Generalised Bismut-Lichnerowicz Formulae and Quantum Corrections in String Theory

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Outline

1. Bismut-Lichnerowicz formulae and generalised geometry
2. Romans' 6d gravities and string dualities
3. Towards dualitiy manifest higher derivative corrections
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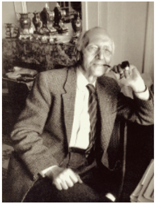
Bismut-Lichnerowicz formulae and generalised geometry

Lichnerowicz formula

Spineurs harmonique (harmonic spinors)[A. Lichnerowicz '63]



SÉANCE DU 1er JUILLET 1963. 7

MÉMOIRES ET COMMUNICATIONS

DES MEMBRES ET CORRESPONDANTS DE L'ACADÉMIE.

GÉOMÉTRIE DIFFÉRENTIELLE. — *Spineurs harmoniques.* Note (*) de M. **André Lichnerowicz.**

Étude des 1-spineurs harmoniques sur une variété spinorielle compacte. Un cas de nullité du Â-genre de Hirzebruch, d'après Atiyah-Singer.

Standard Lichnerowicz formula

3. La connexion riemannienne de V_n induit sur le fibré principal $\mathcal{S}(V_n)$ une connexion infinitésimale appelée connexion spinorielle canonique. Soit ∇ l'opérateur de dérivation covariante dans cette connexion, dérivation qui est compatible avec l'opération $\sim$. Si ψ est un 1-spineur, nous introduisons l'opérateur de Dirac P défini par

$$P\psi = \gamma^\alpha \nabla_\alpha \psi,$$

et qui vérifie :

$$P\mathcal{B} = -\mathcal{B}P. \quad (6)$$

A l'aide de l'identité de Ricci, le *laplacien* $\Delta = P^2$ d'un 1-spineur peut être mis sous la forme

$$\Delta\psi = -\nabla^\rho \nabla_\rho \psi + \frac{1}{4} R\psi, \quad (7)$$

où R est la courbure riemannienne scalaire de la variété.

Bismut-Lichnerowicz formula

• A physicist familiar version is given in components as

$$\left(\nabla\!\!\!/^{\,2} - \nabla_\mu \nabla^\mu\right)\epsilon = -\frac{1}{4} R\epsilon \,.$$

• ∇ here is the covariant derivative with respect to the torsion free Levi-Civita connection and R is the scalar curvature.

• A version of the Lichnerowicz formula with closed three-form torsion H is found due to Bismut.

• The Bismut version have a pair of operators such that the difference of their squares is again tensorial, i.e. there is no derivative piece applying on the spinor on the right hand side.

Bismut-Lichnerowicz formula

A local index theorem for non Kähler manifolds [J-M. Bismut '89]

Let now B be a smooth section of $\Lambda^3(T^*M)$. Of course, ${}^cB$ acts like ${}^cB \otimes 1$ on $F \otimes \xi$. Let $\|B\|$ be the norm of B in $\Lambda^3(T^*M)$.

Theorem 1.3. *The following identity holds*

$$(D^L + {}^cB)^2 = -\sum_1^n (\nabla^L_{e_i} + {}^c(i_{e_i}B))^2 + \frac{K}{4} + {}^c((\nabla^\xi)^2) + {}^c(dB) - 2\|B\|^2 . \qquad (1.13)$$

- In local coordinates

$$(\nabla\!\!\!/^H)^2 - (\nabla^H)^\mu \nabla^H_\mu \epsilon = -\frac{1}{4} R\epsilon + \frac{1}{48} H^{\mu\nu\rho} H_{\mu\nu\rho}\epsilon$$

- The operators look like

$$\nabla^H_\mu \epsilon = \nabla_\mu \epsilon + \frac{1}{8} H_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon$$

$$\nabla\!\!\!/^H \epsilon = \nabla\!\!\!/ \epsilon + \frac{1}{24} H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon$$

Note that the Dirac operator is no longer the trace of the covariant derivative with torsionful connection.

Why and how are these related to SUGRA/String?

- Bismut's formula captures the physics of the ten-dimensional NS sector (without dilaton) —the action and the equations of motion to order of lowest derivatives.
- The operators can be identified as supersymmetry variations of the gravitino ψ_μ and the dilatino ρ. (no dilaton terms, no RR-Flux terms)
- Interestingly enough, it is the inclusion of the dilaton that requires a truly generalised treatment.
- TYPE II SUSY variation including dilatons are

$$\delta\psi_\mu = D_\mu\epsilon = \nabla_\mu\epsilon + \frac{1}{8}H_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon\,,$$

$$\delta\rho = D\epsilon = \nabla\!\!\!/\,\epsilon + \frac{1}{24}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon - \gamma^\mu\partial_\mu\phi\epsilon\,.$$

- However, difference of squares of them does not provide tensorial quantities.

Generalised geometry

Hint from generalised tangent bundle

- In using Bismut-Lichnerowicz formula with dilaton, add extra terms when doing $D^\mu D_\mu \epsilon$

$$2\nabla_\mu \phi D^\mu \epsilon = 2\nabla_\mu \phi \left(\nabla^\mu \epsilon + \frac{1}{8} H^\mu{}_{\nu\rho} \gamma^{\nu\rho} \epsilon \right) .$$

- The formula then yields the bosonic NS action S, and the Bianchi identity for H

$$-\frac{1}{4} S \epsilon - \nabla_{[\mu} H_{\nu\rho\sigma]} \gamma^{\mu\nu\rho\sigma} \epsilon \, .$$

- The key here is having a different action of the same operator on a spinor and a vector-spinor. The above-mentioned pair of operators is then seen as a component of the same "generalised Levi-Civita connection" D_M on $T \oplus T^*$. [Waldram et al. '2011]

Generalised geometry and more

Generalised complex geometry [N. Hitchin '2000 and M. Gualtieri '2004]
GCG unites the geometric data with the antisymmetric two form field and puts diffeomorphisms and gerbe gauge transformations on equal footing

$$
\begin{aligned}
(g, B, \phi) &\longrightarrow (g + \mathcal{L}_v g, B + \mathcal{L}_v B, \phi + \mathcal{L}_v \phi) \\
(g, B, \phi) &\longrightarrow (g, B - d\lambda, \phi)\,.
\end{aligned}
$$

• For TYPE II supergravities, notably, a generalised vanishing torsion condition on a generalised metric-connection yields a notion of a generalised Ricci tensor which fully captures the physics and geometrize the $O(d, d)$ structure. [Waldram et al. '2011]

• The passage to non-closed torsion H requires an extension of the generalised tangent bundle to $T \oplus T^* \oplus \mathfrak{g}$ and the inclusion of gauge fields, eventually leading to a generalised complex description of heterotic strings. [Waldram et al. '2014]

Limitations and new insights from a step back

- In ordinary complex generalised geometry, the RR sector fields does not enter operators. They are mathematically spinors of the generalised tangent bundle.
- Fully geometrise both the NS and RR sector is possible. Exceptional generalised geometry describe M-theory on internal d-dimensional space of a generic supersymmetric compactification, where the generalised tangent bundle corresponds to a representation of an exceptional $E_{d(d)}$ group. [Hull, Waldram et al.]
- It is still, however, an open question whether generalised complex geometry can capture the structure of stringy perturbative corrections to the effective theories.
- Almost only work with maximal SUSY.
- Continuous version of U-dualities ($E_{d(d)}$) is manifest in the exceptional setup. How about other dualities?
- At this point one would like to dream that the suitable versions of the Lichnerowicz formula underlie every supersymmetric theory.

Limitations and new insights from a step back

- In fact, generalised geometry structure is not the unique way out for the missing piece needed for tensoriality. [forthcoming paper '2023]
- SUSY variation of gravitino $\delta\psi_\mu = D_\mu \epsilon$ defines an operator sending spinor to vector-spinor

$$\begin{aligned} \mathcal{C}^\infty(S) &\longrightarrow \mathcal{C}^\infty(S \otimes T^*M) \\ \epsilon &\longmapsto D\epsilon\,. \end{aligned}$$

- The spinor bilinear form $<\epsilon, \chi> = \int \mathrm{e}^{-2\phi}\sqrt{-g}\,\bar{\epsilon}\chi$ and $<<\psi, \eta>> = \int \mathrm{e}^{-2\phi}\sqrt{-g}\,\bar{\psi}^{\bar{a}}\eta_{\bar{a}}$ allow as to compute the formal adjoint of D, i.e. an operator $D^\dagger : \mathcal{C}^\infty(S \otimes T^*M) \longrightarrow \mathcal{C}^\infty(S)$ such that

$$<< D\epsilon, \psi >> = < \epsilon, D^\dagger\psi >$$

- Tensoriality is restored by suitable choice of integration measure in the bilinear form.

Bismut-Lichnerowicz with Bochner Laplacian

- For instance, for $D_\mu \epsilon = \nabla_\mu \epsilon + \frac{1}{8} H_{\mu\nu\rho}\gamma^{\nu\rho}$ one computes

$$D_\mu^\dagger \psi^\mu = -\nabla_\mu \psi^\mu - \frac{1}{8} H_{\mu\alpha\beta}\gamma^{\alpha\beta}\psi^\mu + 2\partial_\mu \phi \psi^\mu$$

- In fact, we can reinterpret the Bismut-Lichnerowicz formula as

$$\left(D_\mu^\dagger D^\mu + D^\dagger D\right)\epsilon = (\text{tensorial supergravity quantities})\,\epsilon\,.$$

- The squares above are analog of $\nabla^\dagger \nabla$ of Bochner Laplacian on general vector bundles, where $\nabla^\dagger$ is specified by the fiber metric. It differs to the usual connection Laplacian by a sign.
- How about adding gauge fields within the gravity multiplet? It is not a problem in some specific dimensions and we would like to look at $6d$.
- Minimal input: SUSY, and assume that SUSY is order by order.

Romans' 6d gravities and string dualities

The multiplet and supergravities

- Six-dimensional $\mathcal{N} = (1,1)$ gravity multiplet [Strathdee'87] consists of one graviton $g_{\mu\nu}$, four vectors a_μ and ${A_\mu}^I$ with field strengths $f_{\mu\nu}$ and $F^I_{\mu\nu}$, with I runs through 1, 2 and 3 respectively, a 2-form tensor field $B_{\mu\nu}$, a scalar ϕ, two symplectic-Majorana gravitini $\psi_{\mu i}$ and two symplectic-Majorana spin-$\frac{1}{2}$ fields χ_i.
- As shown by Romans [Romans'85], the same field content can be interpreted as degrees of freedom of different supergravity theories in 6d. Some of which are related by duality that can be traced back to string dualities in ten dimensions. His 6d $\mathcal{N} = 4^0$ supergravity can be think of as IIA string reduced on $K3$ with all the 6d vector multiplets removed. The dual theory, called $\mathcal{N} = \tilde{4}^0$, describe the gravity part of Heterotic string on T^4.
- In Romans' $D = 6$ $\mathcal{N} = (1,1)$ gauged supergravity, the four vectors are $SU(2) \times U(1)$ Yang-Mills fields. For simplicity we just want to include the gravi-photon $f_{\mu\nu}$.

Generalised Bismut-Lichnerowicz in six dimensions

- Including the field strength of a_μ the **heterotic** operators (SUSY variations) read

$$D_\mu\epsilon := \nabla_\mu\epsilon + \frac{1}{8}H_{\mu\alpha\beta}\gamma^{\alpha\beta}\epsilon + \frac{1}{2}\gamma^\nu f_{\mu\nu}\epsilon$$

$$D\epsilon := \nabla\!\!\!/\,\epsilon + \frac{1}{24}\gamma^{\mu\nu\rho}H_{\mu\nu\rho}\epsilon - \gamma^\mu\partial_\mu\phi\epsilon + \frac{1}{4}\gamma^{\mu\nu}f_{\mu\nu}\epsilon$$

- One gets

$$\begin{aligned}\left(D^\dagger D + D^\dagger_\mu D_\mu\right)\epsilon = \big(&-\frac{1}{4}R - \nabla^2\phi + (\partial\phi)^2 - \frac{1}{8}f_{\mu\nu}f^{\mu\nu} + \frac{1}{48}H_{\mu\nu\rho}H^{\mu\nu\rho}\\ &+ \frac{1}{24}\nabla_\mu H_{\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma} + \frac{1}{4}\nabla_\mu f_{\nu\rho}\gamma^{\mu\nu\rho} - \frac{1}{16}f_{\mu\nu}f_{\rho\sigma}\gamma^{\mu\nu\rho\sigma}\big)\epsilon\,,\end{aligned}$$

where the $\gamma^{(4)}$ part yields the Bianchi identity

$$\nabla_{[a}H_{bcd]} = \frac{3}{2}f_{[ab}f_{cd]}\,,$$

which in differential form language reads

$$dH = f\wedge f\,.$$

Dualising the heterotic version to IIA version

Duality map between six SUGRAs

The map is explicitly given as [Liu and Minasian '13]

$$\begin{aligned} \phi^{\mathsf{HET}} &= -\varphi^{\mathsf{IIA}} \\ g^{\mathsf{HET}} &= \mathrm{e}^{2\phi} g^{\mathsf{IIA}} = \mathrm{e}^{-2\varphi} g^{\mathsf{IIA}} \\ H^{\mathsf{HET}} &= \mathrm{e}^{2\phi} \star \tilde{H}^{\mathsf{IIA}} = \mathrm{e}^{-2\varphi} \star \tilde{H}^{\mathsf{IIA}} \,. \end{aligned}$$

We also need to rescale fermions to keep tensoriality: $\psi_\mu \longrightarrow \mathrm{e}^{-\varphi}\psi_\mu$.

Operators change under the duality map

$$\begin{aligned} D_\mu \epsilon &\equiv \mathrm{e}^{\varphi} \left(\nabla_\mu \epsilon - \frac{1}{2}\gamma_{\mu\nu}\partial^\nu \varphi \epsilon - \frac{1}{24}\gamma_7 \gamma_{\mu\nu\rho\sigma}\tilde{H}^{\nu\rho\sigma}\epsilon + \frac{1}{2}\mathrm{e}^{\varphi} f_{\mu\nu}\gamma^\nu \epsilon \right) \\ D\epsilon &\equiv \mathrm{e}^{\varphi} \left(\nabla\!\!\!/ \epsilon - \frac{3}{2}\partial\!\!\!/ \varphi \epsilon + \frac{1}{24}\gamma_7 \tilde{H}_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon + \frac{1}{4}\mathrm{e}^{\varphi} f_{\mu\nu}\gamma^{\mu\nu}\epsilon \right) \,. \end{aligned}$$

IIA Bismut-Licherowicz with dilaton

- The canonical choice for the measure is $\mathrm{e}^{-4\varphi}\sqrt{-g}$, which is also obtained by direct duality transformation from the heterotic measure

$$\mathrm{e}^{-2\phi}\sqrt{-g^{\mathsf{HET}}} \longrightarrow \mathrm{e}^{4\phi}\sqrt{-g^{\mathsf{IIA}}} = \mathrm{e}^{-4\varphi}\sqrt{-g}\,.$$

- The result
  Note that we multiply the whole thing by -4 to compare with the standard normalisation

$$\begin{aligned}
&\mathrm{e}^{2\varphi}\Big(R + \frac{1}{12}\tilde{H}^2 + \frac{1}{2}\mathrm{e}^{2\varphi}f^2 - 8(\partial\varphi)^2 + 6\nabla^2\varphi \\
&\quad - \gamma^{\mu\nu\rho}\mathrm{e}^{\varphi}\nabla_\mu f_{\nu\rho} \\
&\quad + \gamma^{\mu\nu\rho\sigma}\left[-\frac{1}{6}\mathrm{e}^{2\varphi}\nabla_\mu\left(\mathrm{e}^{-2\varphi}(\star\tilde{H}_{\nu\rho\sigma})\right) + \frac{1}{4}\mathrm{e}^{2\varphi}f_{\mu\nu}f_{\rho\sigma}\right]\Big)\,,
\end{aligned}$$

Interpretation of the dual Lichnerowicz

- The 0-form part together with the measure gives

$$\int \mathrm{e}^{-2\varphi}\sqrt{-g}\,dx\left(R+\frac{1}{12}\tilde{H}^2+\frac{1}{2}\mathrm{e}^{2\varphi}f^2-8(\partial\varphi)^2+6\nabla^2\varphi\right)$$

$$=\int \mathrm{e}^{-2\varphi}\sqrt{-g}\,dx\left(R+\frac{1}{12}\tilde{H}^2+\frac{1}{2}\mathrm{e}^{2\varphi}f^2+4(\partial\varphi)^2\right)$$

$$=\int\left(\mathrm{e}^{-2\varphi}\left[R+\frac{1}{2}\star\tilde{H}\wedge\tilde{H}+4\star d\varphi\wedge d\varphi\right]+\star f\wedge f\right)\,.$$

- The 4-form part is: $\mathrm{e}^{4\varphi}\left(-d(\mathrm{e}^{-2\varphi}\star\tilde{H})+f\wedge f\right)$.
  We can multiply it by $\mathrm{e}^{-4\varphi}$ and then wedge it with the 2-form $\tilde{B}$ to get a top-form:

$$\int -\mathrm{e}^{-2\varphi}\star\tilde{H}\wedge\tilde{H}+\tilde{B}\wedge f\wedge f$$

- These together we get the action

$$\int\left(\mathrm{e}^{-2\varphi}\left[R-\frac{1}{2}\star\tilde{H}\wedge\tilde{H}+4\star d\varphi\wedge d\varphi\right]+\star f\wedge f+\tilde{B}\wedge f\wedge f\right)\,.$$

Towards dualitiy manifest higher derivative corrections

SUSY order by order for heterotic side

• Let us now consider higher order modifications of the SUSY operators. We demand that they still retain a form that is consistent with tensoriality.

$$
\begin{aligned}
\delta\psi_\alpha &= D_\alpha\epsilon = D^0_\alpha\epsilon - 4k(X_2)_{\alpha\beta}\gamma^\beta\epsilon - k((D^0)^\dagger)^{\alpha\beta}\left(K_{\alpha\beta}\epsilon\right), \\
\delta\rho &= D\epsilon = D^0\epsilon - 2k(X_2)_{\alpha\beta}\gamma^{\alpha\beta}\epsilon,
\end{aligned}
$$

where k is a parameter to keep track of derivative count, $(X_2)_{\alpha\beta} = \frac{1}{2}R^-_{\alpha\beta\mu\nu}f^{\mu\nu} + \frac{1}{4}f_{\alpha\beta}f^2$, and

$$
K_{\alpha\beta}\epsilon = 8[D_\alpha, D_\beta]\epsilon = R^+_{\alpha\beta\mu\nu}\gamma^{\mu\nu}\epsilon + 4(\tilde{\nabla}_{[\alpha}f_{\beta]\rho})\gamma^\rho\epsilon + 2f_{\alpha\rho}f_{\beta\sigma}\gamma^{\rho\sigma}\epsilon,
$$

is proportional to the curvature of the D^0_α connection.

• The result is a polyform Θ_1 where the scalar component is the dilaton eom (from which the higher-order Lagrangian can be recovered) and the 3- and 4-forms are Bianchi identities (in particular, they have to be closed).

IIA side and a glimpse of the result

- In principle, one applies the duality map for the corrected operators and do the generalised Bismut-Lichnerowicz. We verified that the next level is consistent with the standard result, see e.g. [Liu and Minasian '13].

- The IIA correction term K is now

$$
\begin{aligned}
K_{\mu\nu}{}^{ab} =& R_{\mu\nu}{}^{ab} - 2\delta^{[a}_{[\mu} R^{b]}_{\nu]} - 4\delta^{[a}_{[\mu}\nabla_{\nu]}\varphi\partial^{b]}\varphi - \delta^{a}_{[\mu}\delta^{b}_{\nu]}\nabla^2\varphi - \frac{1}{2}\tilde{H}_{[\mu}{}^{a\kappa}\tilde{H}_{\nu]\kappa}{}^{b} \\
&+ \nabla_{[\mu}(\star\tilde{H})_{\nu]}{}^{ab} + 2(\star\tilde{H})_{\mu\nu}{}^{[a}\nabla^{b]}\varphi + 2\delta^{[a}_{[\mu}(\star\tilde{H})_{\nu]}{}^{b]\lambda}\nabla_{\lambda}\varphi\,.
\end{aligned}
$$

IIA side and a glimpse of the result

- We also define $\hat{H} = \star\tilde{H}$.

- The result is (remember that IIA measure is with $\mathrm{e}^{-4\varphi}$ so this would give one-loop and α' (we used k) mixed corrections)

$$
\begin{aligned}
& k\mathrm{e}^{4\varphi}\Big(-\frac{1}{2}K_{\mu\nu\alpha\beta}R^{\mu\nu}{}_{\lambda\kappa}\gamma^{\alpha\beta\lambda\kappa} + K_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\
& - K^{\mu\alpha\beta\kappa}\nabla_\mu\nabla_\nu\varphi\gamma^{\nu\alpha\beta\kappa} + 2K_{\mu\rho\nu}{}^{\rho}\nabla_\mu\nabla_\nu\varphi \\
& - \frac{1}{4}K_{\mu\nu\alpha\beta}\nabla^\mu\hat{H}^\nu{}_{\rho\sigma}\gamma^{\alpha\beta\rho\sigma} + \frac{1}{2}K_{\mu\nu\rho\sigma}\nabla^\mu\hat{H}^{\nu\rho\sigma} \\
& - K^{\mu\alpha\beta\kappa}\nabla_\mu\varphi\nabla_\nu\varphi\gamma^{\nu\alpha\beta\kappa} + 2K_{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi \\
& - \frac{1}{2}K_{\mu\nu\alpha\beta}\gamma^{\mu\nu\alpha\beta}(\partial\varphi)^2 + K_{\mu\nu}{}^{\mu\nu}(\partial\varphi)^2 \\
& - \frac{1}{2}\hat{H}^{\mu\nu\rho}K_{\mu\alpha\beta\kappa}\nabla_\rho\varphi\gamma^{\nu\alpha\beta\kappa} + \hat{H}^{\mu\nu\rho}K_{\mu\nu}\nabla_\rho\varphi \\
& - \frac{1}{2}\hat{H}^{\mu\nu\rho}K_{\mu\nu\alpha\beta}\nabla_\kappa\varphi\gamma^{\rho\alpha\beta\kappa} + \hat{H}^{\mu\nu\rho}K_{\mu\nu\rho\sigma}\nabla_\sigma\varphi \\
& + \frac{1}{8}\hat{H}^\mu{}_{\nu\rho}\hat{H}_{\mu\sigma\tau}K^{\nu\sigma}{}_{\alpha\beta}\gamma^{\rho\tau\alpha\beta} + \frac{1}{4}\hat{H}^\mu{}_{\nu\rho}\hat{H}_{\mu\alpha\beta}K^{\nu\rho\alpha\beta}\Big)
\end{aligned}
$$

Summary and discussion

Summary and discussion

- Bismut-Lichnerowicz formula can be generalised which offers more than the common generalised complex geometry.
- The can be a (String/String) duality manifest generalised complex geometry structure that we will leave for future work.
- Uplift to 10d in consideration.
- Gauged Bismut-Lichnerowicz formula should be straight-forward to establish which describes gauged supergravities
- Gauge multiplets are not yet added in our $6d$ story and it could be plugged-in a way like the HET-generalised geometry, which deserves more attentions. (cf. transitive Courant Algebroid [Waldram '14])
- Thanks a lot for your attention!