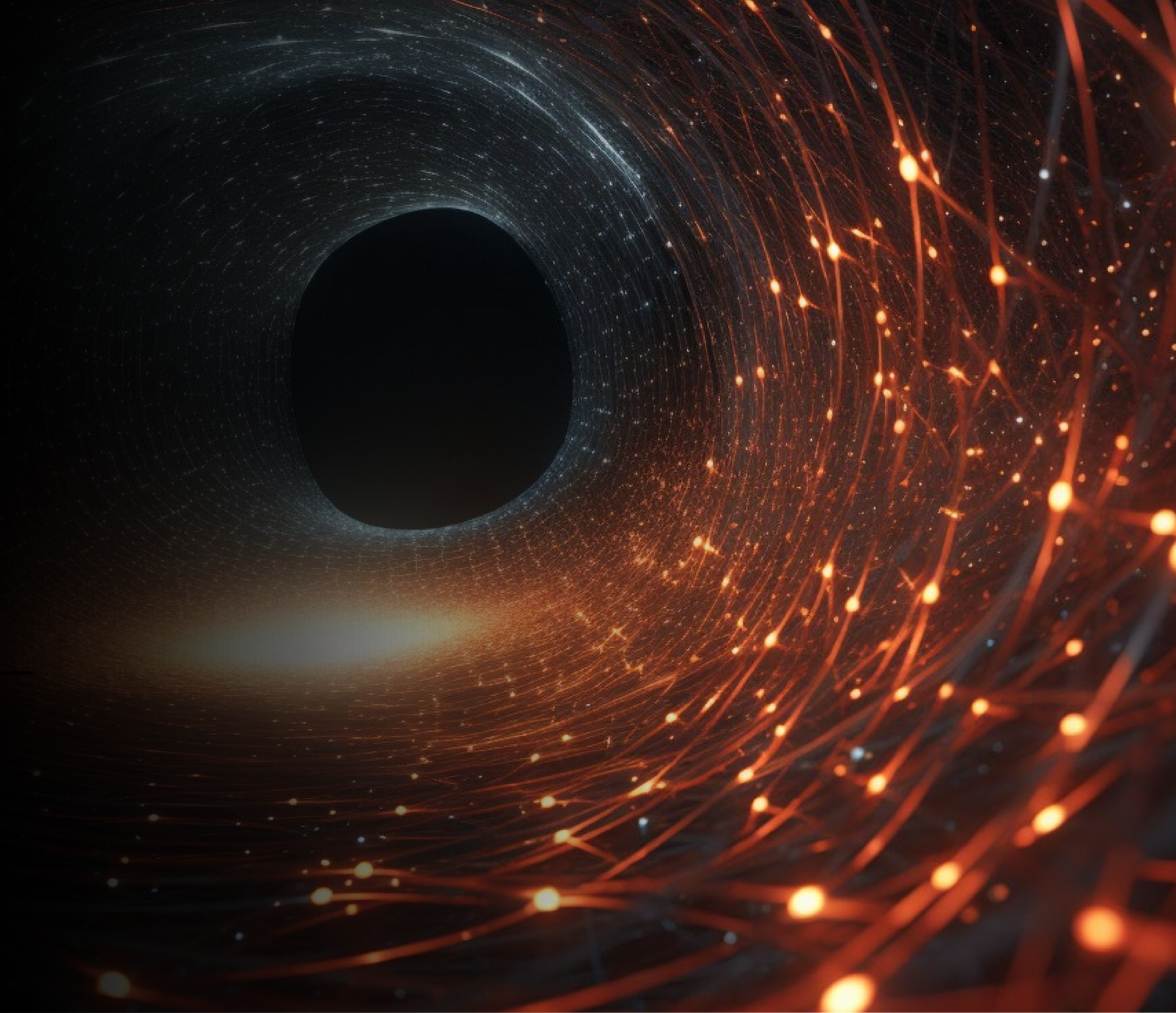
Emergent Ensemble Symmetries in Holography & Swampland

Jacob M. Leedom

String Phenomenology 2023

arXiv: 2305.10224+

with Meer Ashwinkumar
& Masahito Yamazaki



Wormholes & Ensembles

$$Z = \int DgD\phi \exp\left[- S[g,\phi;\lambda] + \frac{1}{2}\iint \Delta_{ij}\ \mathcal{O}_i(x)\mathcal{O}_j(y)\right]$$

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[Coleman,'88]
[Giddings & Strominger,'88,'89]

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Wormholes induce an average over an ensemble of theories

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Wormholes induce an average over an ensemble of theories

Vast Literature on their role in the Landscape, Swampland, & Beyond:

[Loges,Shiu,Van Riet, '23]
[Guidetti,Righi,Venken,Westphal, '22]
[Andriolo,Shiu,Soler,Van Riet, '22]
[Andriolo,Huang,Noumi,Ooguri,Shiu, '20]
[McNamara & Vafa, '20]
[Hebecker,Mangat,Theisen,Witkowski, '16]
[Montero,Valenzuela,Uranga, '15]
[Brown,Cottrell,Shiu,Soler, '15]

& more!

Wormholes & Ensembles

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Problematic:

- **Statistical interpretation of couplings**
- **No sign of ensembles in top-down holography, e.g. $AdS_5 \times S^5$**

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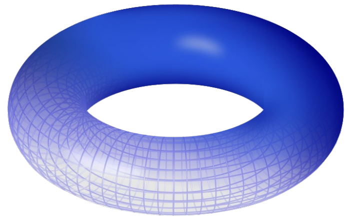
Holography of Narain Ensembles



$$\mathrm{AdS}_3 \leftrightarrow \mathrm{CFT}_2$$

$$Z_{\mathrm{Bulk}}[\phi(\mathrm{boundary}) = J] = Z_{\mathrm{CFT}}[J]$$

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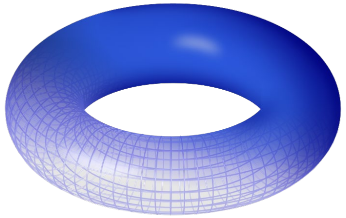

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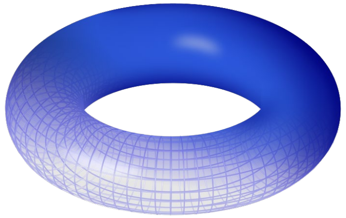

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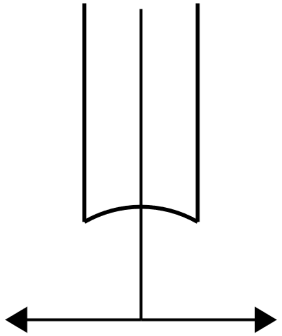

2D Narain CFTs with $(c, \bar{c}) = (p, p)$

$$\mathcal{M} = O(p,p,\mathbb{Z})\backslash O(p,p,\mathbb{R})/O(p) \times O(p)$$

$$Z_{\mathrm{CFT}}[m;\tau] = \frac{\vartheta_{p/2}(m;\tau)}{|\eta(\tau)|^{2p}}$$

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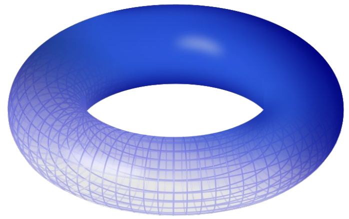

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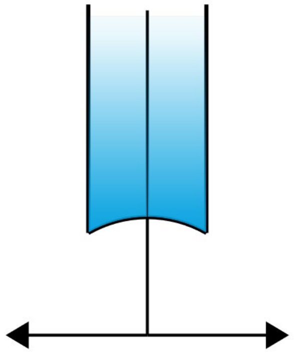

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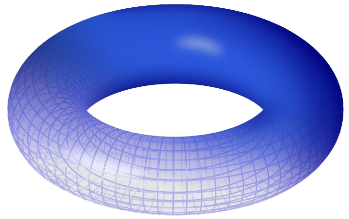

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3D Abelian Chern-Simons with no non-trivial Wilson lines

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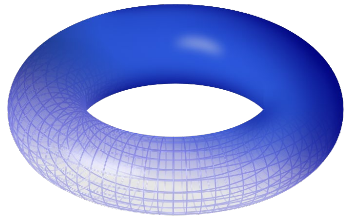

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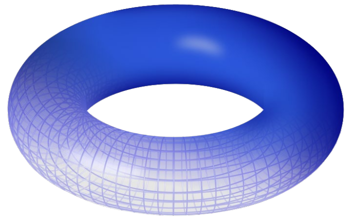

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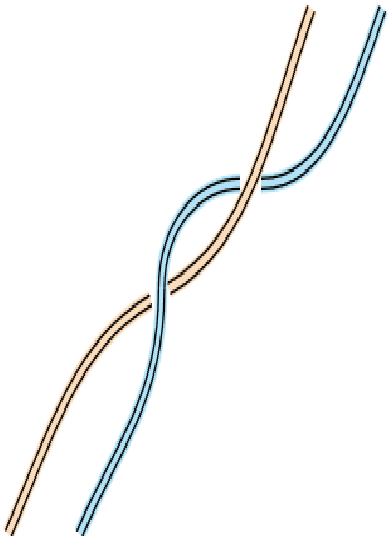

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Wilson lines -> anyons

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Emergent Ensemble Symmetries

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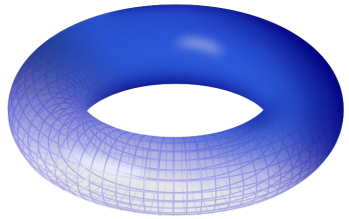

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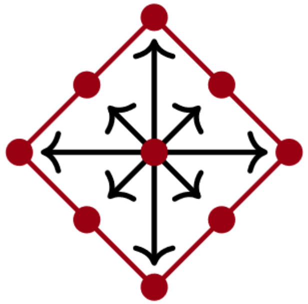
After ensemble averaging -> global symmetries emerge that act on anyons



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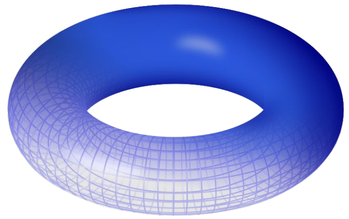

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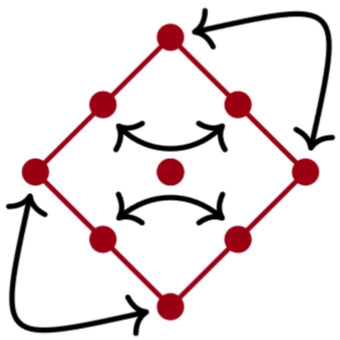

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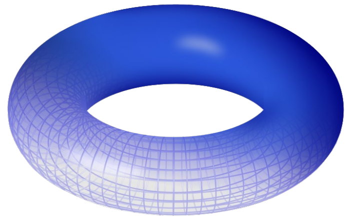

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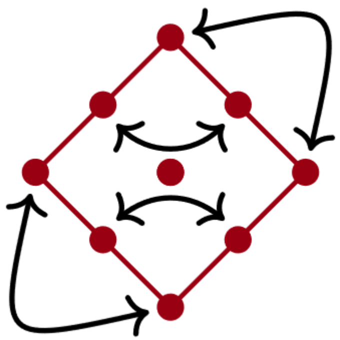

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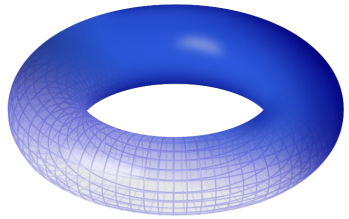

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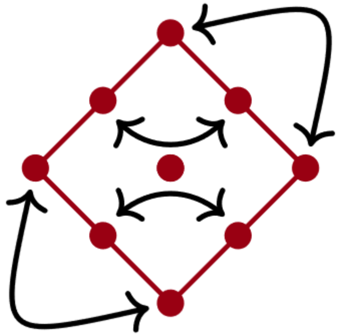

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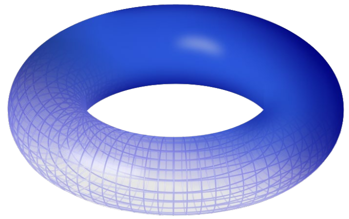

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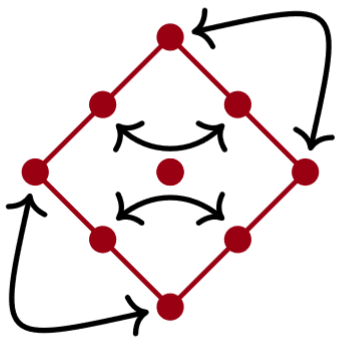

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To have any hope, we need to break or gauge the symmetry

Breaking Emergent Ensemble Symmetries

The answer: even more number theory

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Roelcke-Selberg Spectral Decomposition:

$$f(\tau) = \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=\frac{1}{2}} (f, E_s)E_s(\tau)ds + \sum_{n=0}^{\infty}(f,\nu_n)\nu_n(\tau)$$

[Terras]

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For the non-square integrable Narain partition functions:

[Zagier, '81]
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Ensemble Average

Ensemble Average displays emergent global symmetry, the full theta function does not:

Classical Symmetries -> T-Duality

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How should we interpret the additional terms?

Breaking Emergent Ensemble Symmetries

Theta functions furnish basis of wavefunctions for 3D Maxwell-Chern-Simons

$$\Psi_\alpha \propto \frac{\vartheta_\alpha(\tau, \xi(A); m)}{\eta^p(\tau)\bar{\eta}^q(\bar{\tau})}$$

Arise from string compactifications on AdS_3 x M_7 – [Gukov,Martinec, Moore, Strominger, '04]

- Argue for duality: Topological limit of MCS & Irrational CFT
- Suggests a duality **before** ensemble averaging, and averaging is a coarse-grained description
- Then additional terms in spectral decomposition bring us back to UV completion

What did we learn?

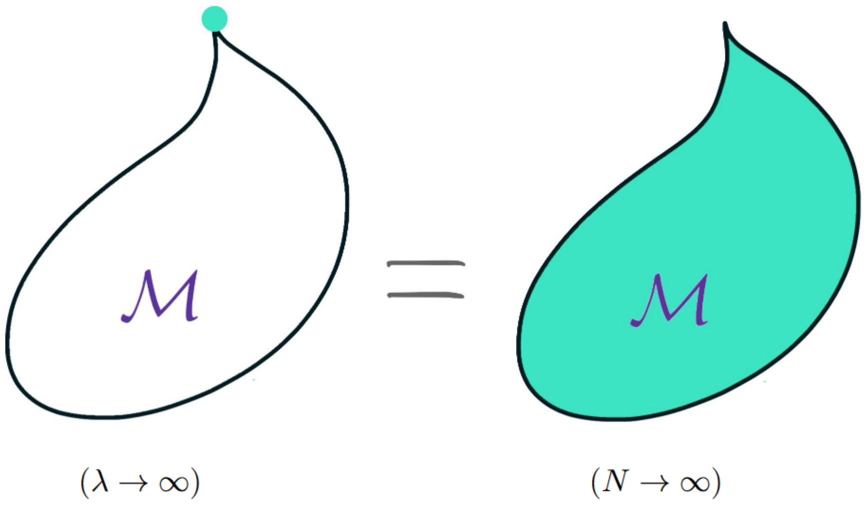
Ensemble averaged theories can have global symmetries after averaging

There is a mathematically well-defined method to break these via the spectral decomposition of theta functions

This seems to suggest a natural embedding in string compactifications:

- Ensemble averages sit as an effective description inside standard holography
- Similar to N=4 SYM story of [Collier & Perlmutter, '22]:

Large N, Large λ $\longleftrightarrow$ Large N, Ensemble Averaged



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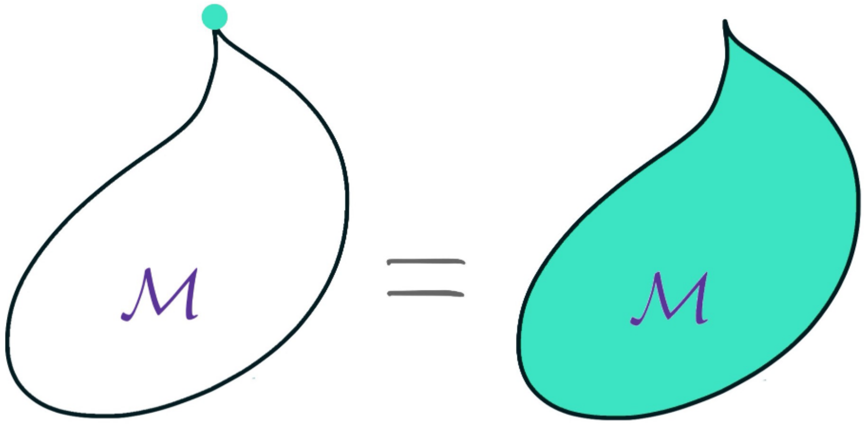

$(\lambda \to \infty)$ $(N \to \infty)$

$$\mathcal{O}_{SUGRA} = \mathcal{O}(\lambda \to \infty) = \langle \mathcal{O} \rangle$$

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Implications for wormhole physics:

- Wormholes -> Ensemble Averaged theories = coarse grained description
- Corrections -> deviation from average, return to QG. Interpret as half-wormholes?

Akin to corrections in JT-SYK to restore factorization

[Saad,Shenker,Yao,'21]+....

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Ongoing Work:

- General Orbifolded Narain CFTs -> alternative UV interpretation? [Benini+,'22]
- Physics Interpretation of the Spectral Decomposition

Backup Slide #1

$$S_{Narain} = \int d^2\sigma \left\{ G_{MN}\partial^a X^M \partial_a X^N + B_{MN}\epsilon_{ab}\partial^a X^M \partial^b X^N \right\}$$

- Compact Bosons:

$$\begin{pmatrix} X_L^M \\ X_R^M \end{pmatrix} \sim \begin{pmatrix} X_L^M \\ X_R^M \end{pmatrix} + \mathcal{E}L$$

- $L \in \mathbb{Z}^{2D}$
- $\mathcal{E}$: Narain Vielbein

- Operators:

$$J^M = \partial X^M$$
$$\bar{J}^M = \bar{\partial} X^M$$
$$\mathcal{V}_{(p_L,p_R)} = :e^{ip_L\cdot X_L + ip_R\cdot X_R}:$$

OPE:

$$\mathcal{V}_{(p_L,p_R)}(z)\mathcal{V}_{(k_L,k_R)}(w) \sim (z-w)^{p_L\cdot k_L}(\bar{z}-\bar{w})^{p_R\cdot k_R}\mathcal{V}_{(p_L+k_L,p_R+k_R)}(w) + \cdots$$

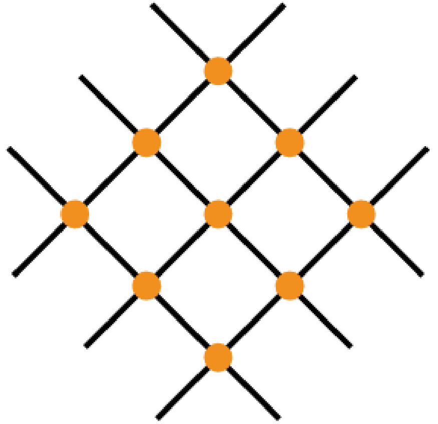
Closure requires an infinite lattice of operators

Backup Slide #2

Operators → Lattice → Partition Function → Quadratic Forms

$$\Lambda_{Narain} := \{\lambda = (p_L, p_R) = \mathcal{E}\ell \,|\, \ell = (n^i, w_i) \in \mathbb{Z}^{2D}\}$$

Discrete Data: Λ

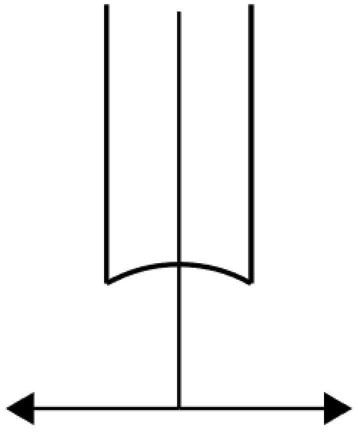


$$\Lambda := \left\{\ell \in \mathbb{Z}^{2D} \mid Q[\ell] \in 2\mathbb{Z}\right\}$$

$$Z_D(\tau; m) = \frac{\vartheta_D(\tau; m)}{|\eta(\tau)|^{2D}}$$

$$\vartheta_D(\tau; m) = \sum_{\lambda \in \Lambda_{Narain}} q^{\frac{p_L^2}{2}} \bar{q}^{\frac{p_R^2}{2}}$$

Continuous Data: $\mathcal{M}_N$



$$\mathcal{M}_N := O(D, D; \mathbb{Z}) \Big\backslash \frac{O(D, D; \mathbb{R})}{O(D) \times O(D)}$$

$$H = \mathcal{E}^T \mathcal{E} \in \mathfrak{h}_Q$$

$$HQ^{-1}H = Q$$

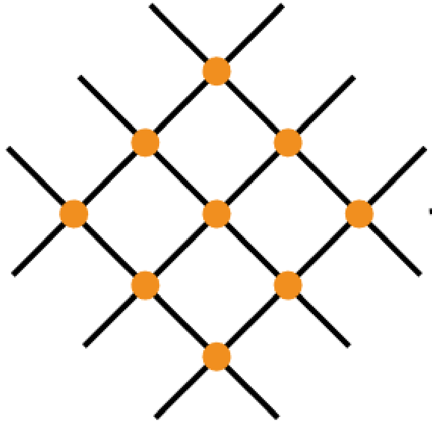
$$H[\ell] := \ell^T H \ell = p_L^2 + p_R^2$$

Backup Slide #3

Operators → Lattice → Partition Function → Quadratic Forms

$$\Lambda_{Narain} := \{\lambda = (p_L, p_R) = \mathcal{E}\ell | \ell = (n^i, w_i) \in \mathbb{Z}^{2D}\}$$

Discrete Data: Λ

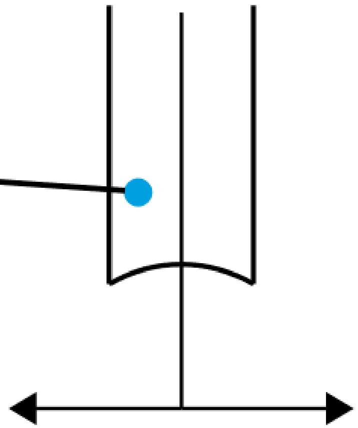


$$\Lambda := \left\{\ell \in \mathbb{Z}^{2D} \mid Q[\ell] \in 2\mathbb{Z}\right\}$$

$$Z_D(\tau; m) = \frac{\vartheta_D(\tau; m)}{|\eta(\tau)|^{2D}}$$

$$\vartheta_D(\tau; m) = \sum_{\ell \in \Lambda} \exp\left[\pi i \tau_1 Q[\ell] - \pi \tau_2 H[\ell]\right]$$

Continuous Data: $\mathcal{M}_N$



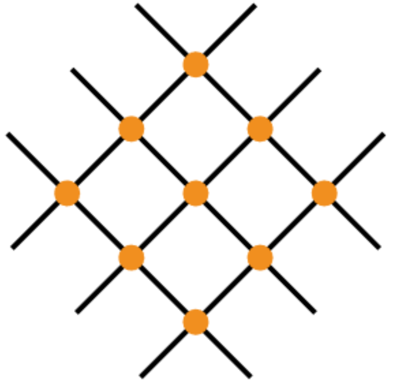
$$\mathcal{M}_N := O(D, D; \mathbb{Z}) \Big\backslash \frac{O(D, D; \mathbb{R})}{O(D) \times O(D)}$$

$$H = \mathcal{E}^T \mathcal{E} \in \mathfrak{h}_Q$$

Operators → Lattice → Partition Function → Quadratic Forms

$$\Lambda_{Narain} := \{\lambda = (p_L, p_R) = \mathcal{E}\ell \,|\, \ell = (n^i, w_i) \in \mathbb{Z}^{2D}\}$$

Discrete Data: Λ



$$\Lambda := \left\{\ell \in \mathbb{Z}^{2D} \mid Q[\ell] \in 2\mathbb{Z}\right\}$$

$$\langle Z_D(\tau; m)\rangle = \int_{\mathcal{M}_N} \frac{\vartheta_D(\tau; m)}{|\eta(\tau)|^{2D}}[dm]$$

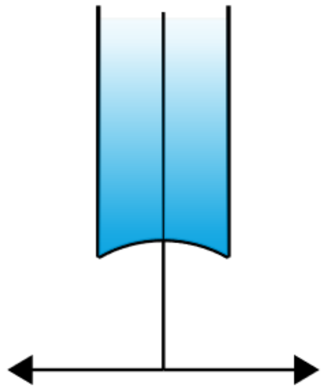
Siegel-Weil Theorem

$$\langle Z_D(\tau; m)\rangle = \frac{E_{D/2}(\tau)}{\tau_2^{D/2}|\eta(\tau)|^{2D}}$$

Poincaré Sum

$$\frac{E_{D/2}(\tau)}{\tau_2^{D/2}|\eta(\tau)|^{2D}} = \sum_{\Gamma_\infty\backslash\Gamma} \frac{1}{|\eta(\gamma\cdot\tau)|^{2D}}$$

Continuous Data: $\mathcal{M}_N$



$$\mathcal{M}_N := O(D,D;\mathbb{Z})\Big\backslash \frac{O(D,D;\mathbb{R})}{O(D)\times O(D)}$$

$$H = \mathcal{E}^T\mathcal{E} \in \mathfrak{h}_Q$$

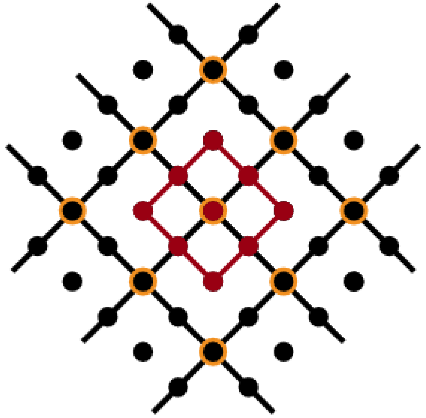
- $E_s(\tau)$: Real-Analytic Eisenstein Series

$$E_s(\tau) := \sum_{(c,d)=1} \frac{\tau_2^s}{|c\tau + d|^{2s}} = \sum_{\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma\cdot\tau)^s$$

- $E_s(\gamma\cdot\tau) = E_s(\tau) \quad \forall\gamma \in \Gamma := \mathrm{PSL}_2(\mathbb{Z})$
- $\Gamma_\infty = \left\{\gamma \in \Gamma \,\middle|\, \gamma = \begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\right\}$

Backup Slide #4

Discrete Data: Λ



$$\Lambda := \{\ell \in \mathbb{Z}^{p+q} \mid Q[\ell] \in 2\mathbb{Z}\}$$

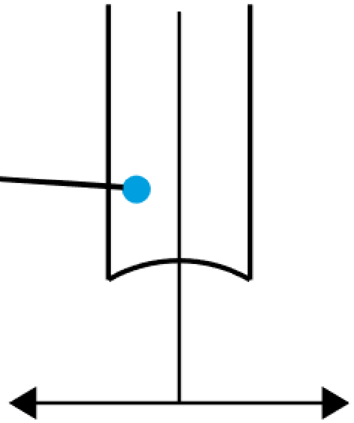
$$\Lambda^* := \{x \in \mathbb{R}^{p+q} \mid Q[x,\ell] \in \mathbb{Z}\}$$

$$\mathcal{D}_\Lambda := \Lambda^*/\Lambda$$

$$Z_\alpha(\tau; m) = \frac{\vartheta_\alpha(\tau; m)}{\eta^p(\tau)\bar{\eta}^q(\bar{\tau})}$$

$$\vartheta_\alpha(\tau; m) = \sum_{\ell \in \Lambda} \exp\left[\pi i \tau_1 Q[\ell+\alpha] - \pi \tau_2 H[\ell+\alpha]\right]$$

Continuous Data: $\mathcal{M}_\Lambda$



$$\mathcal{M}_\Lambda := O_\Lambda(p,q;\mathbb{Z}) \Big\backslash \frac{O(p,q;\mathbb{R})}{O(p)\times O(q)}$$

$$\mathfrak{h}_\Lambda := \{H \in \mathrm{GL}(p+q;\mathbb{R}) \mid HQ^{-1}H = Q\}$$

$$\vartheta_\alpha(g\cdot\tau; m) = \sum_{\beta \in \mathcal{D}_\Lambda} \mathcal{U}_{\alpha,\beta}(g,\tau)\, \vartheta_\beta(\tau; m)$$

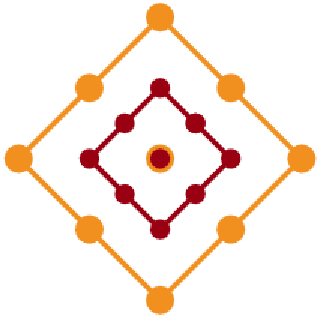
- $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\mathcal{U}_{\alpha,\beta}(g,\tau) := (c\tau + d)^{\frac{p}{2}} (c\bar{\tau} + d)^{\frac{q}{2}} \lambda_{\alpha,\beta}(g)$$

$$\lambda_{\alpha,\beta}(g) := \frac{e^{-\frac{\pi i \sigma}{4}}}{\sqrt{|\mathcal{D}_\Lambda|}}\, c^{-\frac{p+q}{2}} \sum_{\ell_c \in \Lambda/(c\Lambda)} e^{\frac{\pi i}{c}\left(aQ[\ell_c+\alpha] - 2Q[\ell_c+\alpha,\beta] + dQ[\beta]\right)}$$

Backup Slide #5

Discrete Data: Λ



$$\Lambda := \{\ell \in \mathbb{Z}^{p+q} \mid Q[\ell] \in 2\mathbb{Z}\}$$

$$\Lambda^* := \{x \in \mathbb{R}^{p+q} \mid Q[x,\ell] \in \mathbb{Z}\}$$

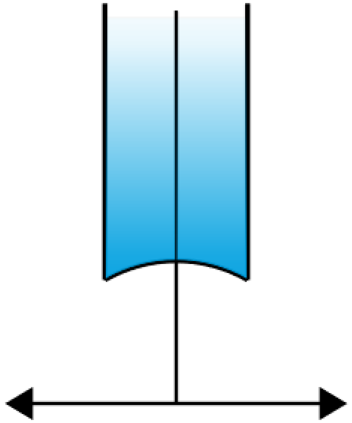
$$\mathcal{D}_\Lambda := \Lambda^*/\Lambda$$

$$\langle Z_\alpha(\tau;m)\rangle = \int_{\mathcal{M}} \frac{\vartheta_\alpha(\tau;m)}{\eta^p(\tau)\bar{\eta}^q(\bar{\tau})}[dm]$$

$\downarrow$ Siegel-Weil Theorem

$$\langle Z_\alpha(\tau;m)\rangle = \frac{E_\alpha(\tau)}{\eta^p(\tau)\bar{\eta}^q(\bar{\tau})}$$

Continuous Data: $\mathcal{M}_\Lambda$



$$\mathcal{M}_\Lambda := O_\Lambda(p,q;\mathbb{Z}) \Big\backslash \frac{O(p,q;\mathbb{R})}{O(p)\times O(q)}$$

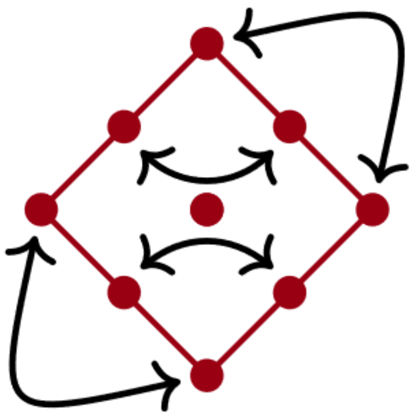
$$E_\alpha(\tau) := \delta_{\alpha\in\Lambda} + \sum_{\substack{(c,d)=1 \\ c>0}} \frac{\gamma_\alpha(c,d)}{(c\tau+d)^{\frac{p}{2}}(c\bar{\tau}+d)^{\frac{q}{2}}}$$

$$\gamma_\alpha(c,d) := e^{i\pi\sigma/4}|\mathcal{D}_\Lambda|^{-\frac{1}{2}}c^{-\frac{p+q}{2}} \sum_{\ell'\in\Lambda/c\Lambda} \exp\left(-i\pi\frac{d}{c}Q[\ell'+\alpha]\right)$$

Backup Slide #6

[Delmastro, Gomis, ,'19]

$$S_{CS} = \int Q_{IJ}\, A^I dA^J\, d^3x$$



$\mathcal{D}_\Lambda = \Lambda^*/\Lambda$

$\alpha \to Y_\pm \alpha$

- Symmetries of a TQFT are automorphisms of its data
  $\Longrightarrow$ permutations of the anyons $\mathcal{D}_\Lambda$ that leave TQFT data invariant modulo gauge transformations (and maybe complex conjugation)
- TQFT data for Abelian TQFTs is completely determined by finite Abelian group $\mathcal{D}_\Lambda$ and topological spins
- 0-form Global Symmetries: $Y_\pm : \mathcal{D}_\Lambda \to \mathcal{D}_\Lambda$

Unitary	**Anti-Unitary**
$Y_+(\alpha \times \beta) = Y_+(\alpha) \times Y_+(\beta)$ $\theta(Y_+(\alpha)) = \theta(\alpha)$ $B(Y_+(\alpha), Y_+(\beta)) = B(\alpha, \beta)$	$Y_-(\alpha \times \beta) = Y_-(\alpha) \times Y_-(\beta)$ $\theta(Y_-(\alpha)) = \theta(\alpha)^*$ $B(Y_-(\alpha), Y_-(\beta)) = B(\alpha, \beta)^*$
Classical $Y_\pm \cdot S_{CS} = S_{CS}$	**Quantum** $Y_\pm \cdot S_{CS} \neq S_{CS}$ **but obey Ward Identities**

$$G_0 := \mathrm{Aut}(\mathcal{D}_\Lambda, \theta) \simeq \{Y_\pm \in \mathbb{Z}^{n \times n} | QPQ \pm Y_\pm^T Q Y_\pm = Q\}/\sim$$

- If $P \neq 0$ $\forall$ representatives of $[Y_\pm]$, then the symmetry is quantum

Backup Slide #7

The Kraken:

[Roelcke, '66 & '67]
[Mono, '19]

$$f(\tau,\bar{\tau}) = \sum_{i=1}^{\infty} \langle f, u_i \rangle \, u_i(\tau) + \sum_{j=1}^{N} \langle f, v_j \rangle \, v_j(\tau)$$

$$+ \sum_{k=1}^{n} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle f, E_{\mathfrak{a}_k}\left(\tau, \frac{1}{2} + it\right) \right\rangle E_{\mathfrak{a}_k}\left(\tau, \frac{1}{2} + it\right) dt$$