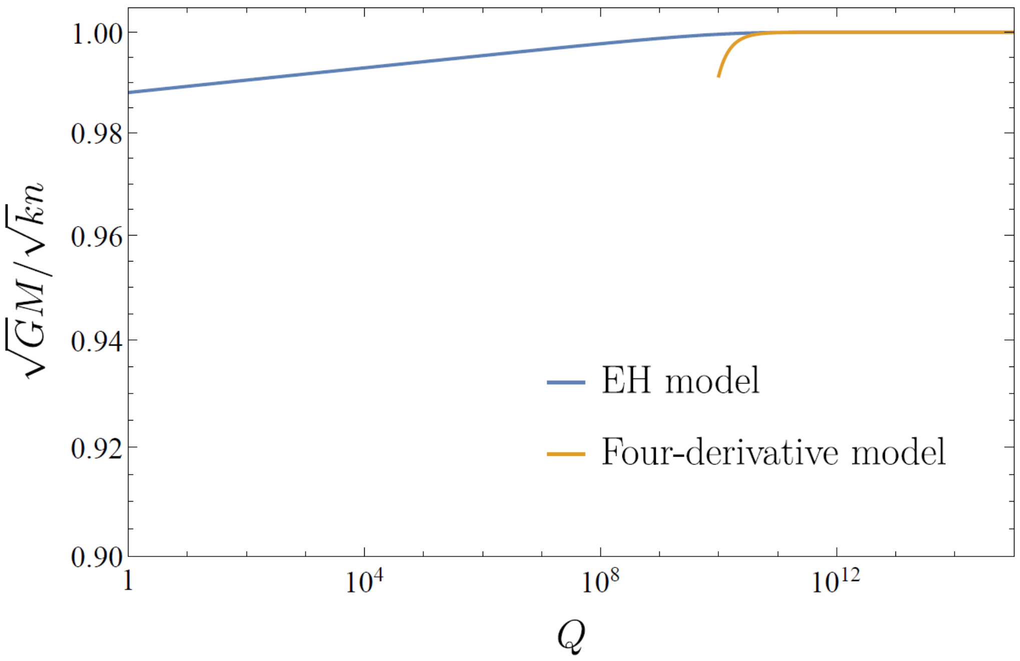
- EH Lagraigian is required for $F_{\mu\nu}F^{\mu\nu}/m^4 \gtrsim 1$



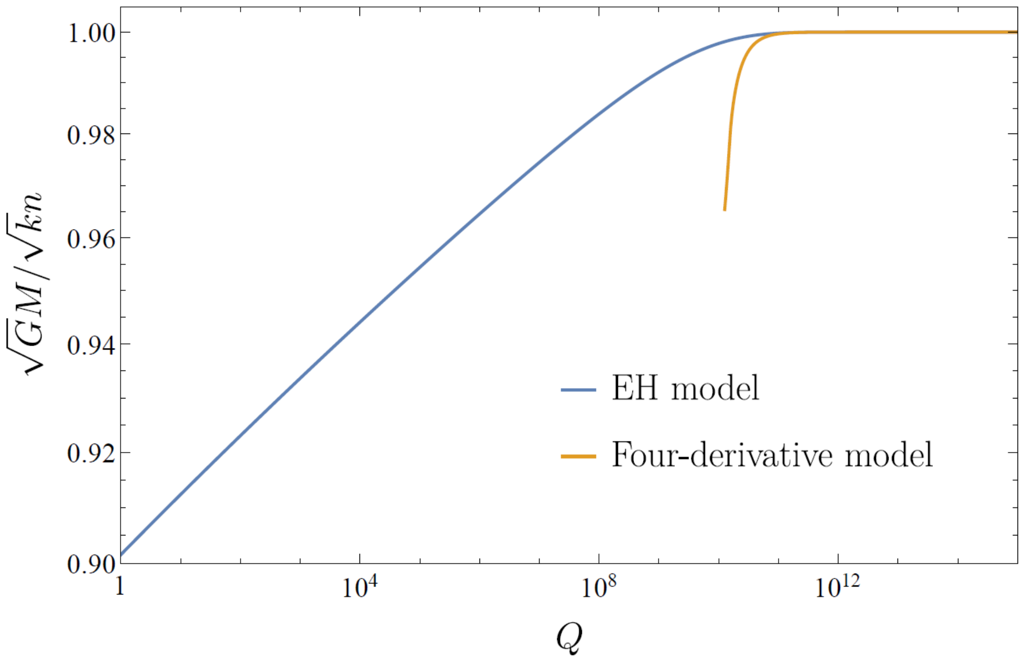
BH extremality in asymptotically flat geometry

Mass-to-charge ratio of extremal BH w/ EH Lagrangian

- EH model vs four-derivative corrections
- $m = 10^{-5} M_{\mathrm{Pl}}, g_e = 1$



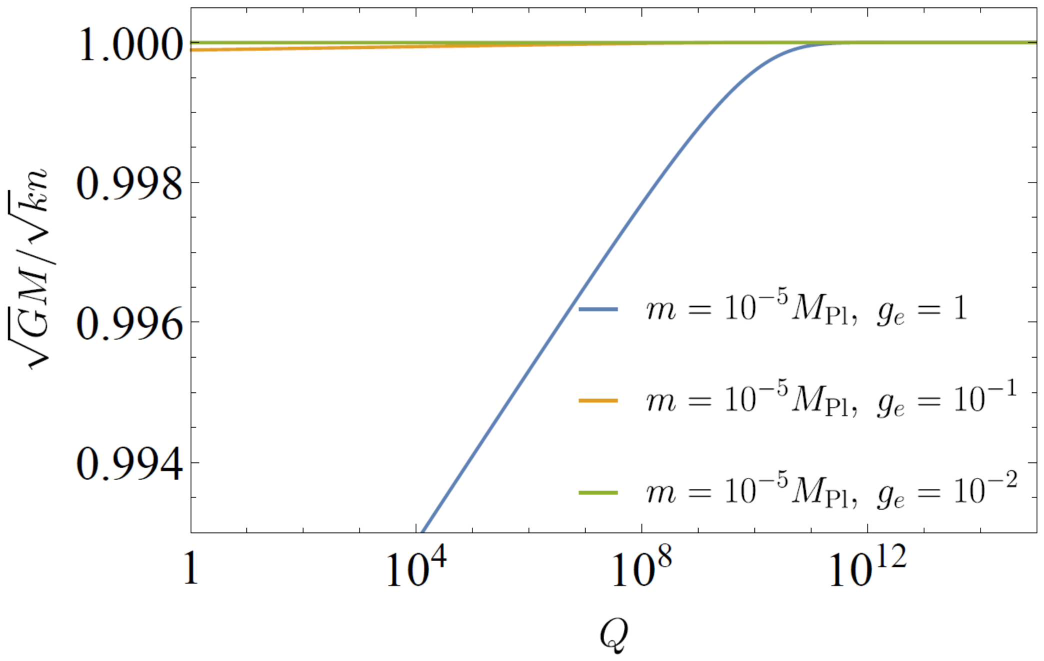
Scalar loop



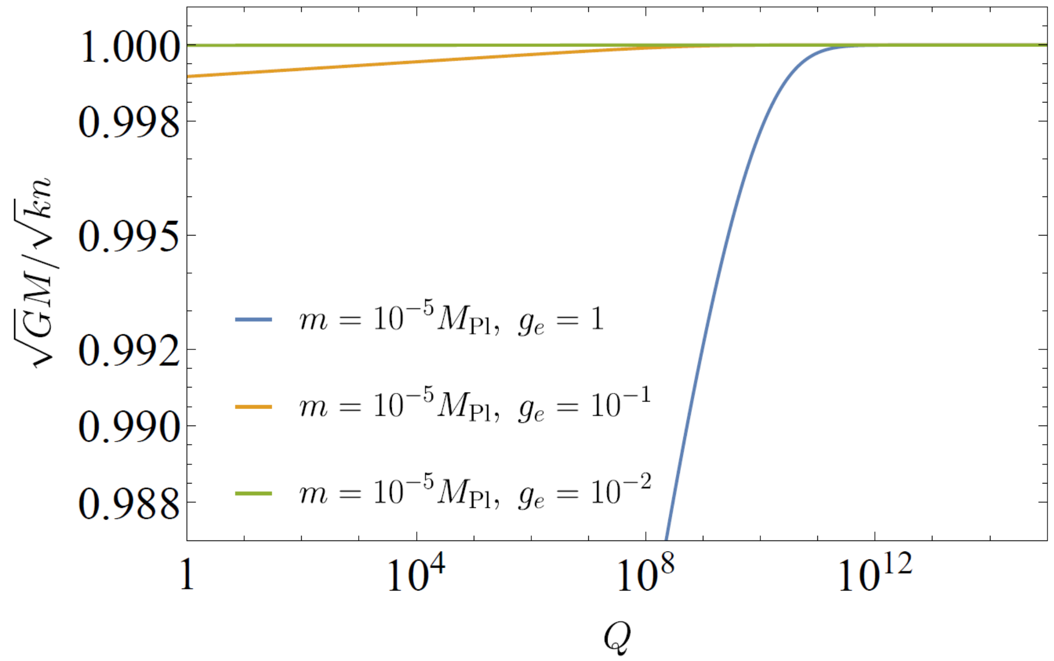
Fermion loop

Mass-to-charge ratio of extremal BH w/ EH Lagrangian

- Gauge coupling dependence



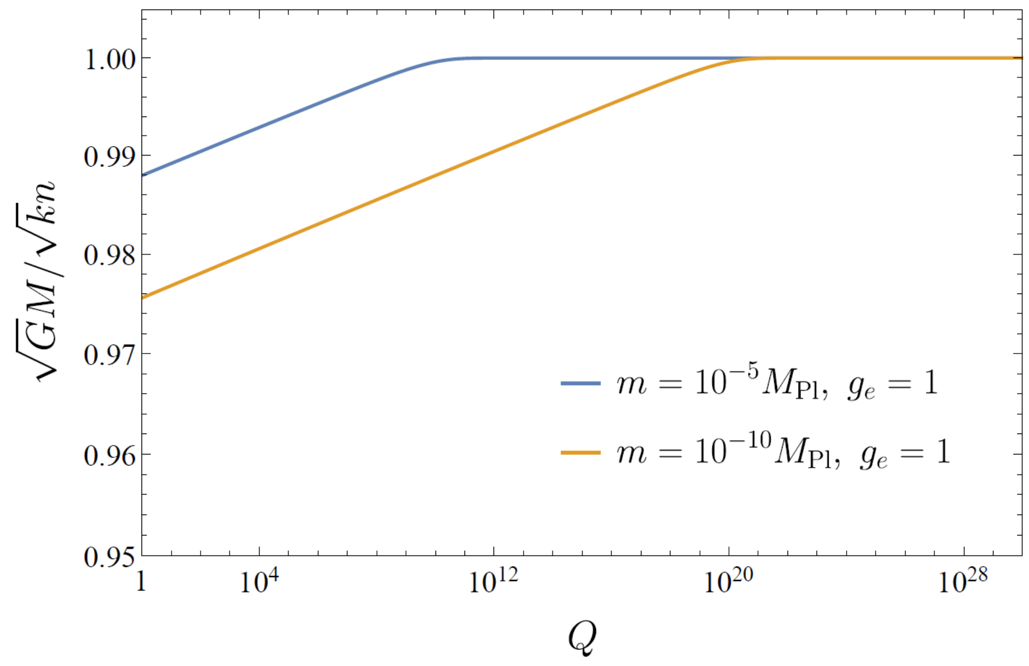
Scalar loop



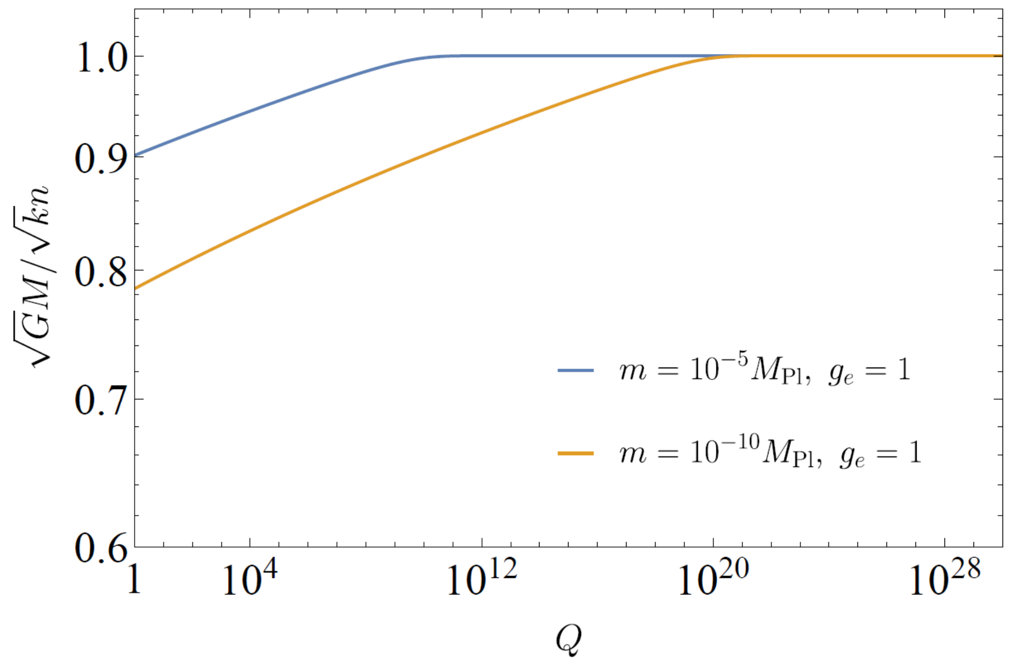
Fermion loop

Mass-to-charge ratio of extremal BH w/ EH Lagrangian

- Charged particle mass dependence



Scalar loop

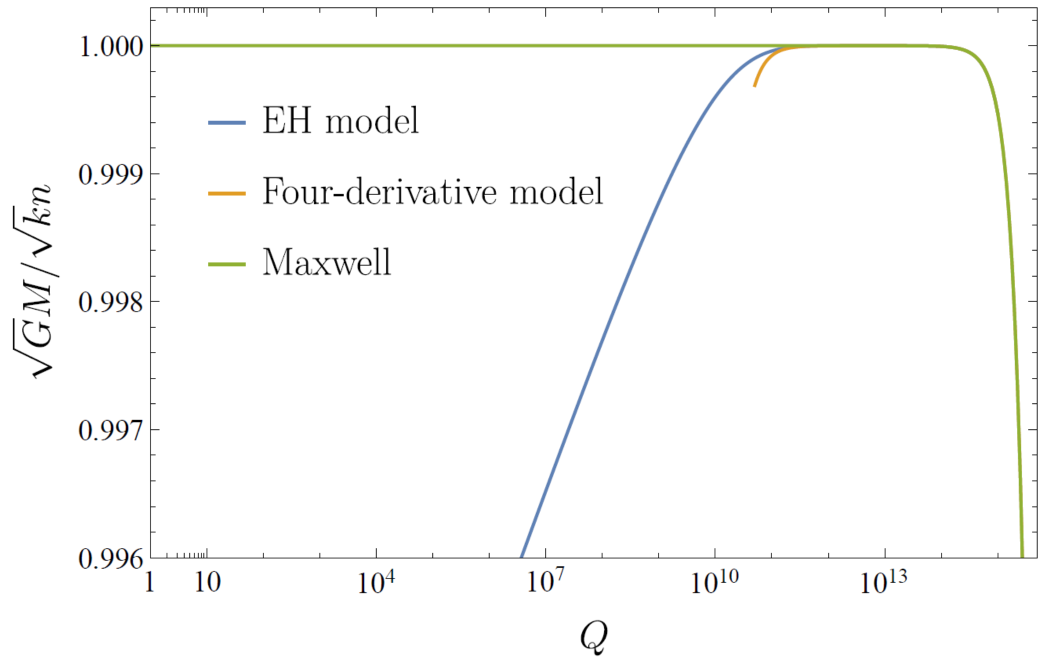


Fermion loop

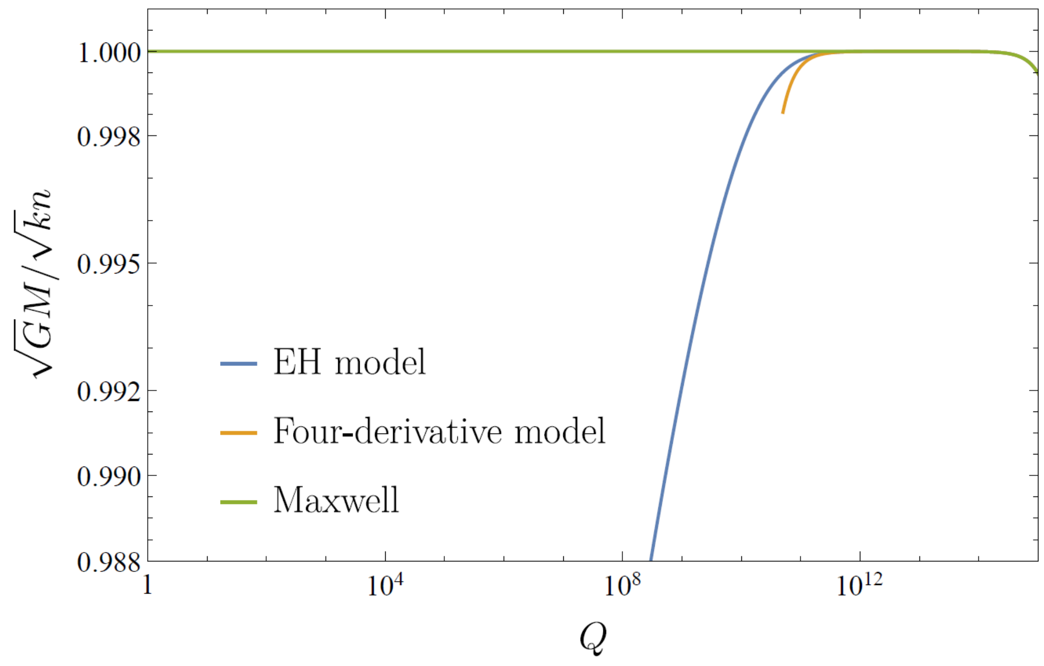
BH extremality in asymptotically dS geometry

Mass-to-charge ratio of extremal BH w/ EH Lagrangian

- $\Lambda = \left(10^{-15} M_{\text{Pl}}\right)^2, m = 10^{-5} M_{\text{Pl}}, g_e = 1$



Scalar loop



Fermion loop

Mass-to-charge ratio of extremal BH w/ EH Lagrangian

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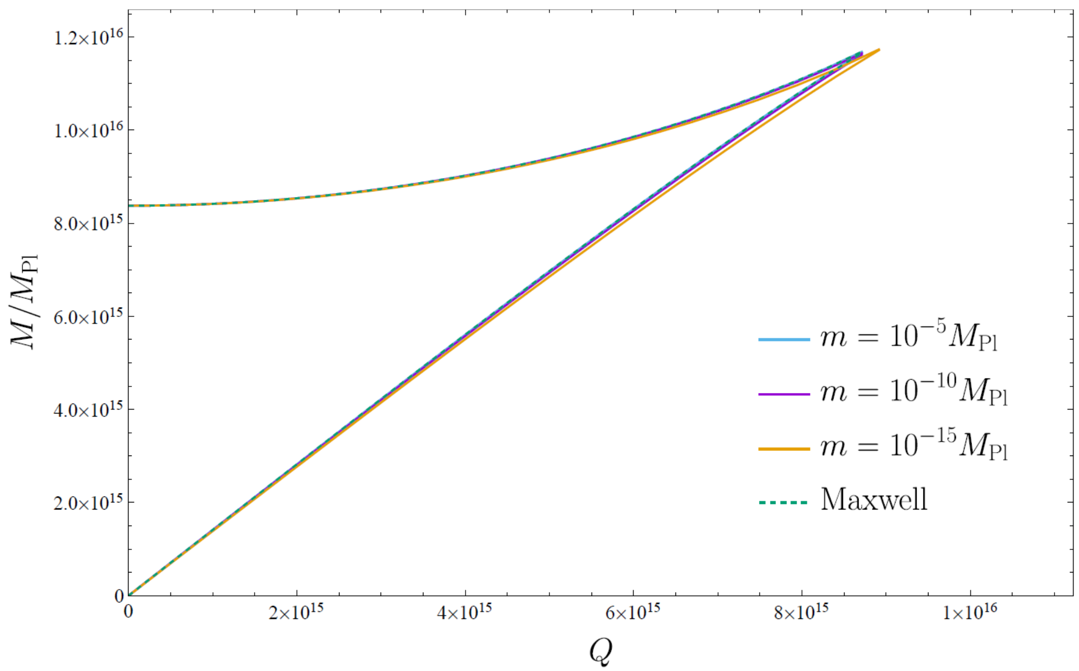
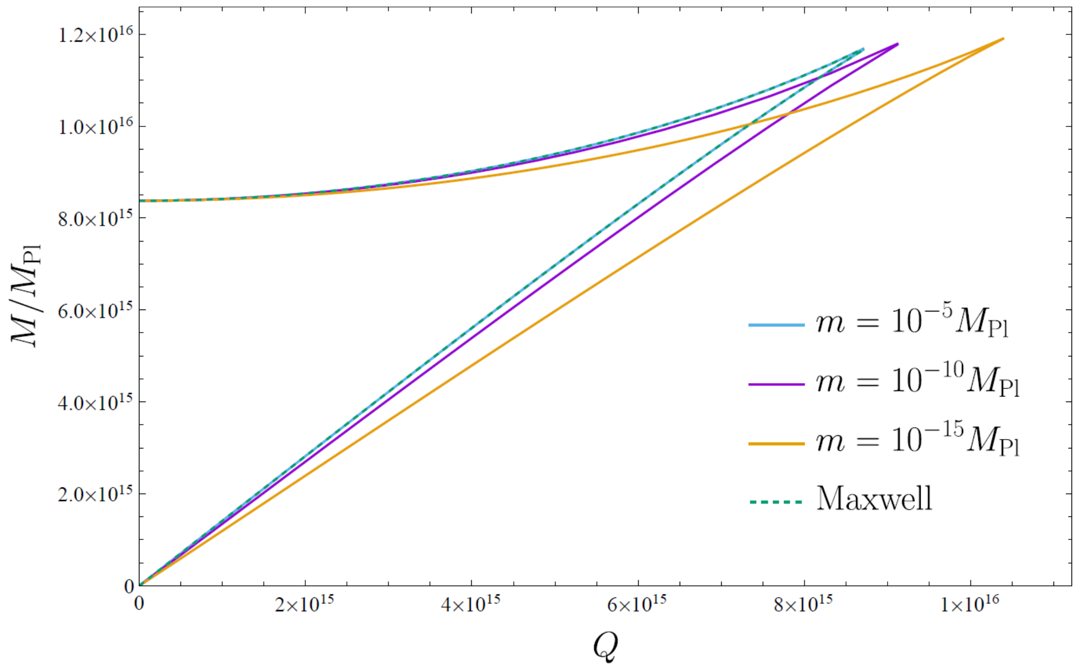


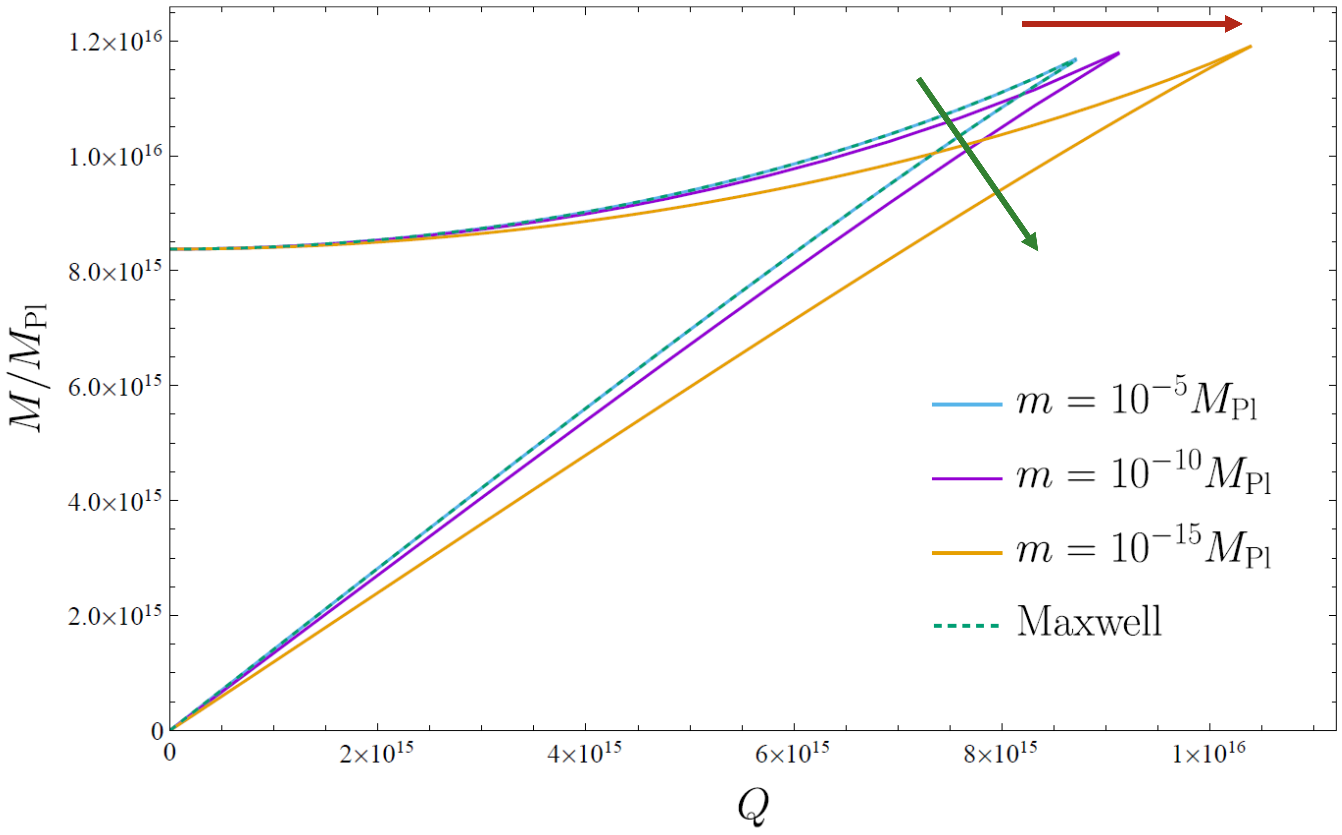
Scalar loop



Fermion loop

Mass-to-charge ratio of extremal BH w/ EH Lagrangian

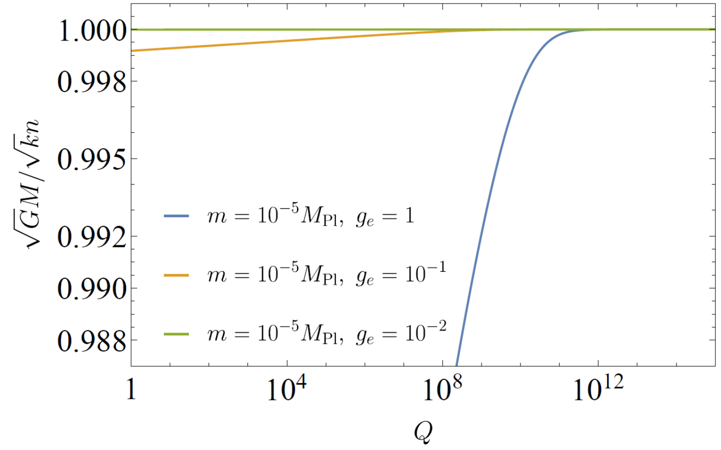
- $\Lambda = \left(10^{-15} M_{\text{Pl}}\right)^2, g_e = 1$



Euler-Heisenberg vs gravity

- The correction to the BH mass-to-charge ratio from EH model

$$\Delta\mu_{\rm EH} \sim -g_e^2 \log \frac{Q_*}{Q} \qquad Q_* \sim g_e \left(\frac{M_P}{m}\right)^2$$



- A gravitational correction [Hamada-Noumi-Shiu '19]

$$\Delta\mu_{\rm grav} \sim \frac{g_e^2 M_P^2}{m^2} Q^{-2} \sim \frac{m^2}{M_P^2}\left(\frac{Q_*}{Q}\right)^2$$

$$\Delta \boldsymbol{L} \sim W_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma},\ R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}, \ldots$$

- Monotonicity

$$\Delta\mu_{\rm EH} + \Delta\mu_{\rm grav} < 0 \quad \longleftrightarrow \quad \frac{Q}{Q_*} \gtrsim \frac{m}{g_e M_P}$$

Analogy of gravitational positivity

- Let us consider the typical energy of the gauge flux around the horizon

$$\mathsf{E} \sim \mathcal{F}^{1/4} \sim \left(\frac{n^2}{r_H^4}\right)^{1/4} \sim \left(\frac{g_e^2 Q^2}{r_H^4}\right)^{1/4} \sim (g_e/Q)^{1/2} M_P$$

- From the monotonicity, we obtain

$$\Delta\mu_{\rm EH} + \Delta\mu_{\rm grav} < 0 \quad \longleftrightarrow \quad \frac{Q}{Q_*} \gtrsim \frac{m}{g_e M_P}$$

- We find the bound on this energy scale

$$\mathsf{E} \lesssim \sqrt{g_e m M_P}$$

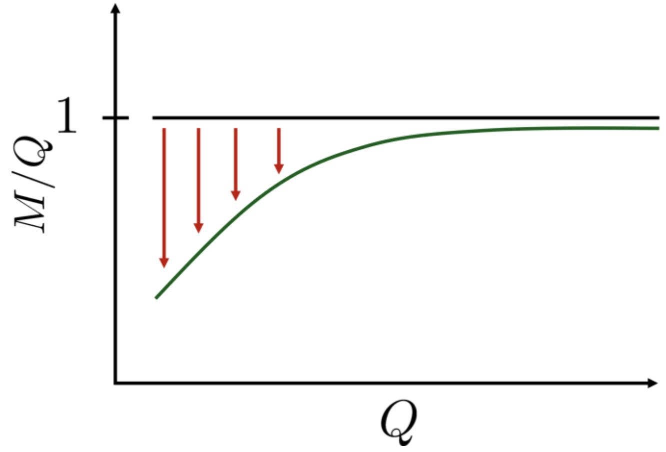
Cf. gravitational positivity bounds in QED
[Alberte-de Rham-Jaitly-Tolley '21]

Summary

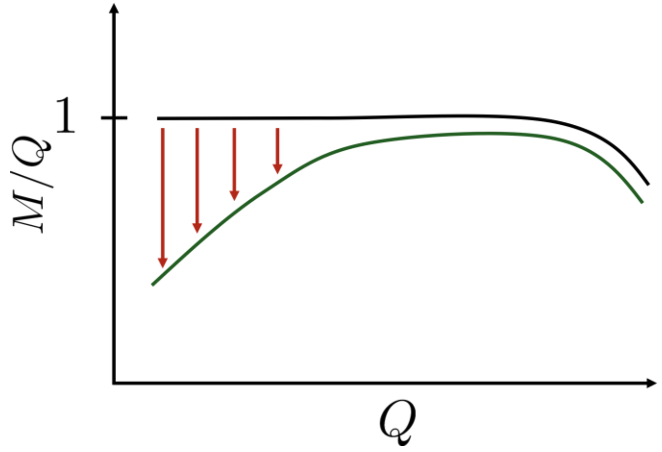
- WGC implies the monotonicity of the extremal curve
- We review the derivation of the BH solution in nonlinear electrodynamics and construct the solution for the Euler-Heisenberg Lagrangian (+ DBI)
- We show the monotonic behavior favored by WGC
- Implications:
  - Deformation of the shark fin in the de Sitter spacetime
  - Analogy of the gravitational positivity

Messages of this talk

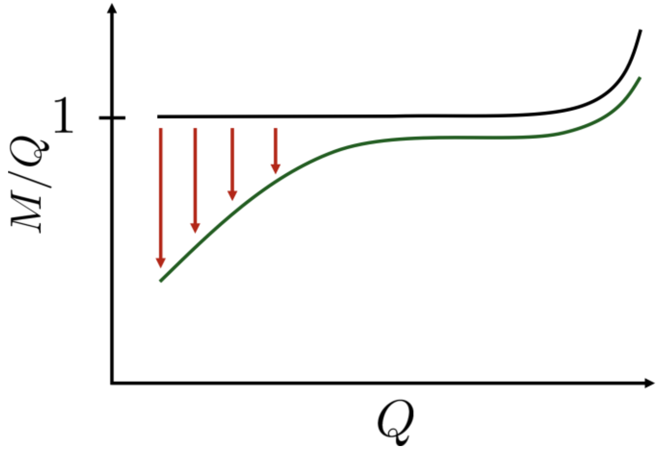
- Black hole mass-to-charge ratio



Asymptotically flat

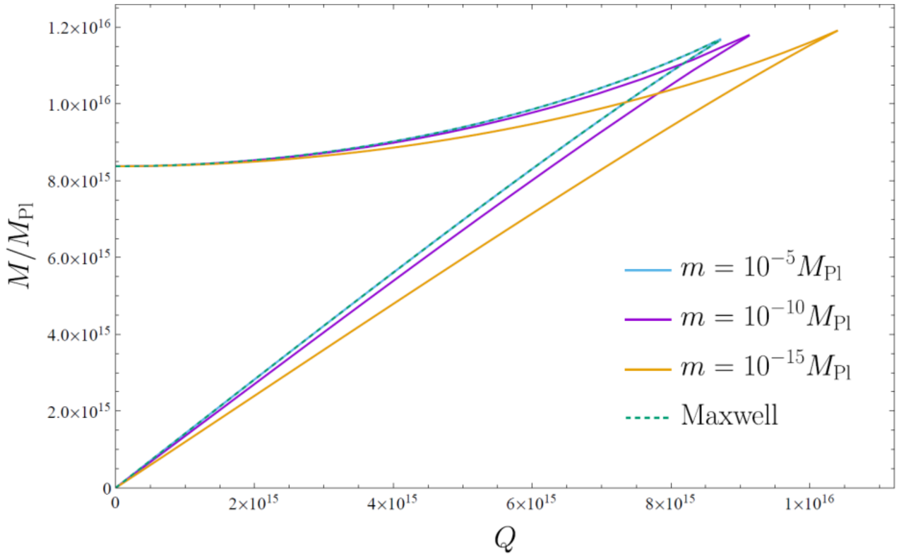


Asymptotically dS



Asymptotically AdS

- De Sitter black hole spectrum



Backup

Notation

- Metric

$$\eta_{ab} = \mathrm{diag}(-1, +1, \ldots, +1), \qquad g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$$

- Action and Lagrangian (density)

$$S = \int \mathrm{d}^D x \mathcal{L} = \int \mathrm{d}^D x \sqrt{-g} \boldsymbol{L}, \quad \mathcal{L} = \sqrt{-g} \boldsymbol{L}$$

- Clifford algebra

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}$$

- Anti-symmetric symbols and tensors

$$\varepsilon^{\mu\nu\rho\sigma} = e\epsilon^{\mu\nu\rho\sigma} = e\epsilon^{abcd} e_a{}^\mu e_b{}^\nu e_c^\rho e_d{}^\sigma,$$

$$\varepsilon_{\mu\nu\rho\sigma} = e^{-1}\epsilon_{\mu\nu\rho\sigma} = e^{-1}\epsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho e^d{}_\sigma$$

$$\epsilon_{\hat{0}\hat{1}\hat{2}\hat{3}} = +1$$

Black hole extremality in Einstein-Maxwell theory

- Metric function is assumed to have the following form

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3} + \frac{8\pi G}{r}\int \mathrm{d}r' r'^2 T^t{}_t$$

- The derivative of $M(r_H, n)$ with respect to the horizon radius is

$$\frac{\partial M}{\partial r_H} = \frac{1}{2G}(1 - \Lambda r_H^2 + 8\pi G r_H^2 T^t{}_t)$$

- On the other hand, the horizon radius of the extremal BH satisfies $f(r_H) = f'(r_H) = 0$

$$f(r_H) = 1 - \frac{2GM}{r_H} - \frac{\Lambda r_H^2}{3} + \frac{8\pi G}{r_H}\int^{r_H} \mathrm{d}r r^2 T^t{}_t = 0$$

$$f'(r_H) = \frac{2GM}{r_H^2} - \frac{2}{3}\Lambda r_H - \frac{8\pi G}{r_H^2}\int^{r_H} \mathrm{d}r r^2 T^t{}_t + 8\pi G r_H T^t{}_t = 0$$

Substituting 1st Eq to 2nd Eq. ⇒ $\partial M/\partial r_H = 0$ is obtained

Phase structures of horizons

- The line of the mass-to-charge ratio disappears at the small Q region
  → The horizon degeneracy does not occur there

- Let us expand $f(r)$ at $r \to 0$ in the following form

$$f(r) = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3} + \frac{8\pi G}{r}\int_\infty^r \mathrm{d}r' r'^2 \boldsymbol{L}\Big(\frac{n^2}{8r'^4}, 0\Big)$$

$$\approx -\frac{2GM}{r} + \frac{8\pi G}{r}\int_\infty^0 \mathrm{d}r' r'^2 \boldsymbol{L}\Big(\frac{n^2}{8r'^4}, 0\Big)$$

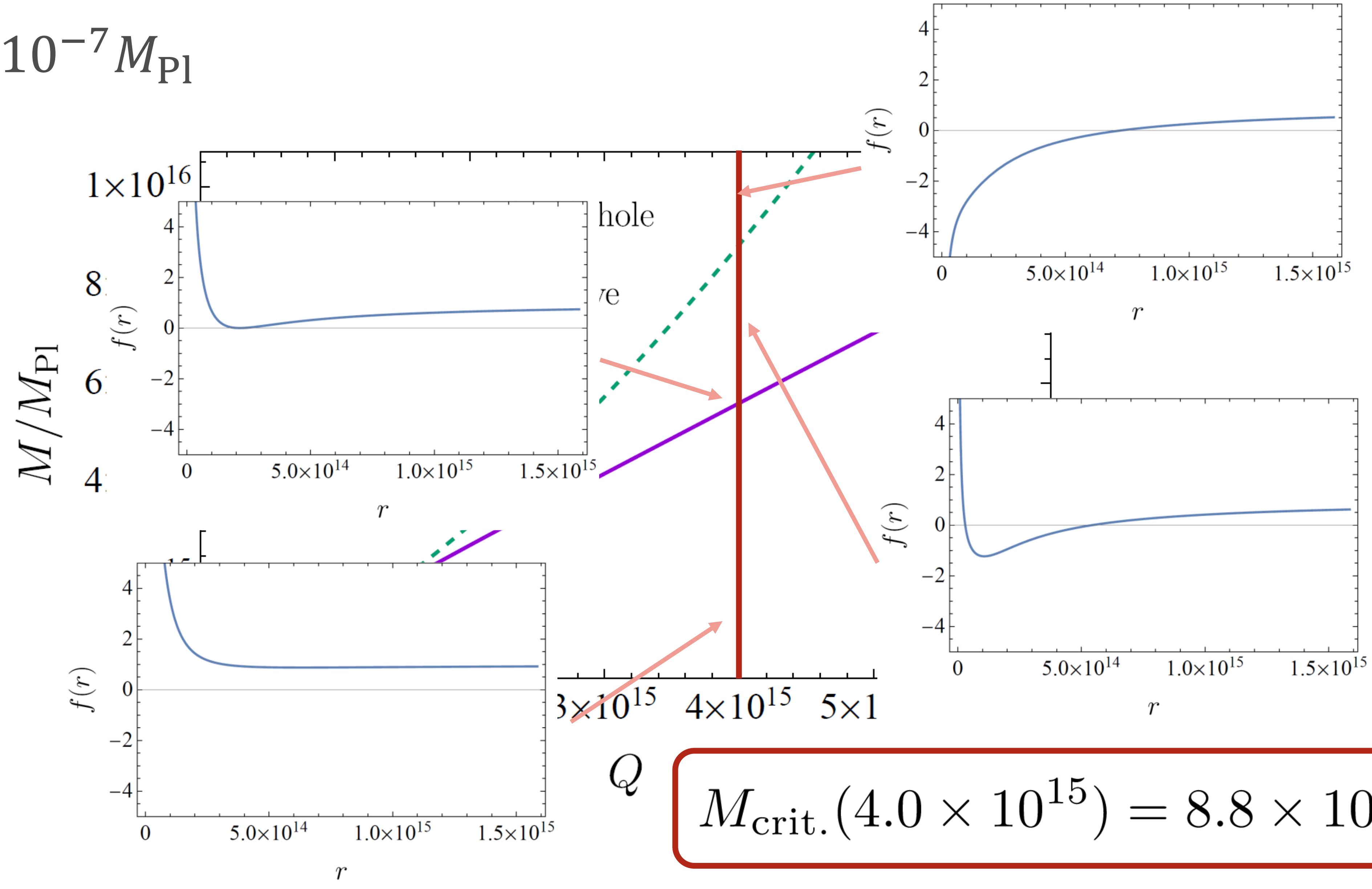
$$=: \frac{2G}{r}\big(M_{\mathrm{crit.}}(Q) - M\big)$$

- Critical mass

$$M_{\mathrm{crit.}}(Q) := 4\pi \int_\infty^0 \mathrm{d}r' r'^2 \boldsymbol{L}\Big(\frac{n^2}{8r'^4}, 0\Big)$$

Phase structures of horizons

- $\Lambda_{\mathrm{DBI}} = 10^{-7} M_{\mathrm{Pl}}$



$$M_{\mathrm{crit.}}(4.0 \times 10^{15}) = 8.8 \times 10^{15} M_{\mathrm{Pl}}$$