KING'S College LONDON

LEVERHULME TRUST

Modular Forms and the Road to Heterotic Vacua

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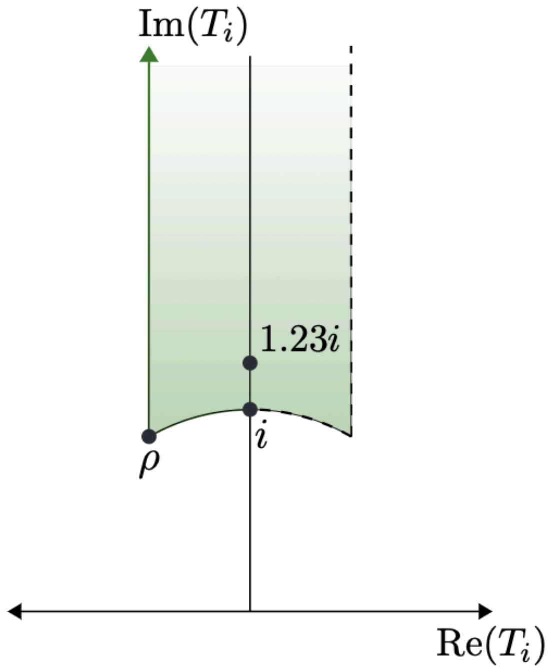
Work in Progress

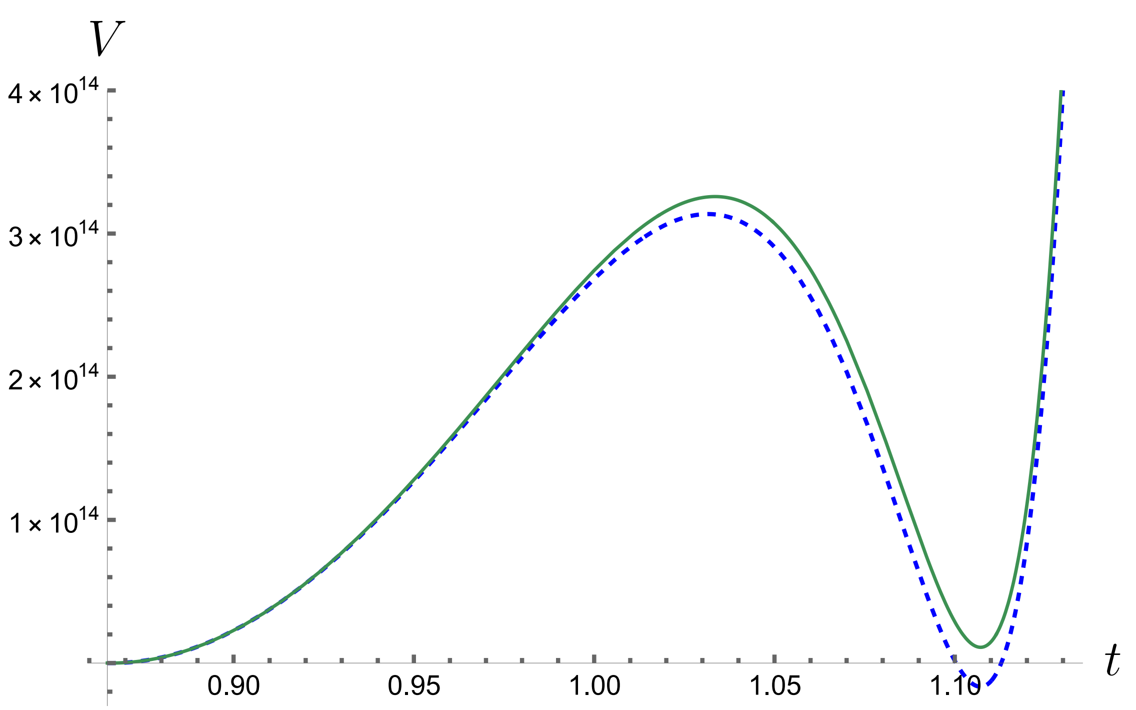
with A. Kidambi, J. M. Leedom and A. Westphal

Parallel Session, StringPheno 2023

From Alexander: $SL(2,\mathbb{Z})$ + stringy instantons



- heterotic string on toroidal orbifold
- field content: $S = 1/g_s^2 + i\theta$ and $T = a + it$
- T transforms under $SL(2,\mathbb{Z})$ as $T \to \dfrac{aT+b}{cT+d}$
- $F_S \neq 0$ via $\delta K_{np} \sim e^{1/g_s}$ [Shenker '90] [Silverstein '94]



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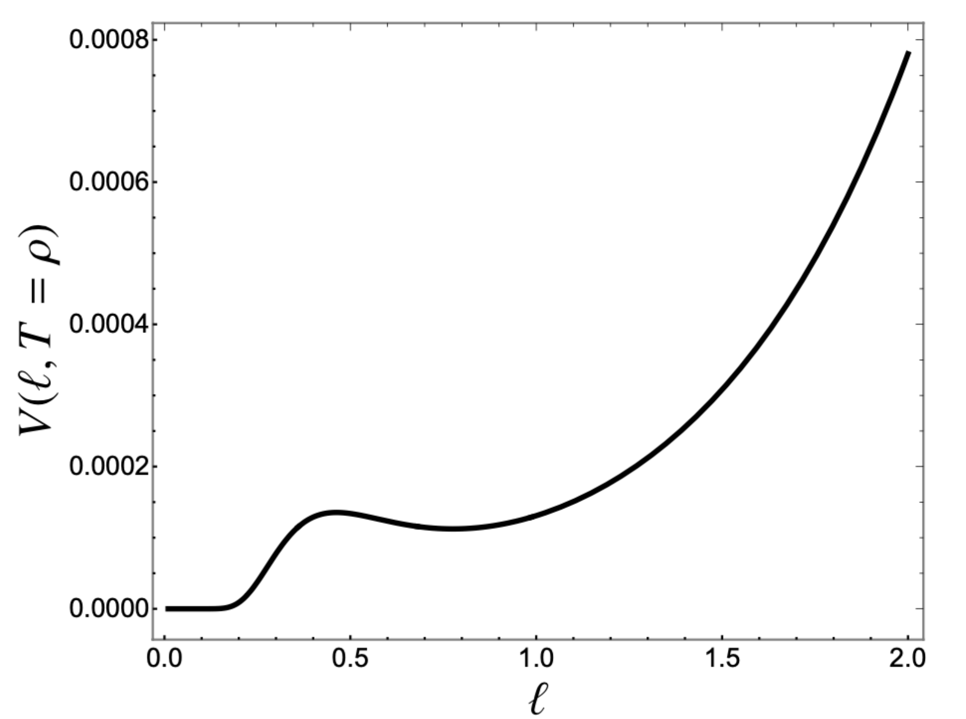
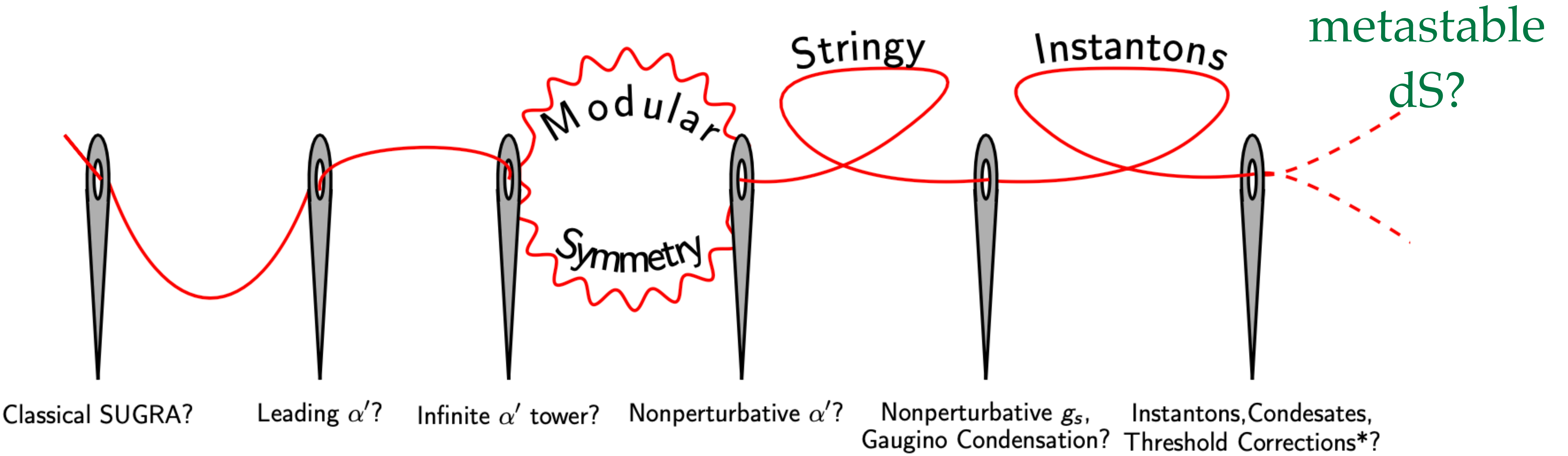


[Leedom, NR, Westphal '22]

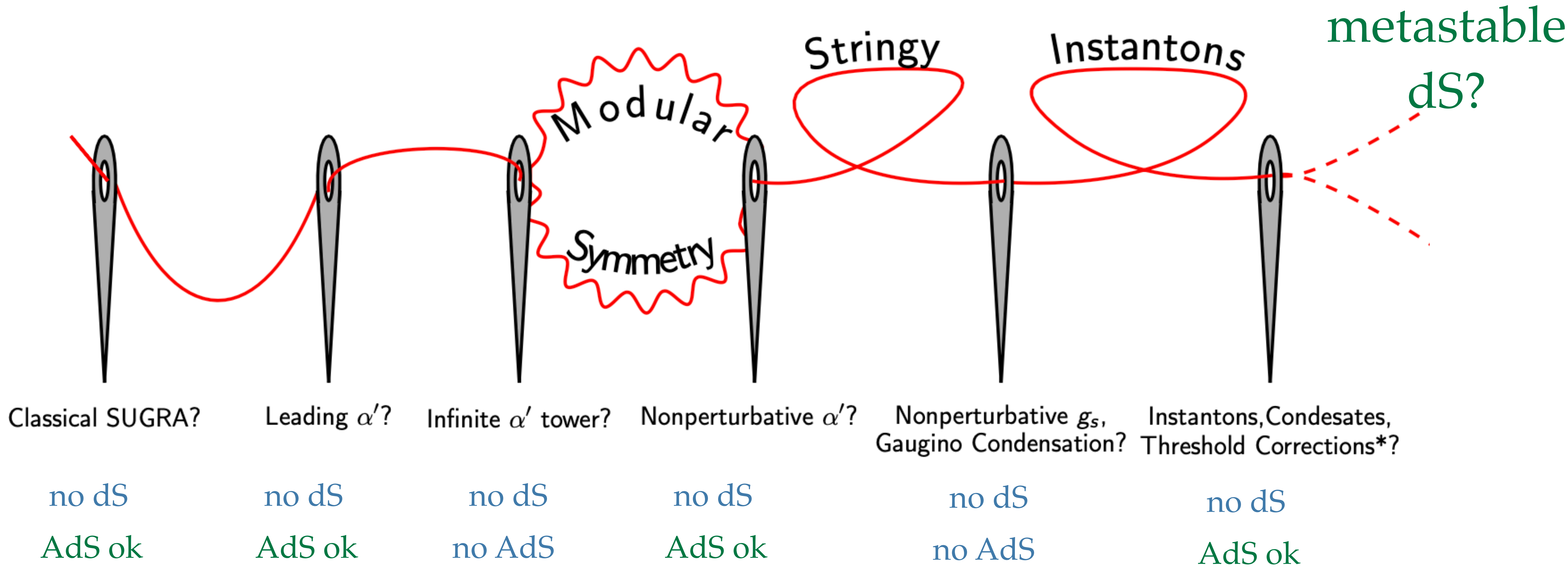
From Alexander: $SL(2,\mathbb{Z})$ + stringy instantons



Classical SUGRA?	Leading α'?	Infinite α' tower?	Nonperturbative α'?	Nonperturbative g_s, Gaugino Condensation?	Instantons,Condesates, Threshold Corrections*?
no dS	no dS	no dS	no dS	no dS	no dS
AdS ok	AdS ok	no AdS	AdS ok	no AdS	AdS ok

[Leedom, NR, Westphal '22]

From Alexander: $SL(2,\mathbb{Z})$ + stringy instantons

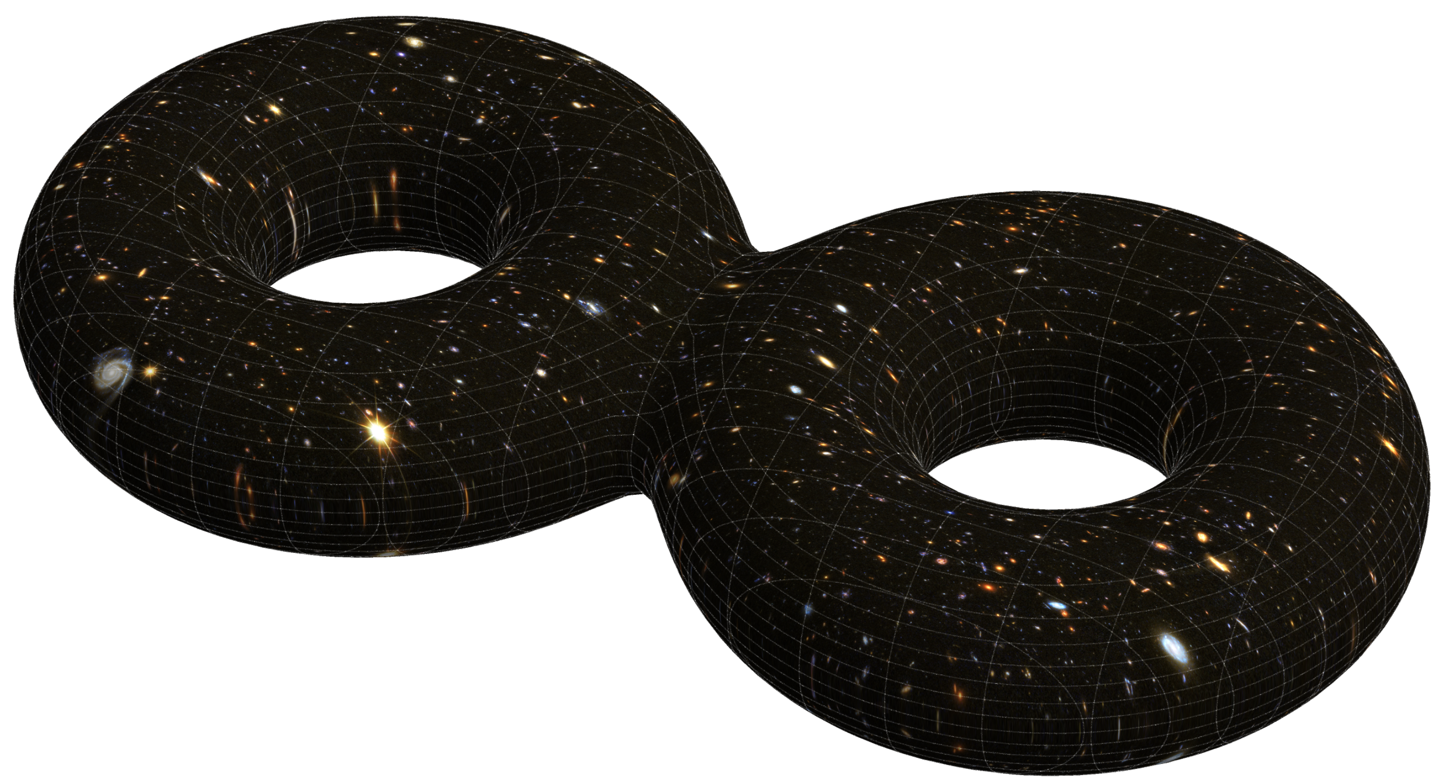


→ many checks to be done, e.g. addition of extra fields

[Leedom, NR, Westphal '22]

Genus 2: $Sp(4,\mathbb{Z})$

- the moduli structure requires an auxiliary genus-2 surface



Genus 2: $Sp(4,\mathbb{Z})$

- the moduli structure requires an auxiliary genus-2 surface

Why considering such enhancement?

$$T = B_{12} + i\sqrt{\det(G)}$$
$$U = \frac{1}{G_{11}}\left(G_{12} + i\sqrt{\det(G)}\right)$$

$$\Downarrow$$

$$SL(2,\mathbb{Z})_T \times SL(2,\mathbb{Z})_U \text{ + mirror symm.}$$

Genus 2: $Sp(4,\mathbb{Z})$

- the moduli structure requires an auxiliary genus-2 surface

Why considering such enhancement?

$$T = B_{12} + i\sqrt{\det(G)} + a_1(-a_2 + Ua_1)$$

$$U = \frac{1}{G_{11}}\left(G_{12} + i\sqrt{\det(G)}\right)$$

$$Z = -a_2 + Ua_1 \qquad \text{Wilson line}$$

[Lopes Cardoso, Lüst, Mohaupt '94]

$$\Downarrow$$

$$Sp(4,\mathbb{Z})$$

The building blocks

$$V = e^{K}\left(K^{a\bar{b}}F_a\bar{F}_{\bar{b}} - 3W\bar{W}\right)$$

- Kahler potential $\quad M = \begin{pmatrix} T & Z \\ Z & U \end{pmatrix}$

$$K_{(2)} = -\ln\left[\det(M - M^{\dagger})\right] = -\ln\left[(T - T^*)(U - U^*) - (Z - Z^*)^2\right]$$

[Lopes Cardoso, Lüst, Mohaupt '94]
[Ferrara, Kounnas, Lüst, Zwirner '91]

under $Sp(4,\mathbb{Z})$:

$$M \rightarrow (AM + B)(CM + D)^{-1}$$

$$K_{(2)} \rightarrow K_{(2)} + \ln\left[\det(CM^{\dagger} + D)\right] + \ln\left[\det(CM + D)\right]$$

with $AD - BC = 1$

The building blocks

$$V = e^{K}\left(K^{a\bar{b}}F_a\bar{F}_{\bar{b}} - 3W\bar{W}\right)$$

- Superpotential

defining $G \equiv K_{(2)} + \ln|W_{(2)}|^2 \;\Rightarrow\; V = e^{G}\left(G_i G^{i\bar{j}} G_{\bar{j}} - 3\right)$

must be a modular function

$\Downarrow$

G must be invariant

$\Downarrow$

$$W_{(2)} \rightarrow \det(CM + D)^{-1} W_{(2)}$$

The building blocks

- Superpotential

$$W_{(2)} = \frac{\Omega(S) H_{(2)}(M)}{\Xi(M)}$$

The building blocks

- Superpotential

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threshold corrections

flashback to $SL(2,\mathbb{Z})$: $\Delta_a(T) \sim b_a \ln\left(Im(T)^2|\eta(T)|^4\right)$

[Dixon, Kaplunovsky, Louis '91]
[Kaplunovsky, Louis '95]

ring of $SL(2,\mathbb{Z})$: $\mathscr{M} \cong \mathbb{Z}\left[G_4, G_6, \Delta \equiv \eta^{12}\right] / \left(E_6^2 = 1728(E_4^2 - \Delta)\right)$

The building blocks

- Superpotential

$$W_{(2)} = \frac{\Omega(S) H_{(2)}(M)}{\Xi(M)}$$

threshold corrections

for $Sp(4,\mathbb{Z})$:

$$\Delta_a(M) \sim b_a \ln\left(\det(Im(M))^n |\vartheta(M)|^2\right)$$

[Mayr, Stieberger '95]

ϑ constructed from the ring:

[Igusa '79]

$$\mathscr{M} \cong \mathbb{C}\left[G_4, G_6, \chi_{10}, \chi_{12}, \chi_{35}\right] / \left(\chi_{35}^2 = \mathscr{P}\right)$$

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$$\mathscr{M} \cong \mathbb{C}\left[G_4, G_6, \chi_{10}, \chi_{12}, \chi_{35}\right] / \left(\chi_{35}^2 = \mathscr{P}\right)$$

- without singularities in $\mathscr{F}$: $\Delta_a \sim b_a \ln\left(\det(Im(M))^{12} |\chi_{12}(M)|^2\right)$

The building blocks

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non-perturbative corrections

flashback to $SL(2,\mathbb{Z})$:

ring of $SL(2,\mathbb{Z})$: $\mathscr{M} \cong \mathbb{Z}\left[G_4, G_6, \Delta \equiv \eta^{12}\right] / \left(E_6^2 = 1728(E_4^2 - \Delta)\right)$

construct the most general modular function

$$H(T) = \left(\frac{G_4(T)}{\eta^8(T)}\right)^n \left(\frac{G_6(T)}{\eta^{12}(T)}\right)^m \mathscr{P}(j(T))$$

[Rademacher, Zuckerman '38]

The building blocks

- Superpotential

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non-perturbative corrections

for $Sp(4,\mathbb{Z})$: $\mathscr{M} \cong \mathbb{C}\left[G_4, G_6, \chi_{10}, \chi_{12}, \chi_{35}\right] / \left(\chi_{35}^2 = \mathscr{P}\right)$ [Igusa '79]

→ construct the most general modular function

$$H_{(2)}(M) = \sum_{m,n,\ell} \left(\frac{G_4G_6}{\chi_{10}}\right)^m \left(\frac{G_6^2}{\chi_{12}}\right)^n \left(\frac{G_4^5}{\chi_{10}^2}\right)^\ell$$

[Kidambi, Leedom, NR, Westphal WiP]

The building blocks

- Superpotential

$$W_{(2)} = \frac{\Omega(S) H_{(2)}(M)}{\Xi(M)}$$

non-perturbative corrections

for $Sp(4,\mathbb{Z})$: $\mathscr{M} \cong \mathbb{C}\left[G_4, G_6, \chi_{10}, \chi_{12}, \chi_{35}\right] / \left(\chi_{35}^2 = \mathscr{P}\right)$ [Igusa '79]

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[Kidambi, Leedom, NR, Westphal WiP]

$\Rightarrow$ we have all the ingredients to compute V !

Minima

[Kidambi, Leedom, NR, Westphal WiP]

$$V = e^K \left(K^{a\bar{b}} F_a \bar{F}_{\bar{b}} - 3W\bar{W} \right)$$

without computing V, we prove:

- all 6 fixed points σ_i are extrema: since V is a Siegel modular function, $\nabla V|_{\sigma_i} = 0$

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- these extrema are always either Minkowski or AdS minima because $F_M = 0$

$$V = 0 \quad \text{if} \quad W = 0$$
$$V < 0 \quad \text{if} \quad W \neq 0$$

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$\Rightarrow$ can we uplift requiring $F_S \neq 0$ as in [Leedom, NR, Westphal '22] ?

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Thank you

Backup: extrema from the fixed points

- without explicitly computing V:
  are the fixed points extrema?

  gradient: $\nabla V \rightarrow (CM+D)^T \nabla V (CM+D)$

  6 fixed points σ_i, $i = 1,\ldots,6$, each with a stabiliser group Σ_{σ_i}
  s.t. $\gamma_\alpha \cdot \sigma_i = \sigma_i$ for all matrices $\gamma_\alpha \in \Sigma_{\sigma_i} \subset Sp(4,\mathbb{Z})$

  $\Rightarrow$ at the fixed points: $(C\sigma_i + D)^T \nabla V|_{M=\sigma_i} (C\sigma_i + D) = \nabla V(\gamma_\alpha \cdot \sigma_i) = \nabla V(\sigma_i)$

  $\rightarrow$ vectorisation: given a $m \times n$ matrix A,
  construct a $mn \times \mathbb{I}$ column vector vec(A) as
  $\text{vec}(A) = (a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn})^T$

  $P^T A P = A \longrightarrow (P^T \otimes P^T)\text{vec}(A) = \text{vec}(A)$

  $(\mathbb{I} - P^T \otimes P^T)\text{vec}(A) = 0 \quad \Rightarrow$ eigenvalue problem

  but, we have α matrices $\forall\ \sigma_i \Rightarrow$ diagonalisation problem

Backup: minima

defining $G \equiv K_{(2)} + \ln |W_{(2)}|^2 \Rightarrow V = e^G \left(G_i G^{i\bar{j}} G_{\bar{j}} - 3 \right)$

must be a modular function

$\Downarrow$

G must be invariant

$\Downarrow$

$$\nabla G|_{\sigma_i} = 0$$

$$\Rightarrow \quad \begin{array}{lll} V = 0 & \text{if} & W = 0 \\ V < 0 & \text{if} & W \neq 0 \end{array}$$