The Asymptotic Weak Gravity Conjecture in M-theory

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based mainly on arXiv:2212.09758 [hep-th]
with C. F. Cota, T. Weigand and M. Wiesner

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Weak Gravity Conjecture

Weak Gravity Conjecture

Every gauge theory coupled to gravity has a particle with $\frac{g_{\text{YM}}^2 q^2}{m^2} \geq \frac{g_{\text{YM}} Q^2}{M^2}\Big|_{\text{B.H.}}$.[1]

Repulsive Force Conjecture

States with highest charge-to-mass ratio should be self-repulsive:[2] $F_{\text{Coulomb}} \geq F_{\text{Grav.}} + F_{\text{Yukawa}}$

Tower Weak Gravity Conjecture

The super-extremal states must form an infinite tower.[3]

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[1] N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: hep-th/0601001 (hep-th)

[2] N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: hep-th/0601001 (hep-th); E. Palti, *JHEP* **08**, 034, arXiv: 1705.04328 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: 1906.02206 (hep-th)

[3] B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **02**, 140, arXiv: 1509.06374 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *Phys. Rev. Lett.* **121**, 051601, arXiv: 1802.08698 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: 1906.02206 (hep-th); M. Montero, G. Shiu, P. Soler, *JHEP* **10**, 159, arXiv: 1606.08438 (hep-th); S. Andriolo, D. Junghans, T. Noumi, G. Shiu, *Fortsch. Phys.* **66**, 1800020, arXiv: 1802.04287 (hep-th)

Evidence for Tower WGC

To verify the Tower WGC, one must compute the charge-to-mass ratio of candidates of super-extremal states.

This can be achieved when the charge-to-mass ratio is known:

1. Towers of BPS particles in 4d $\mathcal{N} = 2$,[4] or 5d $\mathcal{N} = 1$ theories.[5]
2. Perturbative heterotic string.[6]
3. F-theory in the limits where $g_{\text{YM}} \longrightarrow 0$.[7]

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[4] T. W. Grimm, E. Palti, I. Valenzuela, *JHEP* **08**, 143, arXiv: 1802.08264 (hep-th); N. Gendler, I. Valenzuela, *JHEP* **01**, 176, arXiv: 2004.10768 (hep-th); B. Bastian, T. W. Grimm, D. van de Heisteeg, *JHEP* **06**, 162, arXiv: 2011.08854 (hep-th).

[5] M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th).

[6] N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: hep-th/0601001 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **02**, 140, arXiv: 1509.06374 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *Phys. Rev. Lett.* **121**, 051601, arXiv: 1802.08698 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: 1906.02206 (hep-th); M. Montero, G. Shiu, P. Soler, *JHEP* **10**, 159, arXiv: 1606.08438 (hep-th); S. Andriolo, D. Junghans, T. Noumi, G. Shiu, *Fortsch. Phys.* **66**, 1800020, arXiv: 1802.04287 (hep-th).

[7] S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **10**, 164, arXiv: 1808.05958 (hep-th); S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **08**, 104, arXiv: 1901.08065 (hep-th); S.-J. Lee, W. Lerche, T. Weigand, *Nucl. Phys. B* **938**, 321–350, arXiv: 1810.05169 (hep-th).

The Asymptotic WGC

The charge-to-mass ratio of non-BPS states is calculated **reliably** only near $g_{\text{YM}} \to 0$, and we referred to such WGC as **asymptotic** WGC.

The asymptotic WGC with minimal supersymmetry, has been studied in F-theory compactification to 6d[8] and 4d theories[9] where

1. No BPS particle states are available, but **non-BPS excitations** of a weakly coupled BPS string are.
2. In 4d, super-extremal states exists in emergent string limits where a BPS string dual to a heterotic string becomes light.
3. In particular, there are cases in 4d, where no particle-like excitations of BPS strings can furnish a tower of super-extremal tower of states.
4. This happens when

$$\frac{\Lambda^2_{\text{WGC}}}{\Lambda^2_{\text{sp}}} \equiv \left.\frac{g^2_{\text{YM}} M^{d-2}_{\text{Pl}}}{\Lambda^2_{\text{sp}}}\right|_{d=4}$$

does not vanish asymptotically.

[8] S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **10**, 164, arXiv: 1808.05958 (hep-th); S.-J. Lee, W. Lerche, T. Weigand, *Nucl. Phys. B* **938**, 321–350, arXiv: 1810.05169 (hep-th).

[9] S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **08**, 104, arXiv: 1901.08065 (hep-th); D. Klaewer, S.-J. Lee, T. Weigand, M. Wiesner, *JHEP* **03**, 252, arXiv: 2011.00024 (hep-th); C. F. Cota, A. **Mininno**, T. Weigand, M. Wiesner, *JHEP* **11**, 058, arXiv: 2208.00009 (hep-th).

Goal of the Presentation

Puzzle for tower WGC in 5d

There are directions in the charge lattice where there is **no** BPS tower.

1. We extend the discussion of the asymptotic WGC to 5d $\mathcal{N} = 1$ theories, given by M-theory on Calabi–Yau 3-fold.
2. We focus on the role of super-extremal **non-BPS towers**, complementing the potential lack of super-extremal BPS states[10] in **all** directions in the charge lattice dual to the gauge groups with a weak coupling limit.
3. Our main interest lies on those directions in the charge lattice that are **not** populated by towers of BPS states, with the goal of showing explicitly that the asymptotic Tower WGC holds along such non-BPS directions.

[10] M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th).

Summary of Results

1. We **refine** the classification of infinite distance limits[11] for tests of the asymptotic WGC.
2. We **classify** the linear combinations of $\mathrm{U}(1)$ factors that become weakly coupled in the respective limit.

Upshots

3. $\mathrm{U}(1)$s can **only** become weakly coupled in a given infinite distance limit if they are obtained from reducing C_3 over
   - a curve that is contained in the **generic fiber** or
   - a curve that is localized in a degenerate fiber associated to a **finite** distance degeneration in the deformation space of the generic fiber.
4. **Any direction** in the charge lattice of five-dimensional M-theory that is **not** populated by a tower of BPS states either carries charge under a $\mathrm{U}(1)$ **without** a weak coupling limit, or there are super-extremal non-BPS states arising as excitations of a **critical string**.

[11] S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **02**, 190, arXiv: 1910.01135 (hep-th).

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M-theory on Calabi–Yau 3-folds

$$S_{5d} = \frac{M_{\text{Pl}}^3}{2} \int_{\mathbb{R}^{1,4}} \left(R \star \mathbb{1} - \mathfrak{g}_{AB} d\Phi^A \wedge \star d\Phi^B \right) - \frac{1}{2g_5^2} \int_{\mathbb{R}^{1,4}} f_{\alpha\beta} F^\alpha \wedge \star F^\beta + \dots ,$$

with

$$M_{\text{Pl}}^3 = 4\pi M_{\text{11d}}^3 \mathcal{V} , \quad \frac{1}{g_5^2} = \frac{M_{\text{Pl}}}{(2\pi)(4\pi)^{1/3}} , \quad f_{\alpha\beta} = \frac{1}{\mathcal{V}^{1/3}} \left(\frac{\mathcal{V}_\alpha \mathcal{V}_\beta}{\mathcal{V}} - \mathcal{V}_{\alpha\beta} \right) = \left(\widehat{\mathcal{V}}_\alpha \widehat{\mathcal{V}}_\beta - \widehat{\mathcal{V}}_{\alpha\beta} \right) .$$

1. Φ^A are $h^{1,1}(X_3) - 1$ scalar fields functions of the Kähler moduli v^a in

$$J = \sum_{\alpha=1}^{h^{1,1}(X_3)} v^\alpha J_\alpha .$$

2. F^α are the field strengths of the $\mathrm{U}(1)$ gauge factors.
3. A generic $\mathrm{U}(1)_C$ for a curve class $C \in H_2(X_3)$ is

$$\mathrm{U}(1)_C = c_\alpha \, \mathrm{U}(1)^\alpha , \quad \text{with } g_{\text{YM},C}^2 = g_5^2 \, c_\alpha f^{\alpha\beta} c_\beta , \quad \text{for } C = c_\alpha \, C^\alpha \in H_2(X_3) , \text{ with } J_\alpha \cdot C^\beta = \delta_\alpha^\beta .$$

Repulsive Force Conjecture

In the weak coupling limit, for a particle of mass M_k, we require[12]

Repulsive Force Conjecture

$$\frac{(M_{\rm Pl} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2} \geq \frac{1}{M_{\rm Pl}^2} \left.\frac{d-3}{d-2}\right|_{d=5} + \frac{1}{4}\frac{M_{\rm Pl}^2}{M_k^4}\left(f^{\alpha\beta} - \frac{1}{3}\hat{v}^\alpha \hat{v}^\beta\right)\partial_\alpha\left(\frac{M_k^2}{M_{\rm Pl}^2}\right)\partial_\beta\left(\frac{M_k^2}{M_{\rm Pl}^2}\right)$$

[12] N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: hep-th/0601001 (hep-th); E. Palti, *JHEP* **08**, 034, arXiv: 1705.04328 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: 1906.02206 (hep-th).

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Repulsive Force Conjecture

$$\underbrace{\frac{(M_{\rm Pl} g_5^2)(Q_\alpha f^{\alpha\beta} Q_\beta)}{M_k^2}}_{F_{\rm Coulomb}} \geq \underbrace{\frac{1}{M_{\rm Pl}^2}\frac{d-3}{d-2}\bigg|_{d=5}}_{F_{\rm Gravitational}} + \underbrace{\frac{1}{4}\frac{M_{\rm Pl}^2}{M_k^4}\left(f^{\alpha\beta} - \frac{1}{3}\hat{v}^\alpha \hat{v}^\beta\right)\partial_\alpha\left(\frac{M_k^2}{M_{\rm Pl}^2}\right)\partial_\beta\left(\frac{M_k^2}{M_{\rm Pl}^2}\right)}_{F_{\rm Yukawa}}$$

[12] N. Arkani-Hamed, L. Motl, A. Nicolis, C. Vafa, *JHEP* **06**, 060, arXiv: hep-th/0601001 (hep-th); E. Palti, *JHEP* **08**, 034, arXiv: 1705.04328 (hep-th); B. Heidenreich, M. Reece, T. Rudelius, *JHEP* **10**, 055, arXiv: 1906.02206 (hep-th).

Weak Coupling Limits

1. The magnetic WGC associates a scale Λ_{WGC} to a gauge theory in d dimensions with (dimensionful) gauge coupling g_{YM}, which is given by
$$\Lambda^2_{\text{WGC}} = g^2_{\text{YM}} M^{d-2}_{\text{Pl}} \, .$$
2. For any $\mathrm{U}(1)_C$, then
$$\Lambda^2_{\text{WGC}}\left(\mathrm{U}(1)_C\right) = g^2_{\text{YM,C}} M^3_{\text{Pl}} = g^2_5 \left(c_\alpha f^{\alpha\beta} c_\beta\right) M^3_{\text{Pl}} \, ,$$
3. and the **weak coupling limit** for the gauge group $\mathrm{U}(1)_C$ now corresponds to the limit

$$\frac{\Lambda^2_{\text{WGC}}\left(\mathrm{U}(1)_C\right)}{\Lambda^2_{\text{QG}}} \longrightarrow 0 \, ,$$

where Λ_{QG} is the quantum gravity cut-off, i.e., the scale at which gravity becomes strongly coupled.

Main Claim

Main Claim

Consider M-theory compactified to five dimensions on a Calabi–Yau X_3. Suppose there exists a primitive charge vector $\mathbf{Q}^0 \in \Lambda_{\mathbf{Q}}$ such that $\{\lambda \mathbf{Q}^0\}_{\lambda \in \mathbb{R}} \cap \Lambda_{\mathbf{Q}}$ is **not** populated by a BPS tower of super-extremal states. Defining $\mathrm{U}(1)_{\mathbf{Q}^0} = Q^0_a \, \mathrm{U}(1)^a$ one of the following two holds:

1. There exists **no** limit in moduli space in which
$$\frac{\Lambda^2_{\text{WGC}}\left(\mathrm{U}(1)_{\mathbf{Q}^0}\right)}{\Lambda^2_{\text{QG}}} \to 0\,,$$
with $\Lambda^2_{\text{WGC}}\left(\mathrm{U}(1)_C\right) = g^2_{\text{YM},C} M^3_{\text{Pl}} = g^2_5 \left(c_\alpha f^{\alpha\beta} c_\beta\right) M^3_{\text{Pl}}$.
2. Alternatively, there does exist a **non-BPS** tower of states along the ray in the charge lattice which is part of the tower of excitations of a **critical string** becoming weakly coupled in the limit, and this tower of states is self-repulsive in the asymptotic limit in the sense of **satisfying condition** of the RFC.

Understanding the Claim

Whenever there **exists** a limit of the type

$$\frac{\Lambda^2_{\text{WGC}}\left(\text{U}(1)_{\mathbf{Q}^0}\right)}{\Lambda^2_{\text{QG}}} \longrightarrow 0$$

either

- there **exists** a BPS tower of super-extremal states along the given ray in the charge lattice, or
- we can identify a **non-BPS** tower of self-repulsive states originating in the excitation spectrum of an asymptotically weakly coupled **critical string**.

1. Since $\Lambda_{\text{QG}} \leq M_{\text{Pl}}$, then

$$Q^0_\alpha f^{\alpha\beta} Q^0_\beta \longrightarrow 0\,.$$

2. We look for **all possible limits** in the 5d moduli space and compare with the scaling of Λ_{QG}.
3. Rescaling of all the Kähler moduli

$$v^\alpha \longrightarrow \lambda \tilde{v}^\alpha\,, \quad \forall \alpha = 1, \ldots, h^{1,1}(X_3)\,,$$

   we can assume that the overall volume $\mathcal{V}$ stays constant along the trajectory in moduli space.
4. Hence it suffices to analyze limits in the **vector moduli space**.

Results

T^2-type Limits

Weakly coupled $\mathrm{U}(1)$**s**: $\mathrm{U}(1)$ associated to the generic T^2-fiber, i.e. KK $\mathrm{U}(1)$ of the lift to F-theory on the base of the CY 3-fold times S^1. These are **BPS states**, reflecting the fact that the generic T^2 fiber is a movable curve.[13]

K3/T^4-type Limits

Weakly coupled $\mathrm{U}(1)$**s**: $\mathrm{U}(1)$s obtained by reducing C_3 over a curve in a generic K3/T^4 fiber of X_3 or curves contained in a degenerate fiber arising at **finite** distance in the fiber moduli space. The charged states can all be interpreted as **excitations of the heterotic string** obtained by wrapping an M5-brane on the generic fiber.

[13] M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th)

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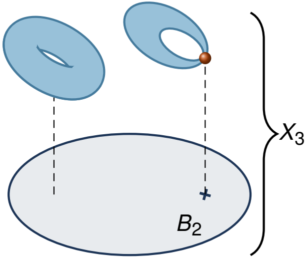
T^2-type Limits



Limits of Type T^2

X_3 admits a torus fibration

$$\pi : X_3 \longrightarrow B_2$$

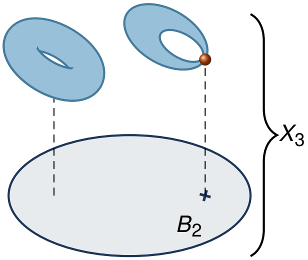
and the weak-coupling limit corresponds to a limit in which the volume of the generic fiber T^2 shrinks as

$$\mathcal{V}_{T^2} \sim \frac{1}{\lambda}\,, \quad \mathcal{V}_{B_2} \sim \lambda\,, \quad \lambda \longrightarrow \infty\,.$$

1. Mori cone generators: $\{C^\alpha\} = \left\{C^a, C_f^i\right\}\,, \quad a = 1, \ldots, h^{1,1}(B_2)\,, \quad i = 1, \ldots, n\,.$
2. Basis of $H^{1,1}(X_3)$: $\{J_\alpha\} = \{J_a, J_i\}$ with $J_a = \pi^*(j_a)\,.$
3. Gauge kinetic matrix

$$f_{\alpha\beta} = \frac{1}{\mathcal{V}^{1/3}}\left(\frac{\mathcal{V}_\alpha \mathcal{V}_\beta}{\mathcal{V}} - \mathcal{V}_{\alpha\beta}\right) = \left(\widehat{\mathcal{V}}_\alpha \widehat{\mathcal{V}}_\beta - \widehat{\mathcal{V}}_{\alpha\beta}\right) \sim \begin{pmatrix} \tilde{f}_{ab}\lambda^{-1} & \tilde{f}_{ai}\lambda^{1/2} \\ \tilde{f}_{ia}\lambda^{1/2} & \tilde{f}_{ij}\lambda^{2} \end{pmatrix}$$

T^2-type Limits



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X_3 admits a torus fibration

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$$\mathcal{V}_{T^2} \sim \frac{1}{\lambda}\,, \quad \mathcal{V}_{B_2} \sim \lambda\,, \quad \lambda \longrightarrow \infty\,.$$

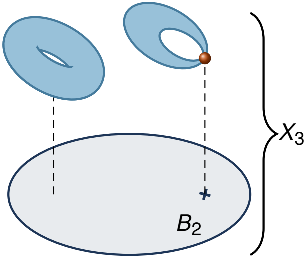
1. $f_{ij} = \lambda^2 \widehat{\widetilde{\mathcal{V}}}_i \widehat{\widetilde{\mathcal{V}}}_j - \sqrt{\lambda}\kappa_{ija}\hat{\tilde{v}}^a + \mathcal{O}(1/\lambda)\,.$
2. Identity: $c_j J_i \cdot \pi^*(j_a) \cdot \pi^*(j_b) = c_i J_j \cdot \pi^*(j_a) \cdot \pi^*(j_b)\,.$
3. U(1)s:

$$\frac{\Lambda^3_{\text{sp,KK}}}{M^3_{\text{Pl}}} = \left(\frac{M^3_{\text{KK}}}{M^3_{\text{Pl}}}\right)^{\frac{n}{n+3}}\Bigg|_{n=1}$$

$$\mathrm{U}(1)^{(i)}_{-} = c_i\,\mathrm{U}(1)^i - c_{i+1}\,\mathrm{U}(1)^{i+1}\,, \text{ with } g^2_{\text{YM}} M_{\text{Pl}} \sim \lambda^{-1/2}\,, \text{ but } \frac{\Lambda^2_{\text{WGC}}(\mathrm{U}(1)^{(i)}_{-})}{\Lambda^2_{\text{sp,KK}}} \sim \frac{\lambda^{1/2}}{\lambda^{1/2}} \sim \text{const.}$$

$$\mathrm{U}(1)_{\mathcal{E}} = \sum_i c_i\,\mathrm{U}(1)^i\,, \text{ with } g^2_{\text{YM}} M_{\text{Pl}} \sim \lambda^{-2}\,, \text{ but } \frac{\Lambda^2_{\text{WGC}}(\mathrm{U}(1)_{\mathcal{E}})}{\Lambda^2_{\text{sp,KK}}} \sim \lambda^{-3/2}$$

T^2-type Limits - BPS Tower



1. The **only** asymptotically weakly coupled linear combination is

$$\mathrm{U}(1)_{\mathcal{E}} = \sum_{i=1}^{n} c_i \, \mathrm{U}(1)^i \text{ with } \frac{\Lambda^2_{\mathrm{WGC}}\left(\mathrm{U}(1)_{\mathcal{E}}\right)}{\Lambda^2_{\mathrm{sp,KK}}} \sim \frac{1}{\lambda^{3/2}} \, .$$

2. This $\mathrm{U}(1)_{\mathcal{E}}$ clearly satisfies the asymptotic tower WGC conjecture because the ray in the charge lattice along $n\mathbf{Q}_{\mathcal{E}}$ is populated by a tower of **BPS particles**.[14]

3. These are the BPS states obtained by wrapping n M2-branes along the generic fiber, which furnish the KK tower for the effective five-dimensional to six-dimensional limit.

4. The BPS invariants, more precisely the genus-zero Gopakumar–Vafa invariants, for the curve $n\mathcal{E}$ coincide with the Euler characteristic of X_3, i.e.,[15]

$$N^0_{n\mathcal{E}} = -\chi(X_3) \, .$$

[15] M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th)

[16] P.-K. Oehlmann, T. Schimannek, *JHEP* **09**, 066, arXiv: 1912.09493 (hep-th); A.-K. Kashani-Poor, *Commun. Math. Phys.* **386**, 1155–1207, arXiv: 1912.10009 (hep-th)

Contents

A Glimpse of the K3-type Limits

Limits of Type K3/T^4

X_3 allows for a surface fibration $\rho : X_3 \longrightarrow \mathbb{P}^1$, with generic fiber **S** being either a K3 surface, or an abelian surface, T^4. In the weak coupling limit, the volume of the surface fiber shrinks as

$$\mathcal{V}_{\mathrm{K3}/T^4} \sim \frac{1}{\lambda^2}\,, \quad \mathcal{V}_{\mathbb{P}^1} \sim \lambda^2\,, \quad \lambda \to \infty\,.$$

Weakly coupled U(1)s obtained reducing C_3 over

1. curves in the **generic K3-fiber**.
2. curves in a degenerate fiber of Kulikov type I,[16] i.e. at **finite distance** in the K3-fiber moduli space.
3. a curve C given by a **specific** linear combination of curves in a degenerate fiber of Kulikov type II or III.[16]

Details

[16] V. S. Kulikov, *Mathematics of the USSR-Izvestiya* **11**, 957; V. S. Kulikov, *Mathematics of the USSR-Izvestiya* **17**, 339; U. Persson, H. Pinkham, *Ann of Math. (2) 113* **01**, 45–66

A Glimpse of the K3-type Limits

$\Sigma \cdot_{\mathrm{K3}} \Sigma \geq 0$

1. Define rays in the charge lattice admitting towers of **BPS states**: M2-branes wrapped n times on Σ give rise to BPS particles[17] with GV invariant $N^0_{n\Sigma} \neq 0$ for $n \in \mathbb{N}$.
2. Such curves lie in the **movable cone** of X_3 because they are movable within the K3-fiber.[18]

$\Sigma \cdot_{\mathrm{K3}} \Sigma < 0$

3. Define rays in the charge lattice which do **not** support a BPS tower.
4. Such curves are **rigid** within the K3-fiber and lie outside the movable cone of X_3.[18]
5. **Existing** of a weak coupling limit, implies that the asymptotic Tower WGC calls for a super-extremal tower of states in these directions.
6. The non-BPS states are excitations of the **emergent heterotic string** which becomes light in the weak coupling limit.[19]

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[17] D. Maulik, R. Pandharipande, in *A celebration of algebraic geometry* (Amer. Math. Soc., Providence, RI, 2013), vol. 18, pp. 469–507

[18] M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th)

[19] S.-J. Lee, W. Lerche, T. Weigand, *JHEP* **02**, 190, arXiv: 1910.01135 (hep-th)

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Conclusions

1. We classified the linear combinations of $\mathrm{U}(1)$ factors that become weakly coupled in the respective limit.

2. $\mathrm{U}(1)$s can **only** become weakly coupled in a given infinite distance limit if they are obtained from reducing C_3 over
   - a curve that is contained in the **generic fiber** or
   - a curve that is localized in a degenerate fiber associated to a **finite** distance degeneration in the deformation space of the generic fiber.
3. **Any direction** in the charge lattice of five-dimensional M-theory that is **not** populated by a tower of BPS states either carries charge under a $\mathrm{U}(1)$ **without** a weak coupling limit, or there are super-extremal non-BPS states arising as excitations of a **critical string**.

Questions

4. What can we say about the tower WGC in the directions where there is **no** tower of states?
5. When $\Lambda_{\text{WGC}} \geq \Lambda_{\text{sp}}$, is the tower required for the validity of the EFT? What kind of constraints can impose to the EFT?

Thank you!

K3-type Limits

Limits of Type K3/T^4

X_3 allows for a surface fibration

$$\rho : X_3 \longrightarrow \mathbb{P}^1$$

with generic fiber **S** being either a K3 surface, or an abelian surface, T^4. In the weak coupling limit, the volume of the surface fiber shrinks as

$$\mathcal{V}_{\mathrm{K3}/T^4} \sim \frac{1}{\lambda^2}\,, \quad \mathcal{V}_{\mathbb{P}^1} \sim \lambda^2\,, \quad \lambda \to \infty\,.$$

1. Mori cone generators: $\{\mathcal{C}^\alpha\} = \left\{\mathcal{C}^0, \mathcal{C}^i, \quad i = 1, \ldots, h^{1,1}(X_3) - 1\right\}$.
2. Basis of $H^{1,1}(X_3)$: $\{J_\alpha\} = \left\{J_0, J_i, \quad i = 1, \ldots, h^{1,1}(X_3) - 1\right\}$.
3. Gauge kinetic matrix:

$$f_{\alpha\beta} \sim \begin{pmatrix} \tilde{f}_{00}\lambda^{-2} & \tilde{f}_{0i}\lambda^{-1/2} \\ \tilde{f}_{i0}\lambda^{-1/2} & \tilde{f}_{ij}\lambda \end{pmatrix}$$

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with generic fiber **S** being either a K3 surface, or an abelian surface, T^4. In the weak coupling limit, the volume of the surface fiber shrinks as

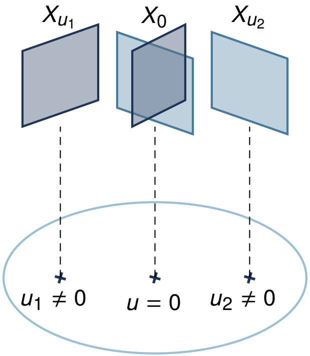
$$\mathcal{V}_{\mathrm{K3}/T^4} \sim \frac{1}{\lambda^2}\,, \quad \mathcal{V}_{\mathbb{P}^1} \sim \lambda^2\,, \quad \lambda \to \infty\,.$$

1. If $f_{ij} = \tilde{f}_{ij}\lambda = \lambda((\hat{\tilde{v}}^0)^2 \eta_{ik}\hat{\tilde{v}}^k \eta_{jl}\hat{\tilde{v}}^l - \hat{\tilde{v}}^0 \eta_{ij}) + \mathcal{O}(\lambda^{-1/2})$ has full rank, then for $\mathrm{U}(1) = Q_i\,\mathrm{U}(1)^i$

$$\frac{\Lambda^2_{\mathrm{WGC}}(\mathrm{U}(1))}{\Lambda^2_{\mathrm{sp}}} \sim \frac{1/\lambda}{1/\lambda(1+\log(\lambda))} \to 0 \qquad \Lambda^2_{\mathrm{sp}} \sim M^2_{\mathrm{het}} \log\left(\frac{M_{\mathrm{Pl}}}{M_{\mathrm{het}}}\right)$$

2. Tower given by excitations of the string given by M5-branes on K3-fiber (i.e. heterotic string), with charge $\mathbf{Q} = (0, Q_i)$.
3. If the fiber is not degenerate, or degenerate of Kulikov Type I, the intersection form η_{ij} (given by the intersection matrix on Λ_0) is **non-degenerate** $\Longrightarrow$ $\tilde{f}_{ij}$ has **full** rank and every fibral curve gives rise to weakly-coupled $\mathrm{U}(1)$s.

K3-type Limits - Degenerations of Type II/III



- Central fiber is reducible:
$$X_0 = \bigcup_{M=1}^{N} X_M \, .$$
- Mori cone generators localized in degenerate Type II/III fibers: $\mathcal{C}^\mu$.

1. The K3-fiber degenerates at **infinite distance** over $p_{II/III} \in \mathbb{P}^1$.
2. Reducible fiber implies divisors D_ρ in the degenerate fibers:
$$J_0 = \sum_\rho a_\rho D_\rho \, .$$
3. Curves C splitting over $p_{II/III} \in \mathbb{P}^1$ are
$$C = \sum_\mu c_\mu \mathcal{C}^\mu + C_{\text{rest}}$$
such that the dual divisors J_μ satisfies $c_\nu J_\mu = c_\mu J_\nu + \sum_\rho \alpha_\rho D_\rho$.
4. However,
$$c_\nu J_\mu \cdot J_0 \cdot J_\alpha = c_\mu J_\nu \cdot J_0 \cdot J_\alpha \, .$$
5. $\tilde{f}_{ij}$ has **reduced** rank and the **only** weak coupling U(1) is given by
$$\mathrm{U}(1)_C = \sum_\mu c_\mu \, \mathrm{U}(1)^\mu + \mathrm{U}(1)_{C_{\text{rest}}} \, , \quad \text{with } \Lambda^2_{\text{WGC}}(\mathrm{U}(1)_C) \sim 1/\lambda \, .$$

K3-type Limits - Weakly Coupled U(1)s

Weakly coupled U(1)s obtained reducing C_3 over

1. curves in the **generic K3-fiber**.
2. curves in a degenerate fiber of Kulikov type I,[20] i.e. at **finite distance** in the K3-fiber moduli space.
3. a curve C given by a **specific** linear combination of curves in a degenerate fiber of Kulikov type II or III.[20]

4. Generic K3-fiber or curves in irreducible degenerate fibers form a sublattice Λ^* of rank $(1, k_{0+\mathrm{I}})$ of $\Gamma^{3,19}$, dual to the polarization lattice Λ of the K3-fibration $\rho : X_3 \longrightarrow \mathbb{P}^1$, i.e. the charge lattice.
5. Let us denote

$$\Lambda^*_{\mathbb{R}} = \Lambda^*_+ \oplus \Lambda^*_- \,,$$

such that $\mathbf{Q}_+ \in \Lambda^*_+$ satisfies $\mathbf{Q}_+^2 > 0$, while $\mathbf{Q}_-^2 < 0$ for $\mathbf{Q}_- \in \Lambda^*_-$.

[20] V. S. Kulikov, *Mathematics of the USSR-Izvestiya* **11**, 957; V. S. Kulikov, *Mathematics of the USSR-Izvestiya* **17**, 339; U. Persson, H. Pinkham, *Ann of Math. (2) 113* **01**, 45–66

K3-type Limits - $\mathbf{Q} \in \Lambda^*_-$

1. The non-BPS states can be identified as excitations of the five-dimensional heterotic string obtained by wrapping an M5-brane on the K3-fiber in M-theory.
2. We identify the coefficients of the elliptic genus of the five-dimensional heterotic string at left-moving excitation level n and with charge vector $\mathbf{Q} \in \Lambda^*_-$ with certain D4-D2-D0 **Donaldson–Thomas** invariants in Type IIA string theory on X_3.
3. The existence of these states then follows from the correspondence with **Noether–Lefschetz numbers**.[21]
4. There exists a distinguished set of non-BPS states at left-moving excitation level n_{L} and with (anti-self-dual) charge vector $\mathbf{Q} \in \Lambda^*_-$ obeying the relation

$$n_{\mathrm{L}} = -\frac{1}{2} Q_i \eta^{ij} Q_j =: -\frac{1}{2}\mathbf{Q}^2 \,,$$

where η^{ij} is the inverse of the intersection form on the K3-fiber.

[21] S. H. Katz, A. Klemm, C. Vafa, *Adv. Theor. Math. Phys.* **3**, 1445–1537, arXiv: hep-th/9910181; R. Pandharipande, R. P. Thomas, *Forum Math. Pi* **4**, e4, arXiv: 1404.6698 (math.AG); D. Maulik, R. Pandharipande, in *A celebration of algebraic geometry* (Amer. Math. Soc., Providence, RI, 2013), vol. 18, pp. 469–507; V. Bouchard, T. Creutzig, D.-E. Diaconescu, C. Doran, C. Quigley, A. Sheshmani, *Commun. Math. Phys.* **350**, 1069–1121, arXiv: 1601.04030 (hep-th).

$Q_+ \neq 0$

K3-type Limits - $\mathbf{Q}_+ = 0$ and $\mathbf{Q}_- \neq 0$

1. We look for states with $n_L = -\frac{1}{2}\mathbf{Q}_-^2$.
2. We consider the elliptic genus of an MSW string[22] obtained from M5-brane on K3-fiber:

$$Z^{(r)}_{K3}(\tau,\bar{\tau},\mathbf{z},\mathcal{B}) = \mathrm{Tr}_{RR} F_R^2 (-1)^{F_R} q^{L_0 - \frac{c_L}{24}} \bar{q}^{\bar{L}_0 - \frac{c_R}{24}} e^{2\pi i z^i Q_i}\ .$$

3. We counted the BPS states obtained by wrapping the MSW string r times on an S^1 such that[23]

$$Z^{(r)}_{K3}(\tau,\bar{\tau},\mathbf{z},\mathcal{B}) = \sum_{\mu\in\Lambda^*/r\Lambda} Z_\mu(\tau)\Theta^*_{\mu,r}(\tau,\bar{\tau},\mathbf{z},\mathcal{B})\,,\ \text{where}$$

- $Z_\mu(\tau) = \sum_{n=0}^{\infty} \Omega(\gamma) q^{n+\mathbf{Q}^2/2r-1}$ is a vector-modular form and $\Omega(\gamma)$ are **Donaldson-Thomas invariants**.
- $\Theta^*_{\mu,r}(\tau,\bar{\tau},\mathbf{z},\mathcal{B}) = \sum_{\lambda\in\mu+\sqrt{r}\Lambda} e^{-\pi i\tau(\lambda+\mathcal{B})_-^2 - \pi i\bar{\tau}(\lambda+\mathcal{B})_+^2 + 2\pi i\left(\lambda+\frac{\mathcal{B}}{2}\right)\cdot z}$ is complex conjugate of Siegel theta series.

4. We use **Noether-Lefshetz theory**[24] to argue that for $n = -\frac{1}{2}\mathbf{Q}^2$, $\Omega(\gamma) \neq 0$.

[22] J. M. Maldacena, A. Strominger, E. Witten, *JHEP* **12**, 002, arXiv: hep-th/9711053 (hep-th).

[23] D. Gaiotto, A. Strominger, X. Yin, *JHEP* **08**, 070, arXiv: hep-th/0607010 (hep-th); V. Bouchard, T. Creutzig, D.-E. Diaconescu, C. Doran, C. Quigley, A. Sheshmani, *Commun. Math. Phys.* **350**, 1069–1121, arXiv: 1601.04030 (hep-th).

[24] D. Maulik, R. Pandharipande, in *A celebration of algebraic geometry* (Amer. Math. Soc., Providence, RI, 2013), vol. 18, pp. 469–507.

K3-type Limits - Tower WGC

1. Consider the excitations of the heterotic string at **left-moving** excitation level n_{L} carrying charges Q_i within the charge lattice Λ^*.
2. The mass is given by

$$M^2_{n_{\text{L}},\mathbf{Q}} = 8\pi(n_{\text{L}} - a)T_s + \Delta_{\text{CB}} = \underbrace{16\pi^2(4\pi)^{-2/3}}_{\omega}\left((n_{\text{L}} - a)\widehat{\mathcal{V}}_{\mathbf{S}} + \frac{1}{4}Q_iQ_j\hat{v}^i\hat{v}^j\right)M^2_{\text{Pl}}$$

3. Let us now analyze in more detail the special subset of states for which

$$n_{\text{L}} = -\frac{1}{2}Q_i\eta^{ij}Q_j$$

4. The RFC evaluates to

$$\text{RHS} \simeq 1 + \frac{\frac{1}{4n_{\text{L}}}Q_iQ_j\hat{v}^i\hat{v}^j}{\widehat{\mathcal{V}}_{\text{K3}} + \frac{1}{4n_{\text{L}}}Q_iQ_j\hat{v}^i\hat{v}^j} - \frac{\omega^2 n_{\text{L}}^2}{\hat{v}^0}\frac{M^4_{\text{Pl}}}{M^4_{n_{\text{L}},\mathbf{Q}}}\frac{1}{4n_{\text{L}}}Q_iQ_j\hat{v}^i\hat{v}^j\left(1 + \frac{1}{2n_{\text{L}}}\left(Q_k\eta^{kl}Q_l\right)\right)$$

$$\text{LHS} \simeq 1 + \frac{\frac{1}{4n_{\text{L}}}Q_iQ_j\hat{v}^i\hat{v}^j}{\widehat{\mathcal{V}}_{\text{K3}} + \frac{1}{4n_{\text{L}}}Q_iQ_j\hat{v}^i\hat{v}^j}$$

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$\mathbf{Q}_+ \neq 0$ and $\mathbf{Q}_- \neq 0$

1. We look for states with $n_{\text{L}} = -\frac{1}{2}\mathbf{Q}_-^2$.
2. If $\mathbf{Q}^2 < 0$, we consider an elliptically fibered K3 and 6d heterotic string wrapped on S^1.
3. For $n_{\text{L}} = 0$, winding number w on the S^1, KK-momentum n_{KK}, and charge Q_{6d} under the 6d gauge group from the M2-brane, we can write
$$\mathbf{Q} = wC_U + n_{\text{KK}}C_T + \mathbf{Q}_{\text{6d}} = \frac{(w + n_{\text{KK}})}{2}C_+ + \frac{(w - n_{\text{KK}})}{2}C_- + \mathbf{Q}_{\text{6d}}\,.$$
4. By duality with M-theory in 5d, the heterotic string at excitation n_{L} is associated to the BPS state obtained by M2-brane on
$$\Sigma = \mathbf{Q} + \frac{n_{\text{L}}}{w}C_T = wC_U + (n_{\text{KK}} + \frac{n_{\text{L}}}{w})C_T + \mathbf{Q}_{\text{6d}}\,.$$
5. Such BPS states **exist** for $\Sigma \cdot_{\text{K3}} \Sigma \geq 0$,[25] implying $n_{\text{L}} \geq -\frac{1}{2}\mathbf{Q}^2$.

Back

[25] M. Alim, B. Heidenreich, T. Rudelius, *Fortsch. Phys.* **69**, 2100125, arXiv: 2108.08309 (hep-th); N. Gendler, B. Heidenreich, L. McAllister, J. Moritz, T. Rudelius, arXiv: 2212.10573 (hep-th).