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A Boltzmann equation approach to string thermodynamics

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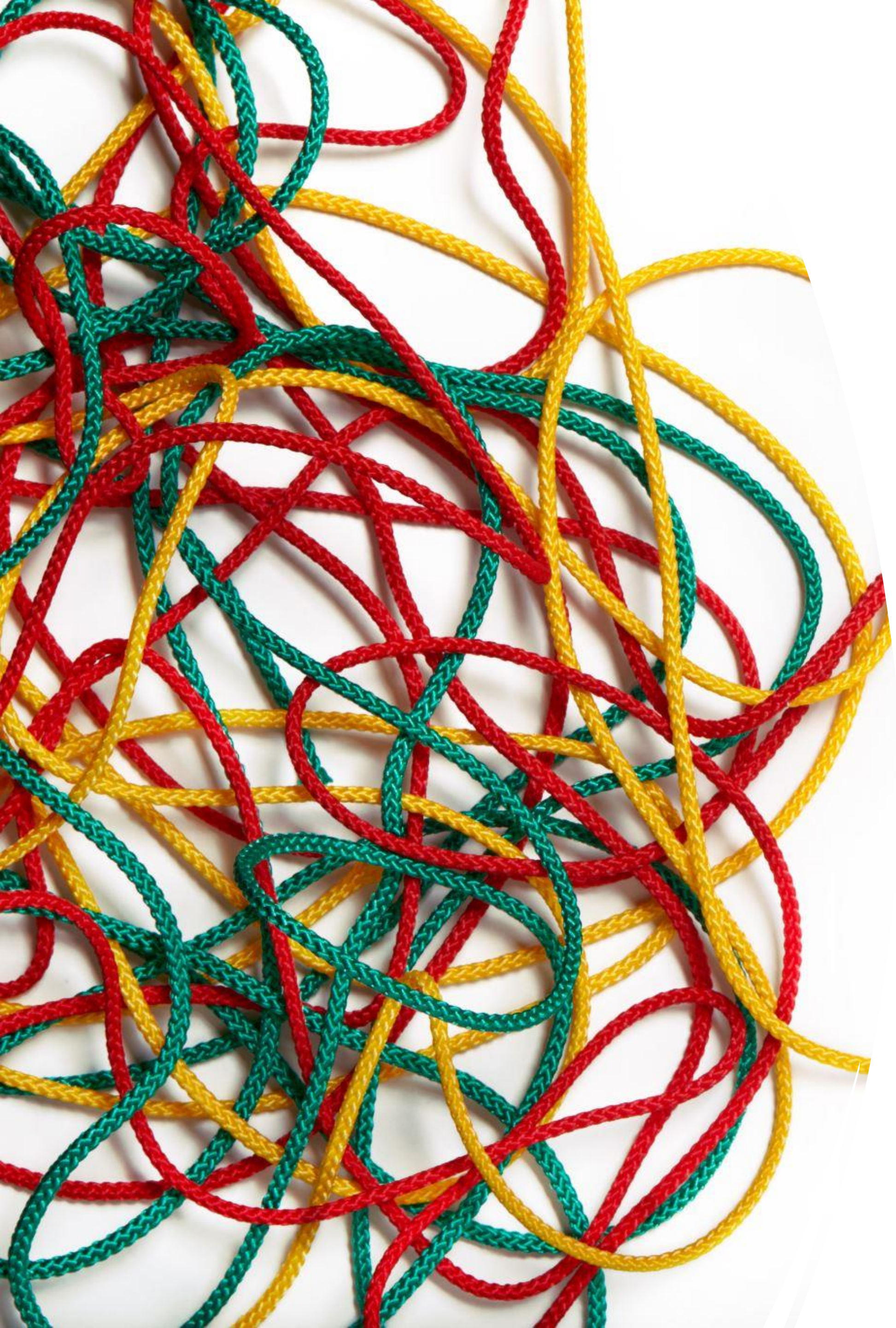
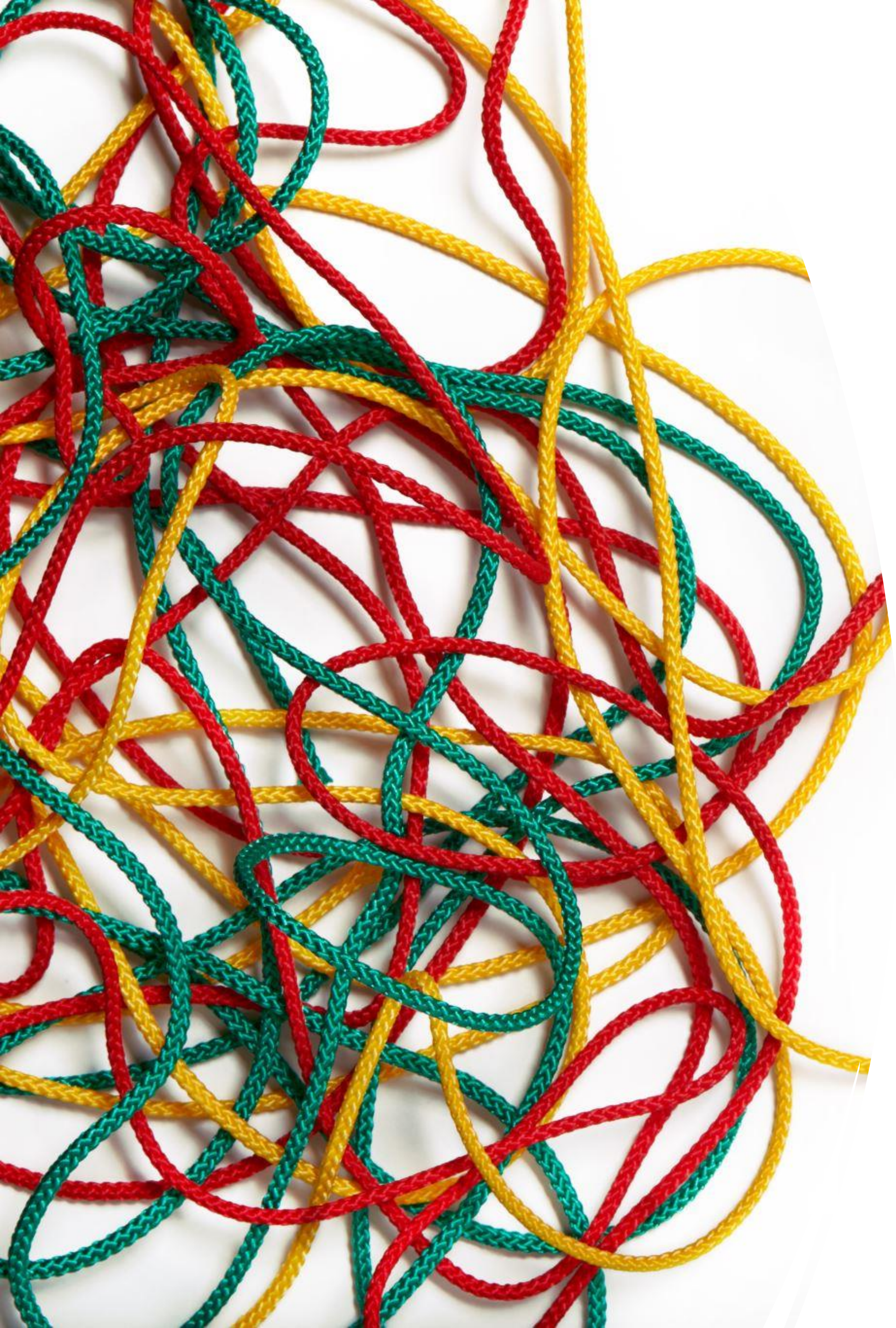


Why string thermodynamics: pheno aspects

Review: Cicoli *et al* ‘23

- Heat up a box of anything: stringy modes will eventually be excited.
- It is common in the early Universe to find energy densities of order one in string units.
- Influences in reheating, possible GW spectrum, moduli problem…
- Most likely out of equilibrium processes are important!



d=0 case: Lowe and Thorlacius'95
d=0 with D-branes: Lee and Thorlacius'97

Why Boltzmann equations

- An invaluable window to out-of-equilibrium physics.
- Equilibrium is a subtle concept in presence of gravity:
  - Gravity is dynamical: Jeans instability.
  - Expanding Universe.
- Knowledge of interaction rates allows to estimate when (and whether) equilibrium is a good approximation.

A phase space for string theory

- Consider a string in a D-dimensional spacetime, with d noncompact directions.
- The phase space is given by the position, noncompact momenta, oscillator level, winding and KK modes.

$$E_l^2 = k^2 + \underbrace{\frac{2}{\alpha'}(N + \tilde{N} - 2) + \sum_{i=1}^{d_c} \left(\frac{n_i}{R_i}\right)^2 + \left(\frac{\omega_i R_i}{\alpha'}\right)^2}_{M^2 = (\mu l)^2}$$

- Hence describe the thermodynamics with a distribution on a generalized phase space $f(r, k, N, \sigma) \equiv f(E_l, l)$

Decay rates: summing over our ignorance

- Interaction rates of highly excited strings are hard to compute.
- In thermodynamics, we are interested in how the average string looks like.
- Consider the averaged semi-inclusive decay rate:

- $N \leftrightarrow V_1 + N_2$

$$\frac{F(N, N_2)}{\mathcal{G}(N)} \equiv \frac{1}{\mathcal{G}(N)} \sum_{\Phi_N} \sum_{\Phi_{N_2}} |\langle \Phi_{N_2} | V_1(k) | \Phi_N \rangle|^2$$

$$F(N, N_2) = \oint_C \frac{dz}{z} z^{-N} \oint_{C_2} \frac{dz_2}{z_2} z_2^{-N_2} \mathrm{Tr}\left[z^{\hat{N}} V_1^\dagger(k,1) z_2^{\hat{N}} V_1(k,1) \right]$$

Amati and Russo'99

Understanding the typical string

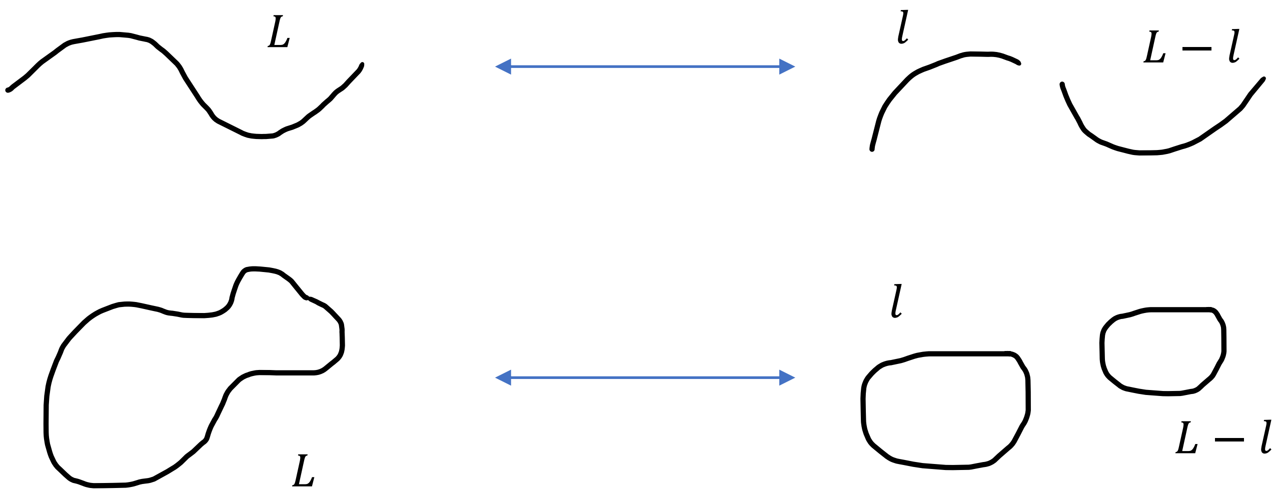
- The string prefers to decay into configurations with small external kinetic energy and with KK and winding mode energies proportional to the length.

Mañes’01
Chen *et al*’05

- It follows that the typical string is described by its length (approximately, level).
- The decay rate looks like:

$$\frac{d\Gamma_o}{dl} \sim g_s\,,$$

$$\frac{d\Gamma_{cl}}{dl} \sim g_s^2 L \left(\frac{L}{l(L-l)}\right)^{d/2}$$



The Boltzmann equation for the typical string

- Recap: we want to describe the thermodynamics of string theory in terms of the average string, well described by the length and the number of non-compact dimensions.
- We have computed the interaction rates, so we can write:

$$\frac{\partial n_c(l)}{\partial t} = \frac{\kappa}{2}\int_{l_c}^{l-l_c} dl' \left(\frac{n_c(l')l'\, n_c(l-l')(l-l')}{V} - l n_c(l) \left(\frac{l}{l'(l-l')}\right)^{d/2} \right) +$$

$$+\kappa \int_{l+l_c}^{\infty} dl' \left(l' n_c(l') \left(\frac{l'}{l(l'-l)}\right)^{d/2} - \frac{l n_c(l)(l'-l) n_c(l'-l)}{V} \right).$$

Consistency check

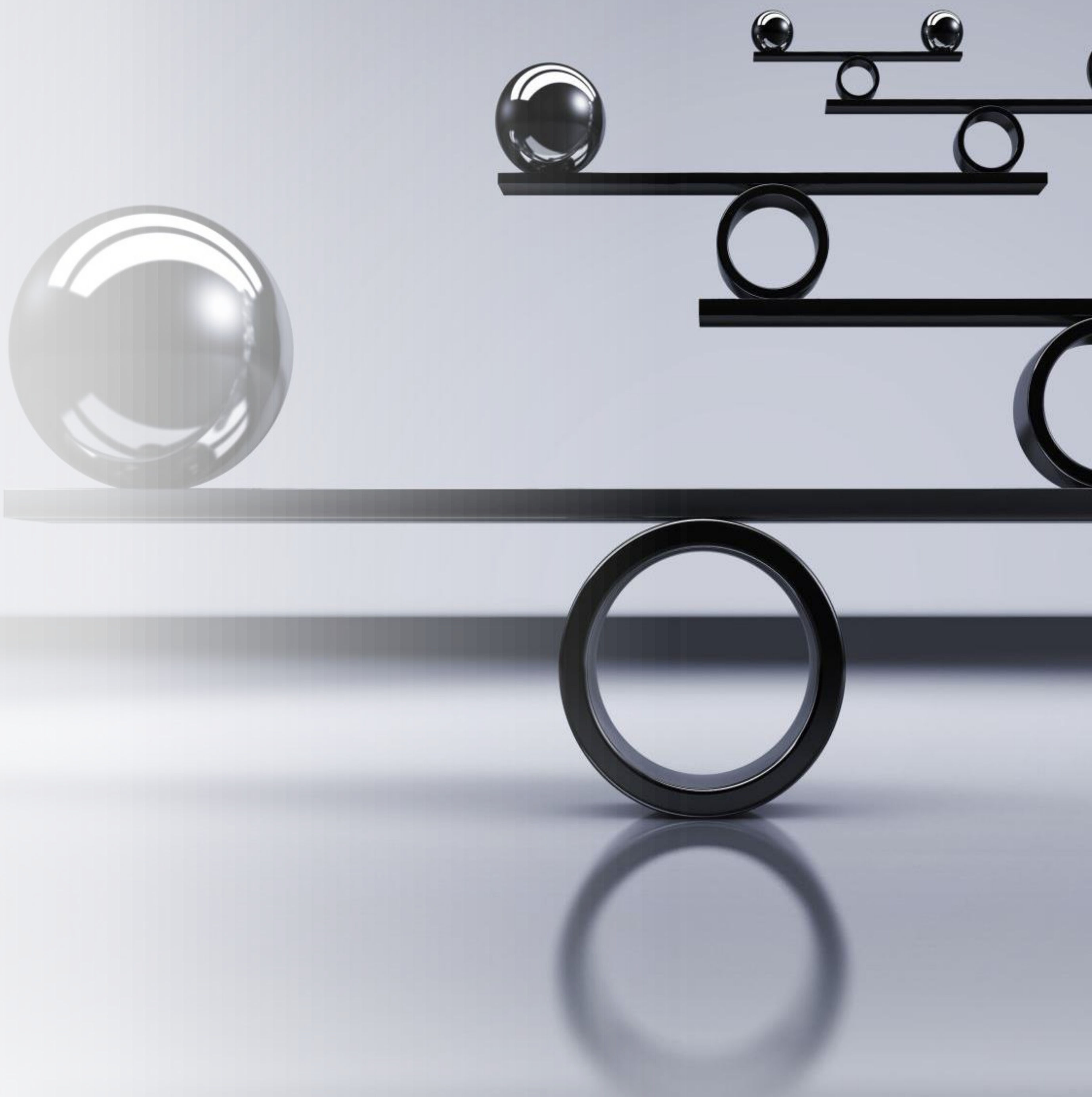
- The equilibrium solution obtained from imposing detailed balance reads

$$n_c(l) = V\,\frac{e^{-l/L}}{l^{1+d/2}}\,,$$

$$1/L = \beta - \beta_H .$$

- These results agree with general considerations in equilibrium thermodynamics, provided winding and KK modes are taken into account.

Deo *et al*’89
Deo *et al*’92



Boltzmann equations with open strings

$$\frac{\partial n_c(l)}{\partial t} = + \frac{b}{2N} V_\perp \frac{n_o(l)}{l^{d/2}} - a\frac{N}{V_\perp} l n_c(l) +$$
$$+ \frac{\kappa}{2}\int_{l_c}^{l-l_c} dl' \left(\frac{n_c(l')l'\, n_c(l-l')(l-l')}{V} - l n_c(l)\left(\frac{l}{l'(l-l')}\right)^{d/2}\right) +$$
$$+ \kappa \int_{l+l_c}^{\infty} dl' \left(\frac{n_c(l')}{l^{d/2}} - \frac{l n_c(l)(l'-l) n_c(l'-l)}{V}\right) +$$
$$+ \kappa \int_{l+l_c}^{\infty} dl' \left(\frac{(l'-l) n_o(l')}{l^{d/2}} - \frac{l n_c(l)(l'-l) n_o(l'-l)}{V}\right).$$

$$\frac{\partial n_o(l)}{\partial t} = + a\frac{N}{V_\perp} l n_c(l) - \frac{b}{2N} V_\perp \frac{n_o(l)}{l^{d/2}}$$
$$+ \int_{l_c}^{l-l_c} dl' \left(\frac{b}{2NV_\parallel} n_o(l') n_o(l-l') - a\frac{N}{V_\perp} n_o(l)\right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(2a\frac{N}{V_\perp} n_o(l') - \frac{b}{NV_\parallel} n_o(l) n_o(l'-l)\right)$$
$$+ \kappa \int_{l_c}^{l-l_c} dl' \left(\frac{l'(l-l') n_c(l') n_o(l-l')}{V} - n_o(l)\frac{l-l'}{l'^{d/2}}\right)$$
$$+ \kappa \int_{l+l_c}^{\infty} dl' \left(\frac{n_o(l')l}{(l'-l)^{d/2}} - \frac{l n_o(l)(l'-l) n_c(l'-l)}{V}\right)$$

Boltzmann equations with D-branes

$$
\begin{aligned}
\frac{\partial n_c(l)}{\partial t} =& \frac{b}{2N}\frac{n_o(l)}{l^{p/2}} - a\frac{N}{V_\perp} l n_c(l) \\
&+ \frac{\kappa}{2}\int_{l_c}^{l-l_c} dl' \left(\frac{n_c(l')l'\, n_c(l-l')(l-l')}{V} - l n_c(l)\left(\frac{l}{l'(l-l')}\right)^{d/2}\right) + \\
&+ \kappa \int_{l+l_c}^{\infty} dl' \left(\frac{n_c(l')}{l^{d/2}} - \frac{l n_c(l)(l'-l) n_c(l'-l)}{V}\right) + \\
&+ \kappa \int_{l+l_c}^{\infty} dl' \int_{l_c}^{l'-l-l_c} \frac{dl_x}{(l_x(l'-l-l_x))^{(d-p)/2}} \left(\frac{n_o(l') l'^{(d-p)/2}}{l^{d/2}} - \frac{l n_c(l)}{V} n_o(l'-l)(l'-l)^{(d-p)/2}\right) .
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial n_o(l)}{\partial t} =& + a\frac{N}{V_\perp} l n_c(l) - \frac{b}{2N}\frac{n_o(l)}{l^{p/2}} \\
&+ \int_{l_c}^{l-l_c} dl' \left(\frac{b}{2NV_\parallel} n_o(l') n_o(l-l') - aN n_o(l)\left(\frac{l}{l'(l-l')}\right)^{(d-p)/2}\right) \\
&+ \int_{l+l_c}^{\infty} dl' \left(2aN n_o(l') \left(\frac{l'}{l(l'-l)}\right)^{(d-p)/2} - \frac{b}{NV_\parallel} n_o(l) n_o(l'-l)\right) \\
&+ \kappa \int_{l_c}^{l-l_c} dl' \int_{l_c}^{l-l'-l_c} \frac{dl_x}{(l_x(l-l'-l_x))^{(d-p)/2}} \left(\frac{l' n_c(l') n_o(l-l')}{V}(l-l')^{(d-p)/2} - \frac{n_o(l)}{l'^{d/2}} l^{(d-p)/2} \right. \\
&+ \kappa \int_{l+l_c}^{\infty} dl' \int_{l_c}^{l-l_c} \frac{dl_x}{(l_x(l-l_x))^{(d-p)/2}} \left(\frac{n_o(l')}{(l'-l)^{d/2}} l'^{(d-p)/2} - \frac{n_o(l)(l'-l) n_c(l'-l)}{V} l^{(d-p)/2}\right)
\end{aligned}
$$

Out of equilibrium: equilibration rates

- Consider perturbations around the equilibrium solution in d=0:

$$\frac{\partial \delta n(l,t)}{\partial t} = -\left(\frac{l^2}{2} + lL\right)\delta n(l,t) + \int_0^l dl'\, l'\delta n(l',t)\left(e^{\frac{-(l-l')}{L}} - 1\right) - E\left(e^{-l/L} - 1\right),$$

Out of equilibrium: equilibration rates

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- We find zero-energy solutions of the form:

$$\delta n(l,t) = \frac{e^{-l/L}}{L} + \sqrt{\frac{\pi(c+tL^2)}{2}}\frac{e^{-\frac{l}{L}+A(t)^2}}{L}\mathrm{Erf}\left(\sqrt{\frac{c+tL^2}{2}}\frac{l}{L} + A(t), A(t)\right),$$

- Qualitatively, find a length-dependent equilibration rate: $\delta n(l,t) \sim \delta n(l,0) e^{-\left(\frac{l^2}{2}+lL\right)t}$

Conclusions and future directions

- We have described string thermodynamics in terms of the *typical* string, described by its length and the number of non-compact directions.
- The equilibrium conditions we find agree with general equilibrium considerations, a non-trivial consistency check of the interactions.
- We are able to probe the out-of-equilibrium regime, showing stability and computing equilibration rates.
- We plan on applying these results to do phenomenology.

A first view of Boltzmann equations

- Boltzmann equations describe the temporal evolution of the probability density.
- Consider the process A ↔ B + C

$$\frac{\partial f_A}{\partial t} = -f_A \Gamma_r (1 \pm f_B)(1 \pm f_c) + f_B f_C \Gamma_l (1 \pm f_A)$$

- A generic contribution to the Boltzmann equation for a string in a specified state looks like:

$$\frac{\partial f(E_l, l)}{\partial t} \supset \sum_{\sigma_x \sigma_y} \int_{\mathbb{R}^p} d^p k_x \, d^p k_y \, \mathcal{A} \left(f_x f_y \left(1 \pm f_l\right) - f_l \left(1 \pm f_x\right) \left(1 \pm f_y\right)\right) \delta(\ldots)$$

Equilibrium and detailed balance

- Standard thermodynamic arguments give the equilibrium distribution:

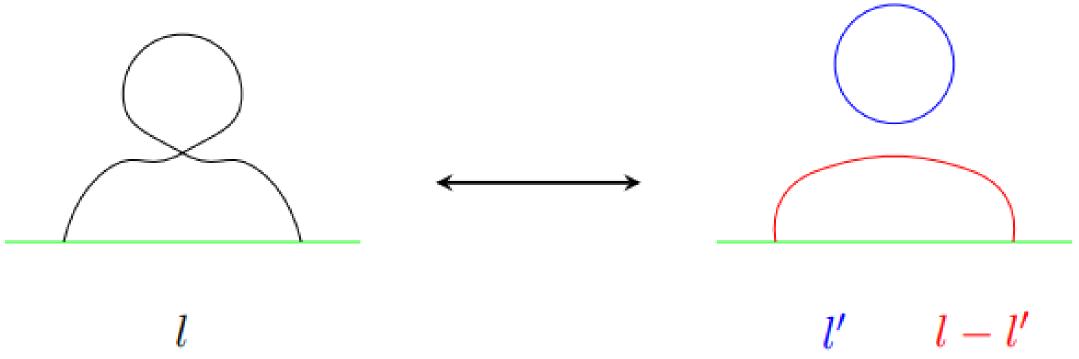
$$f_{eq}(r, k, N, \sigma) = \frac{1}{e^{\beta E} \mp 1}$$

- For given external momentum, all states with same length have the same probability:

$$f_{eq}(r, k, l) = \frac{\mathcal{N}(l)}{e^{\beta E} \mp 1}$$

Semiclassical strings: a random walk interpretation

The random walk interpretation allows us to conjecture the form of other interaction rates:



$$\frac{d\Gamma}{dl} \sim g_s^2 \frac{1}{l'^{d/2}} \int_{l_c}^{l-l_c} dl_x \left(\frac{l-l'}{l_x(l-l'-l_x)} \right)^{(d-p)/2} .$$

Non-trivial check: detailed balance must be satisfied *in every interaction* by an equilibrium solution agreeing with the general case at high energies.