Reducing differential equations for GKZ systems

Arno Hoefnagels,

In collaboration with Thomas Grimm and Guilherme Pimentel

Utrecht University

Introduction

Hypergeometric integrals

- Want to solve integrals of the type:

$$I(z) = \int_\gamma d^n x \, \frac{x_1^{\alpha_1 - 1} \cdots x_n^{\alpha_n - 1}}{f_1(x,z)^{\beta_1} \cdots f_k(x,z)^{\beta_k}}$$

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  - CY periods
  - Feynman integrals
  - Cosmological correlators
  - Many more

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  - Feynman integrals
  - Cosmological correlators
  - Many more
- (Almost always) difficult to solve directly

Recasting as differential equations

- Solution:
  - Find differential equations in z_j satisfied by I
- Many approaches:
  - Picard-Fuchs equations when α_i and β_i are integers
  - Integration by parts identities
  - Gelfand-Kapranov-Zelevinsky (GKZ) systems

[Weinzierl, '22]

[Gelfand, Zelevinsky, Kapranov, '89]
[Saito, Sturmfelds, Takayama, '13]

Useful formulas:

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What is a GKZ system (schematically)?

- Set of differential equations in variables z_j
- Dependent on parameters α_i and β_i
- Solving this system and imposing boundary conditions gives us I

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Why use GKZ systems?

- Easily obtainable solutions (for "sufficiently generic" α_i and β_i)
- Allows systematic study of classes of integrals
- Beautiful geometric properties
- Can use powerful machinery of $\mathcal{D}$-modules

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GKZ systems

Obtaining the differential equations - 1

- Start with easy example:

$$I(z) = \int_\gamma dx_1 dx_2 \, \frac{x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1}}{(z_1 x^2 + z_2 xy + z_3)^{\beta_1}}$$

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  1) Turn exponents of polynomial into vectors:

$$z_1 x^2 y^0 + z_2 x^1 y^1 + z_3 x^0 y^0 \longrightarrow a_1 = (2,0)^T, \; a_2 = (1,1)^T, \; a_3 = (0,0)^T$$

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  2) Turn vectors into (homogenized) matrix

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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  3) $\mathcal{A}$, α_i and β_i determine the differential equations completely!

Obtaining the differential equations - 2

- Differential equation has two parts:
  1) "toric" equations:
     $\partial_1^{u_1} \cdots \partial_m^{u_m} - \partial_1^{v_1} \cdots \partial_m^{v_m}$ with $u_i,\ v_i \in \mathbb{N}$ satisfying $\mathcal{A}u = \mathcal{A}v$
     Trivial in this case since $\mathrm{Ker}(\mathcal{A}) = 0$

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  2) "Euler" equations:

$$\mathcal{A}\begin{pmatrix} z_1\partial_1 \\ z_2\partial_2 \\ z_3\partial_3 \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} z_1\partial_1 + z_2\partial_2 + z_3\partial_3 + \beta_1 \\ 2z_1\partial_1 + z_2\partial_2 + \alpha_1 \\ z_2\partial_2 + \alpha_2 \end{pmatrix}$$

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Solutions

- Easily obtainable basis of solutions
  - For “sufficiently generic” α_i and β_i
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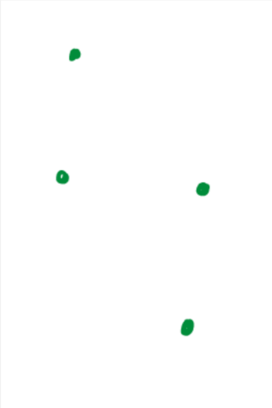

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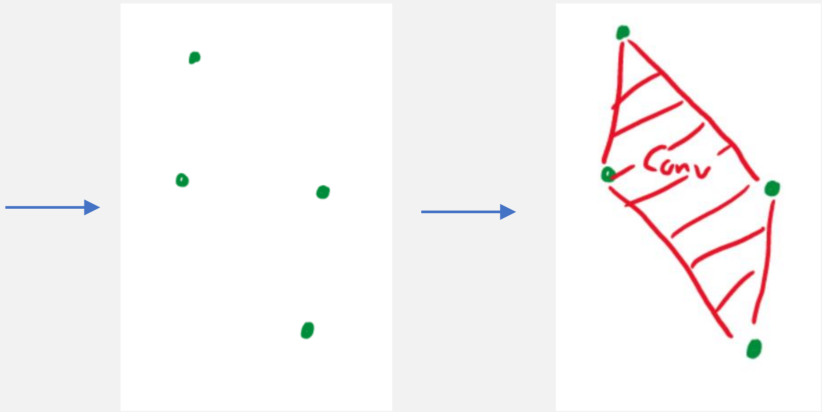

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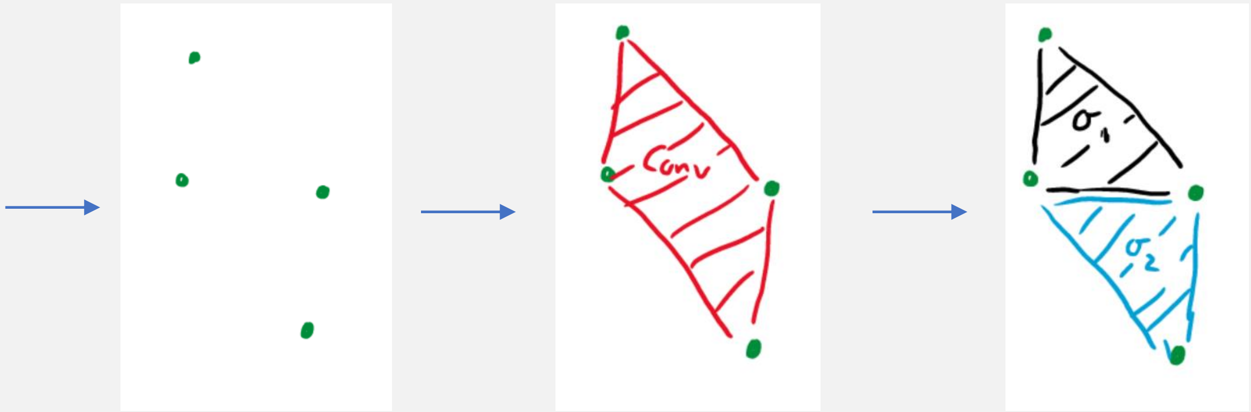

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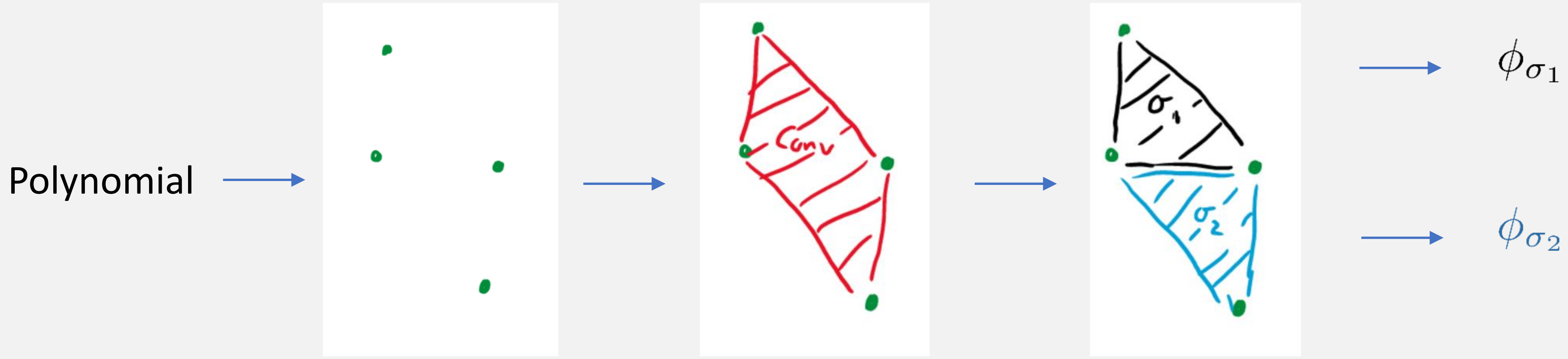

Comments

- Extremely general framework
- Subtleties when $\alpha_i, \beta_i \in \mathbb{Z}$
- Solutions might be difficult to use!
  - Reducing the system: wip with Thomas Grimm and Guilherme Pimentel
- Mathematicians use language of $\mathcal{D}$-modules

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Reductions of differential systems

Why reduce?

- Solutions can involve many summations
- In examples of interest many simplifications seem to happen
- Find when GKZ system “simplifies”

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- Solutions can involve many summations
- In examples of interest many simplifications seem to happen
- Find when GKZ system “simplifies”
- Precisely:
  - GKZ system = $\mathcal{D}$-modules
  - Subsystem = Sub-module
  - So study reducability!

Reducability

- Exist theorems when GKZ systems are irreducible [Beukers, '11] [Schulze, Walther '10]
- Idea: invert these to find subsystems

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Reducability

- Exist theorems when GKZ systems are irreducible [Beukers, '11] [Schulze, Walther '10]
- Idea: invert these to find subsystems
- Key fact: reducability only depends on α_i and β_i

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Algorithm

- Schematic overview:
  1) Identify subsystems using α_i and β_i
  2) Solve enough to obtain basis
  3) Insert subsystems into original system
- Note: last part in practice most difficult

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Thanks for listening!