Topological strings and 5d Wilson loops

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String Pheno ‘23

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Motivation

- For the famous quintic Calabi-Yau threefold M with one Kähler parameter t, its mirror manifold W is described by the orbifold of the hypersurface in $\mathbb{P}^4$

$$P(x,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5$$

- The information of the Yukawa coupling in the A-model M, can be computed from the periods of the B-model W

$$\varpi_j \;=\; \int_{\Gamma^j} \Omega$$

by integrating the holomorphic three-form over three cycles

Motivation

- The Yukawa coupling

$$y_{ttt} = 5 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1 - q^k} \quad , \quad q = e^{2\pi i t}$$

the integral numbers n_k predict the curve counting in enumerate geometry.

- The mirror map

$$t = \frac{1}{2\pi i} \frac{\varpi_1(\psi)}{\varpi_0(\psi)} = \log(z) + 770z + 717825z^2 + \frac{3225308000z^3}{3} + \frac{3947314570625z^4}{2} + 4062154117561404z^5 + \mathcal{O}(z^6)$$

the coefficients are not always integers, but the inverse mirror map has integral coefficients!

Motivation

- The inverse mirror map

$$\frac{1}{z} = (5\psi)^5 = \frac{1}{q} + 770 + 421375q + 274007500q^2 + 236982309375q^3 + 251719793608904q^4 + \mathcal{O}(q^5)$$

- The integrality of the coefficients in the inverse mirror maps seems to be quite general. For many K3 surfaces, such properties have been proven by Lian and Yau in the 90s.
- We focus on the case when the CY3 *M* is non-compact, then the low-energy physics from M-theory compactification is a 5d supersymmetric quantum field theory with 8 supercharges.
- We found the integers in the inverse mirror map count the BPS states of the half-BPS Wilson loop defect in the 5d physics which must be positive integers [Huang, Lee, XW '22]

5d BPS partition function

- 5d $\mathcal{N} = 1$ SYM on omega deformed $\mathbb{R}^4_{\epsilon_{1,2}} \times S^1$
- M-theory compactified on non-compact Calabi-Yau three-fold X
- The BPS states are captured by M2-branes winding on 2-cycles $C \in H_2(X, \mathbb{Z})$
- Schwinger integral of electric particles

$$\mathcal{F}_{\rm BPS} = \sum_{C \in H_2(X,\mathbb{Z})} \sum_{j_L, j_R} N^C_{j_L, j_R} \int_0^\infty \frac{ds}{s} \frac{\mathrm{Tr}_{(j_L, j_R)}(-1)^{\sigma_L + \sigma_R} e^{-sm} e^{-2se(\sigma_L F_+ + \sigma_R F_-)}}{(2\sinh(seF_1/2))(2\sinh(seF_2/2))},$$

Spectrum of dynamic operator

$e^{-(n+1/2)\beta}$

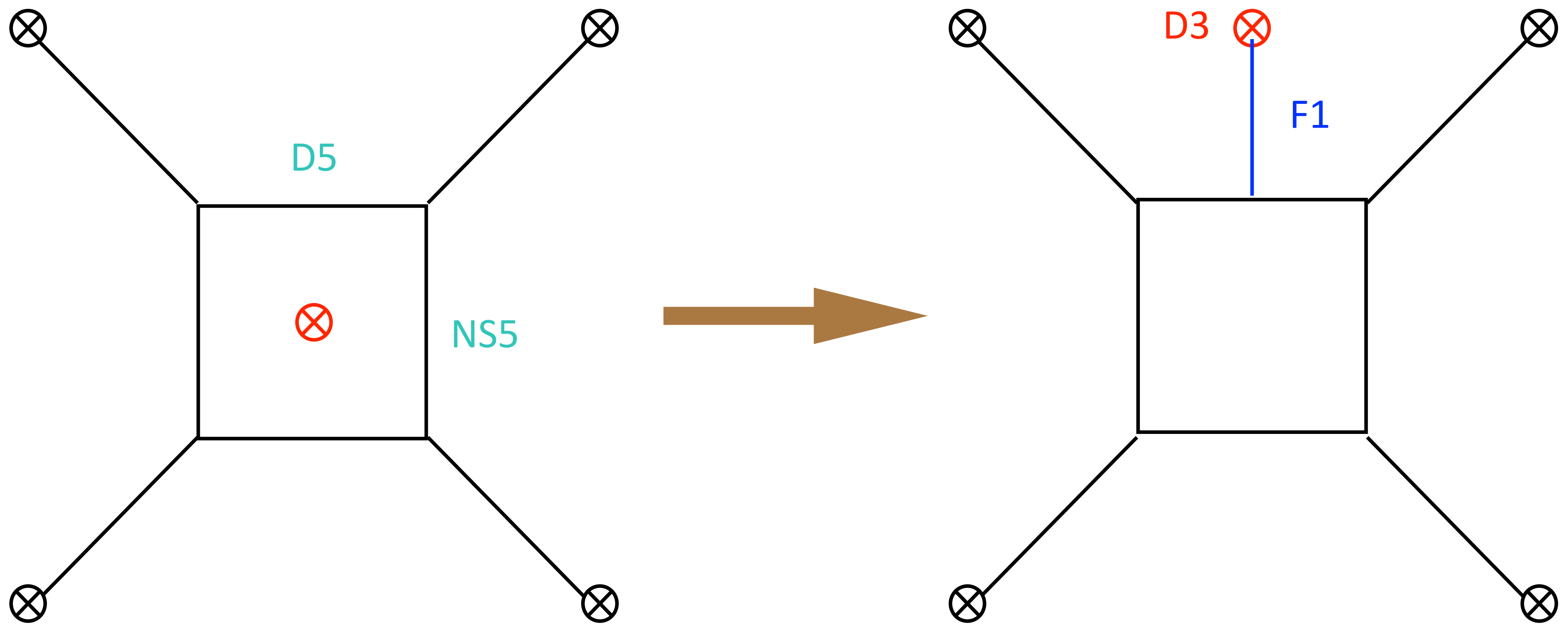
- The BPS expansion

$$\mathcal{F}_{\rm BPS} = \log Z_{\rm BPS} = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{j_L, j_R} (-1)^{2j_L + 2j_R} N^\beta_{j_L, j_R} \frac{\chi_{j_L}(k\epsilon_-)\chi_{j_R}(k\epsilon_+)}{k\left(q_1^{1/2} - q_1^{-1/2}\right)\left(q_2^{1/2} - q_2^{-1/2}\right)} e^{-k\beta \cdot t}$$

Brane realization for the defect

- Half-BPS brane bound states
- D3 branes in type IIB
- F1 string with fixed end point on D3 (stationery)

	0	1	2	3	4	5	6	7	8	9
D5	×	×	×	×	×	×				
NS5	×	×	×	×	×		×			
F1	×						×			
D3	×							×	×	×
7Br	×	×	×	×	×			×	×	×



Codimension 4 defect

5d Wilson loops

- The partition function of Wilson loops can be realized in a similar manner.
- The source particle should be a heavy, electric particle without any dynamic degree of freedom, localized at the origin of the space direction in $\mathbb{R}^4$. Such kind of particles can be realized by adding M2-branes winding on non-compact curves C that go to infinity in the CY3 and then freezing the dynamic degree of freedom

$$Z_{\text{gen}} = \exp\left(\mathcal{F}_{\text{BPS},\{\}} + \mathcal{F}_{\text{BPS},\{\mathsf{C}\}} M\right)$$

with

$$\langle W_{\mathbf{r}_{\mathsf{C}}} \rangle = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{j_L, j_R} (-1)^{2j_L + 2j_R} \widetilde{N}^{\beta}_{j_L, j_R} \chi_{j_L}(\epsilon_-) \chi_{j_R}(\epsilon_+) e^{-\beta \cdot t}$$

5d Wilson loops

- The charge of C determines the representation $\mathbf{r}_C$ of the Wilson loop. We call

$$\langle W_{\mathbf{r}_C}\rangle = \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{j_L,j_R}(-1)^{2j_L+2j_R}\widetilde{N}^{\beta}_{j_L,j_R}\chi_{j_L}(\epsilon_-)\chi_{j_R}(\epsilon_+)e^{-\beta\cdot t}$$

  the BPS expansion of the Wilson loop expectation value. $\tilde{N}^{\beta}_{j_L,j_R}$ here count the BPS states of M2-branes wrapped on the curve class $C+\beta$ in another manifold (which could be CY3), that must be non-negative integers. We call them Wilson loop BPS invariants

- One can verify that in the limit $\epsilon_1,\epsilon_2\to 0$, it reproduces the inverse mirror map

5d Wilson loops

- The B-model quantity, complex structure parameters, under mirror symmetry, are mapped to the genus zero "Wilson loops" observables in the A-model.
- Higher genus contributions can be computed from the holomorphic anomaly equation (HAE) for topological strings [Bershadsky, Cecotti, Ooguri, Vafa '93]
- By using HAE, the integrality of the Wilson loop BPS invariants have been checked for some local CY3's to very high representations. [XW '23]
- The HAE of Wilson loops suggests a generalization of Nakajima and Yoshioka's K-theoretic blowup equations [XW '23]

$$\sum_{n} Z^{\mathbf{S}} Z^{\mathbf{N}} = \left(\sum_{r_k} \langle W_{r_k} \rangle \right) Z$$

Conclusion

- We try to give a physical meaning to the integral coefficients in the inverse mirror map for non-compact Calabi-Yau threefolds. They are closely connected to the Wilson loop observables in 5d SYM.
- We propose a HAE for Wilson loops, that provides a powerful tool for calculating the Wilson loop BPS invariants. Explicit calculations have been done to verify the integrality of these invariants.
- The HAE suggests a generalization of the Nakajima and Yoshioka's K-theoretic blowup equations

Discussion

- For CY3's constructed from local del Pezzo surfaces, a mathematical definition of the genus zero Gromov-Witten invariants for Wilson loops has been given by Lau, Leung, Wu '11. It is interesting to extend their result to higher genus and compare it with our result.

- The calculation in the B-model can be extended to elliptic fibered CY3s.
  - Non-compact CY3: surface operator in 6d and discrete or trivial 2-form symmetries
  - Compact: Add gravity to 6d/5d CFT, BPS states of dynamic objects? closely related to talks by Cvetic, Heckman, Huebner ...

Thank you for your attention!