Cosmological Constant Extrema in the $O(16)\times O(16)$ Heterotic String on S^1

Bernardo Fraiman
(CERN)

Based on upcoming work with
Mariana **Graña** (Saclay), Hector **Parra de Freitas** (Saclay),
and Savdeep **Sethi** (Chicago U.)

arXiv: 2307.xxxxx

String Phenomenology 2023

Institute for Basic Science, Daejeon, Korea

Time	Mon 3	Tue 4	Wed 5	Thu 6	Fri 7
8:30	*Opening Remarks*	Registration			
Chair	McAllister	K. Lee	Yi	Cvetic	Wrase
9:00	Hebecker	Heckman	Cvetic	D. Lust	Quevedo
9:30	Andriot	Bhardwaj	Wang	Wiesner	Moritz
10:00	Wrase	Schlechter	Garcia-Etxebarria	Valenzuela	Marchesano
10:30	Coffee Break				
Chair	Zavala	Raby	Lerche	Padilla	Parameswaran
11:00	Shiu	Nilles	Weigand	Jeong	Im
11:30	Scalisi	Kobayashi	Heidenreich	Parameswaran	Cicoli
12:00	Montero	Gray	S. Lust	Zavala	Faraggi
12:30	Lunch				
13:00					
13:30					
Chair	Hebecker		Quevedo		D. Lust
14:00	Westphal	Parallel 1	Huang	Parallel 1	Raby
14:30	Hamada		Grana		Ibanez
15:00	Farakos		GenHET *Conference Photo*		Martucci
15:30	Coffee Break				

Toroidal compactifications of non-SUSY heterotic strings.

S^1 → matter spectrum

→ cosmological constant

Qualitative characterization of this moduli space and the behavior of the **cosmological constant** (**extrema** and **stability**)

Heterotic strings in 10D:

SUSY:

$SO(32)$

$E_8 \times E_8$

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$\xrightarrow{S^1}$

Equivalent in 9D

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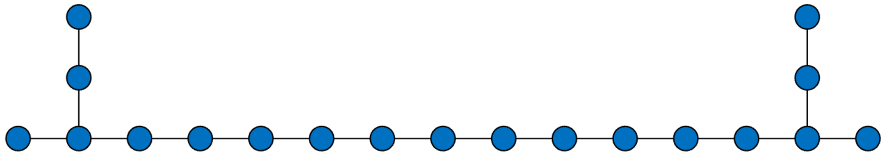
S^1

Equivalent in 9D

Symmetries classified using **EDD**

[F. Cachazo, C. Vafa, '00]

[BF, M. Graña, C. A. Núñez '18]



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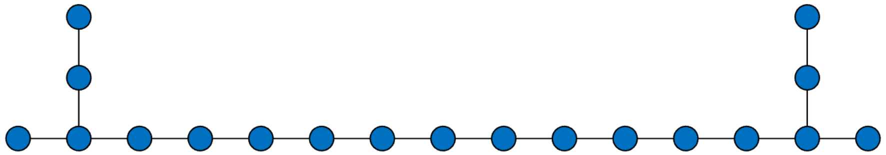

~~SUSY~~: (of rank 16)

[Alvarez-Gaume, Ginsparg, Moore, Vafa ‘86]

[Dixon, Harvey ‘86]

Non-tachyonic:

$\boldsymbol{O(16) \times O(16)}$

Tachyonic:

$SO(32)$ $E_8 \times SO(16)$ $U(16)$

$(E_7 \times SU(2))^2$ $SO(24) \times SO(8)$

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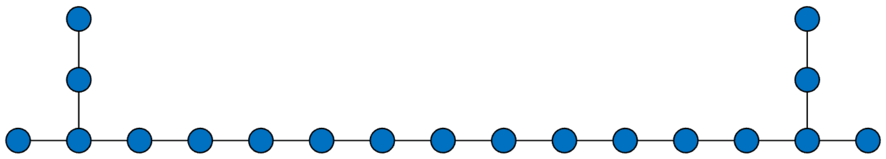

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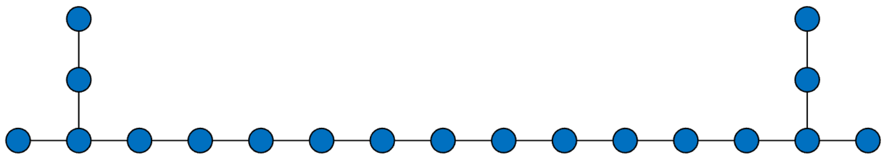

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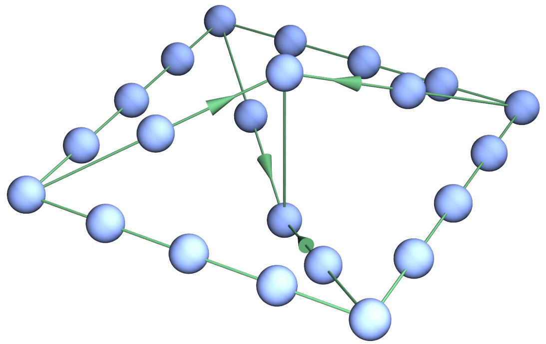
S^1

Equivalent in 9D

Symmetries classified using ***new*** **EDD**

[BF, M. Graña, H. P. de Freitas, S. Sethi WIP]



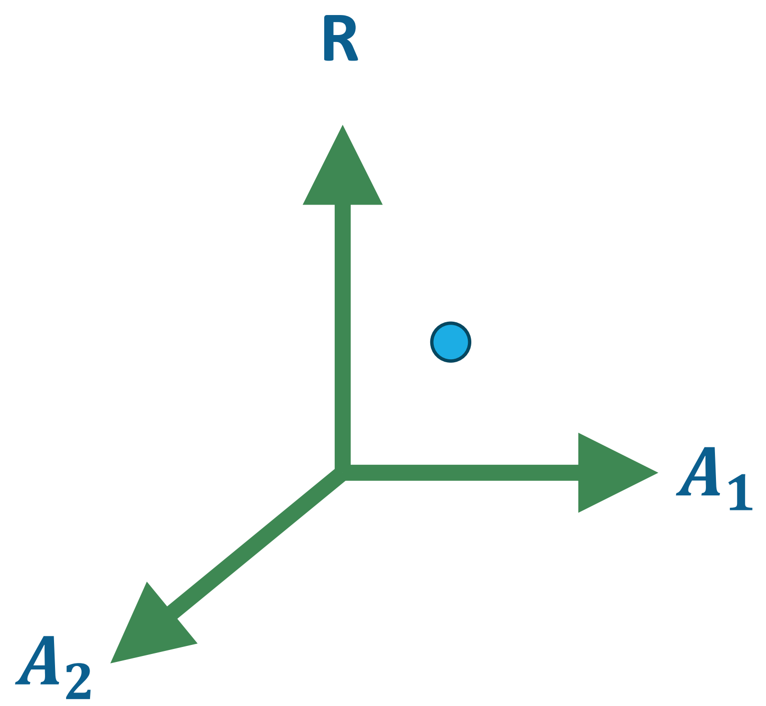
Non-SUSY heterotic on S^1

9D

Classical moduli space:

Radius **R**

16-dimensional Wilson line A_i



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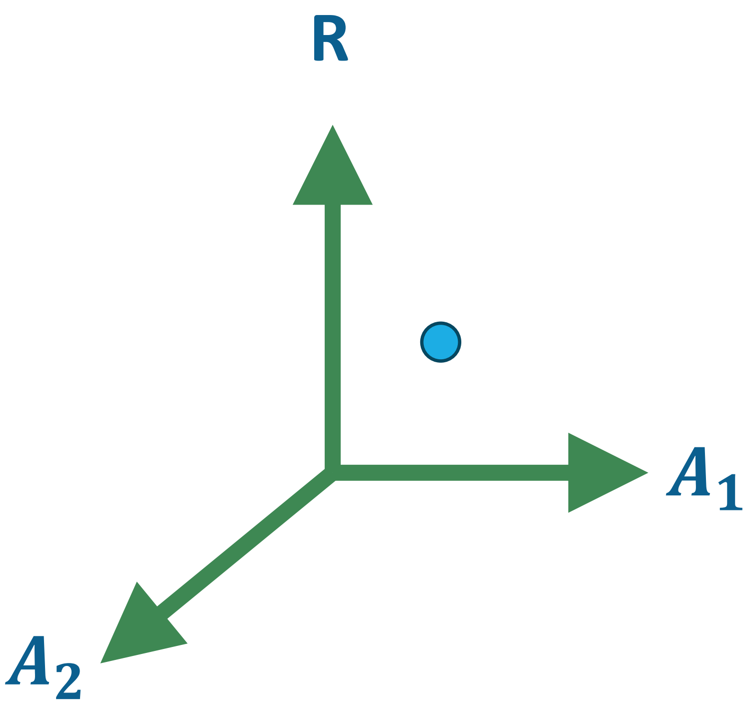

There is a quantum potential for the moduli!

We interpret it as a **cosmological constant $\boldsymbol{\Lambda}$**

$$\boldsymbol{\Lambda}_{1-\mathrm{loop}}(\mathrm{R},\mathrm{A}) = \int \frac{d^2\tau}{\tau_2^2} \boldsymbol{Z}(\tau, R, A)$$

Non-SUSY heterotic on S^1

We are interested in points **extremizing** the one-loop cosmological constant…

What type of extrema?

Maxima
Minima
Saddle points

We don't need to go far to find examples…

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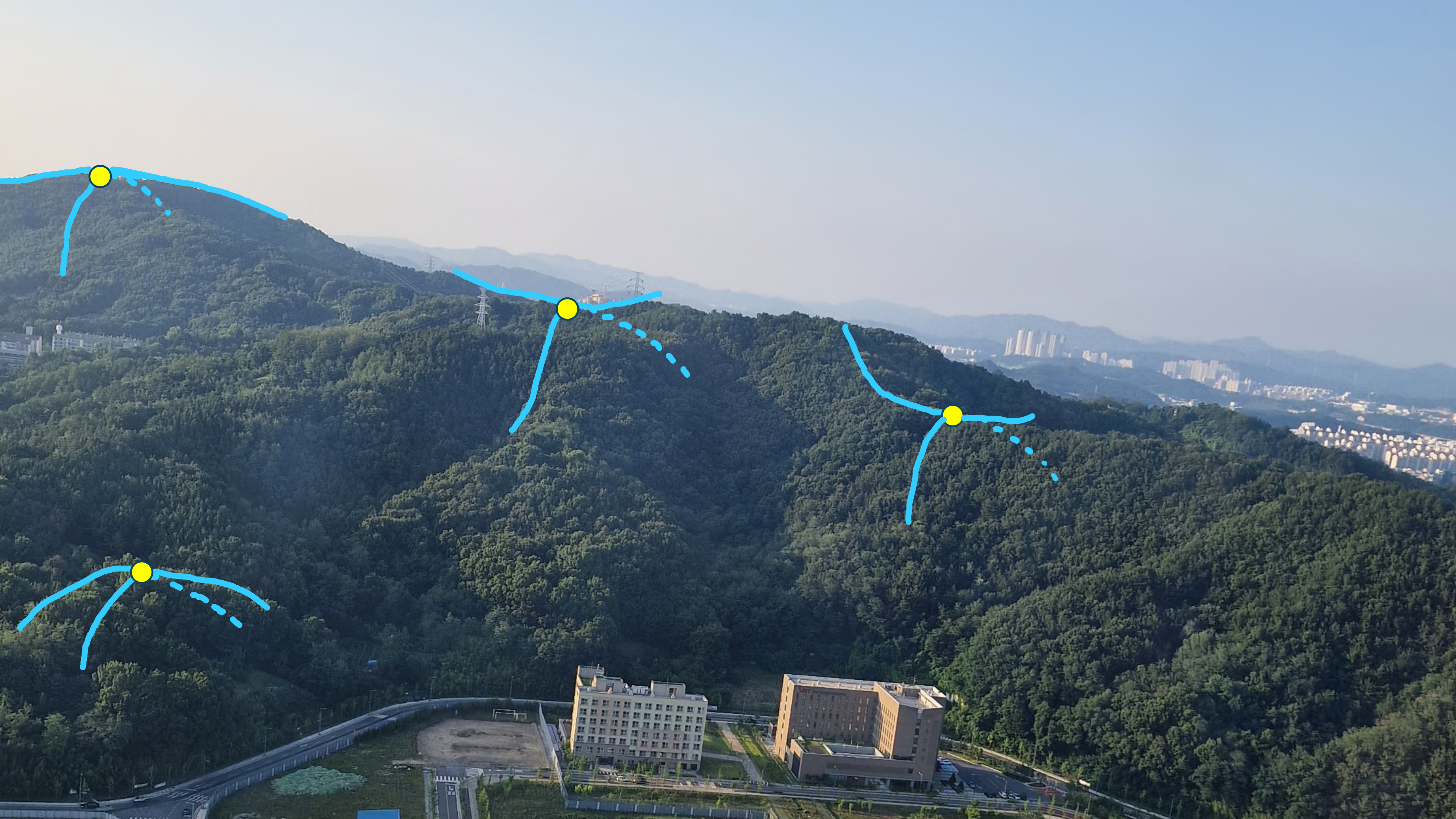
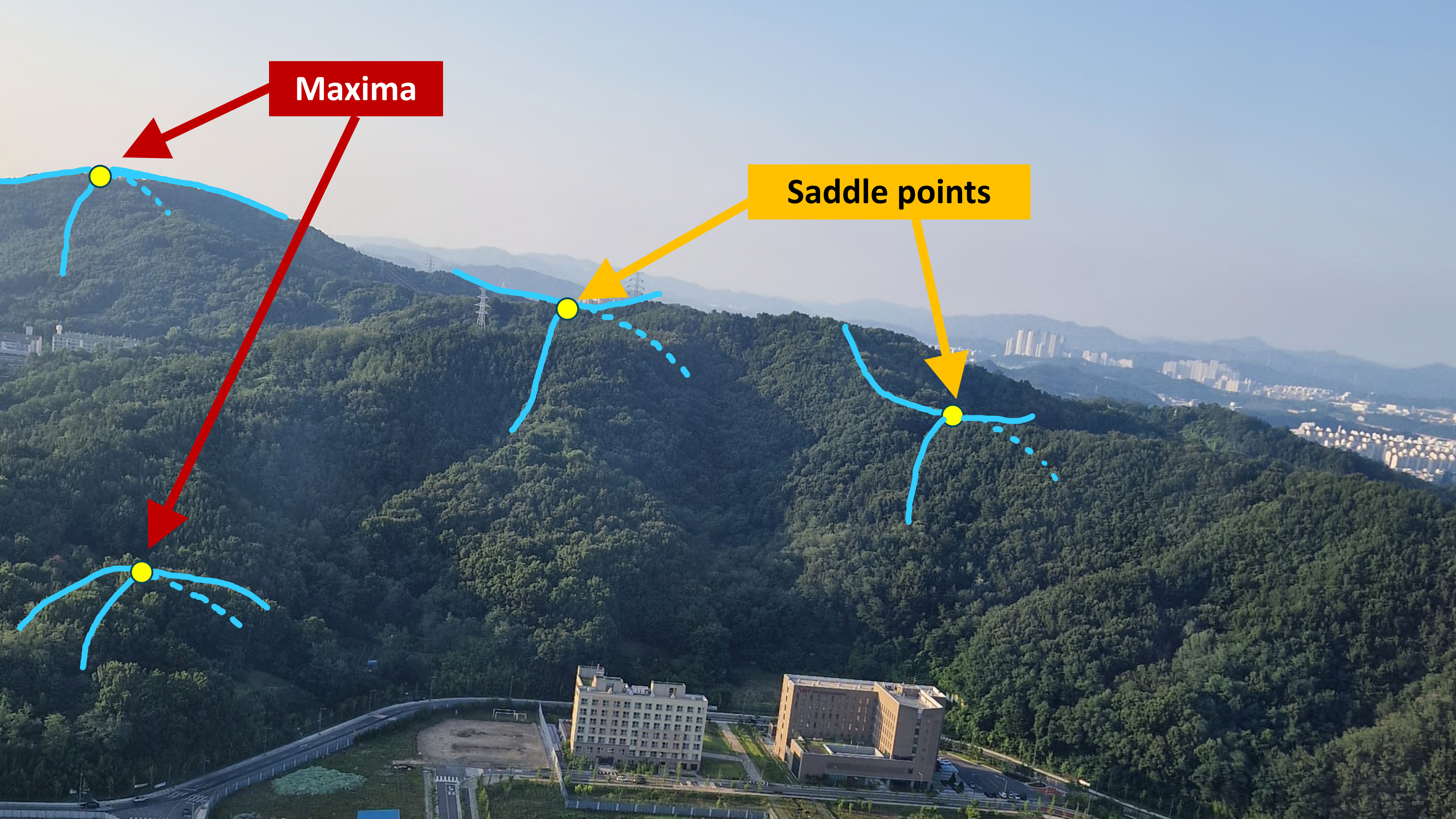
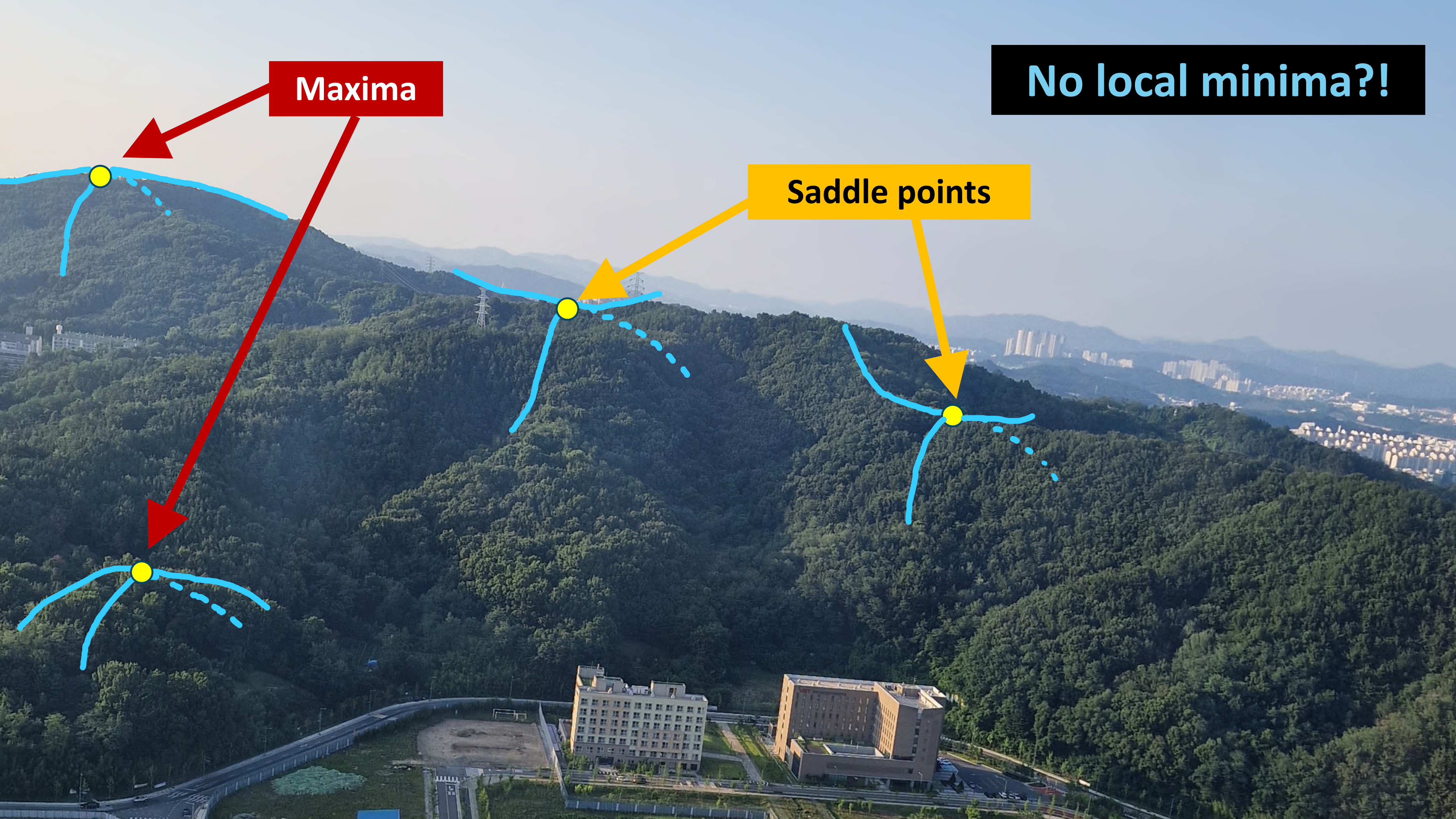



Maxima
Saddle points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No local minima?!
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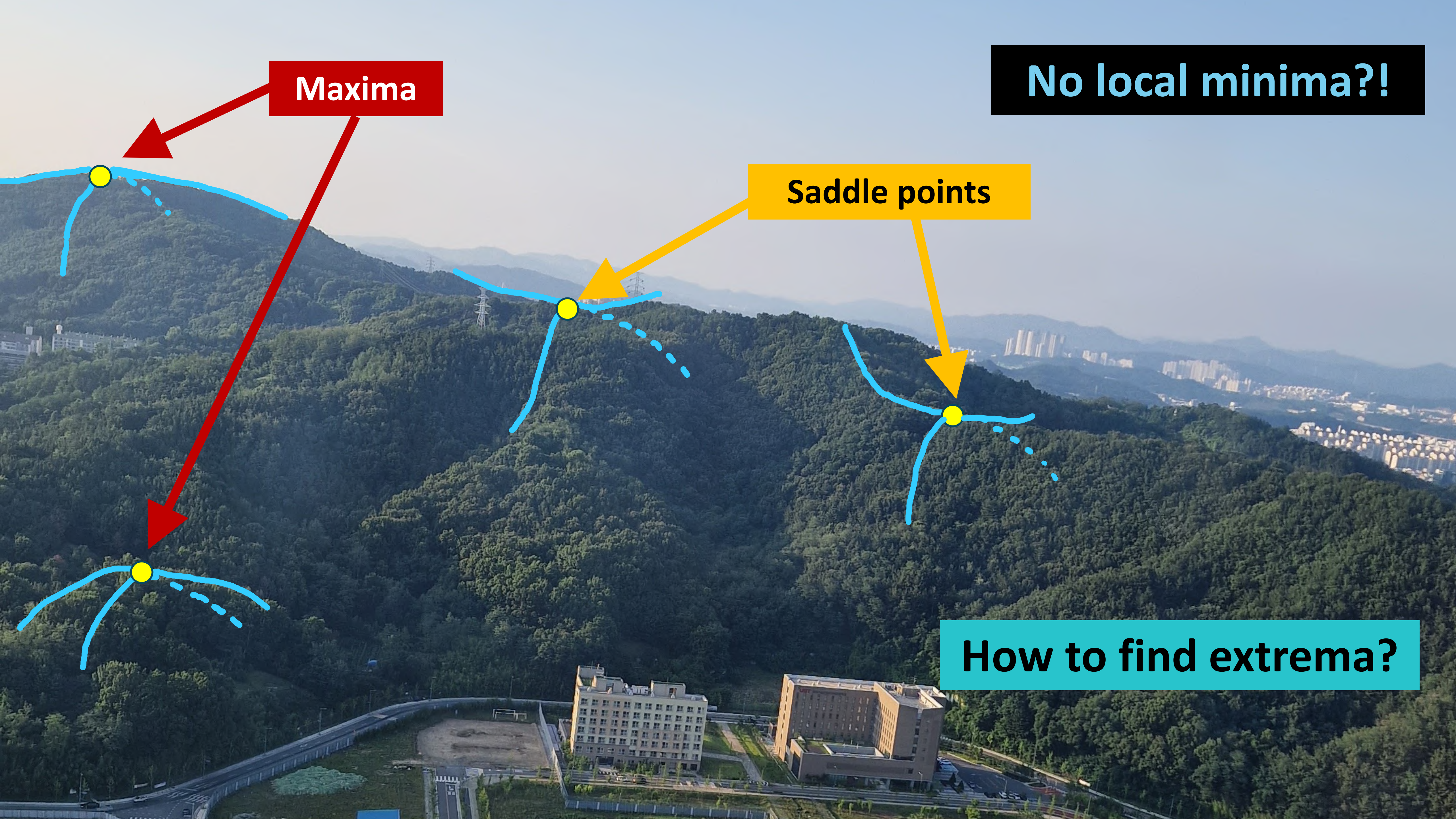
Maxima
No local minima?!
Saddle points
How to find extrema?

How is the (classical) moduli space of **O(16) x O(16) heterotic string on S^1?**

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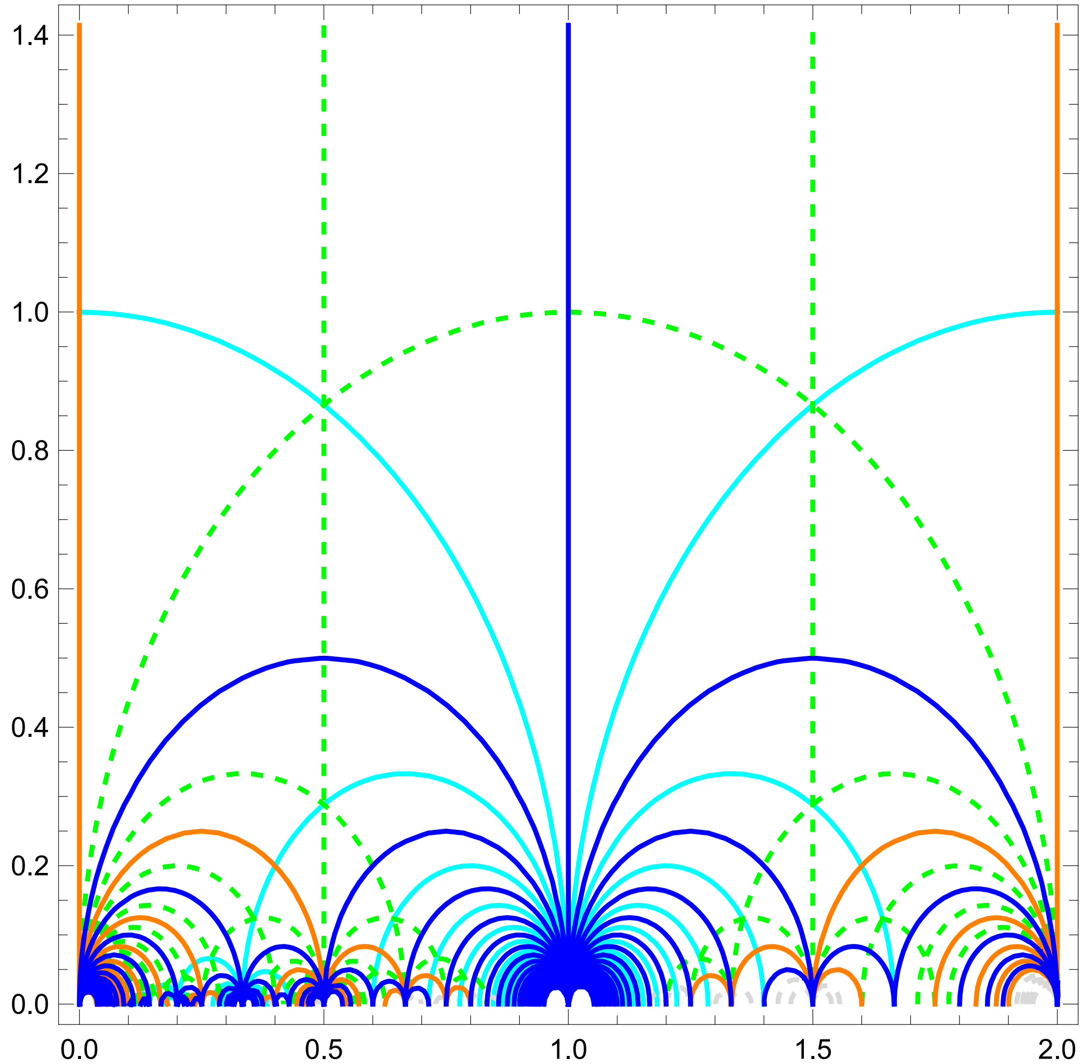
Massless bosons appear at **surfaces/curves/points** (———) (boundaries of fund. region)

Massless fermions appear at **surfaces/curves/points** (- - - -) (not always a boundary)

$$A = (a, 0^7, a, 0^7)$$



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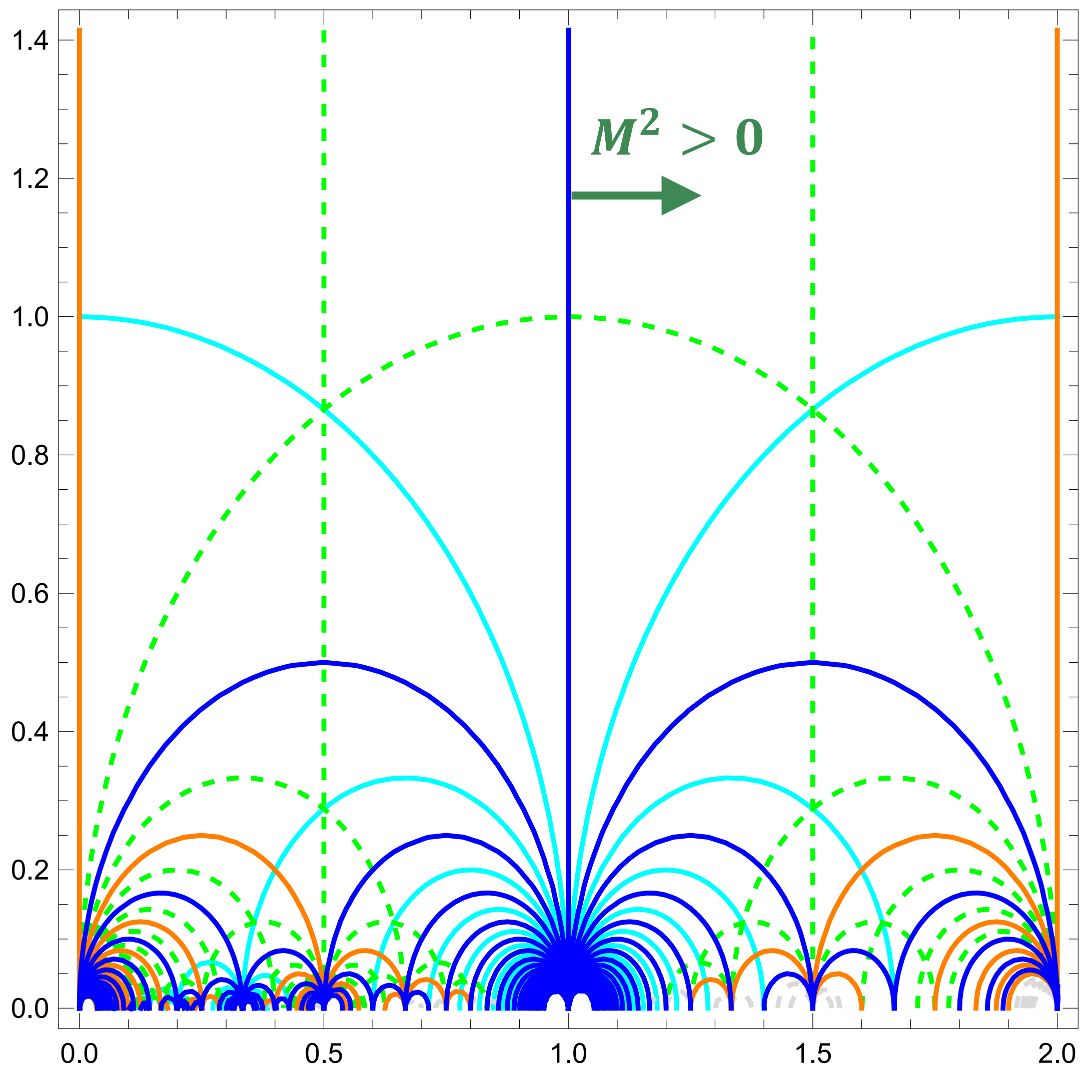

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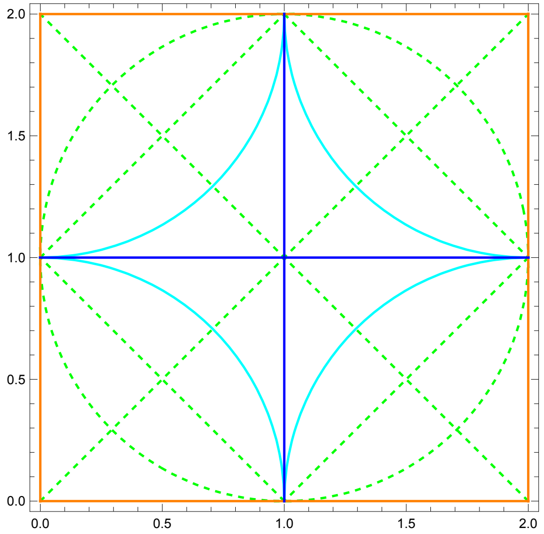

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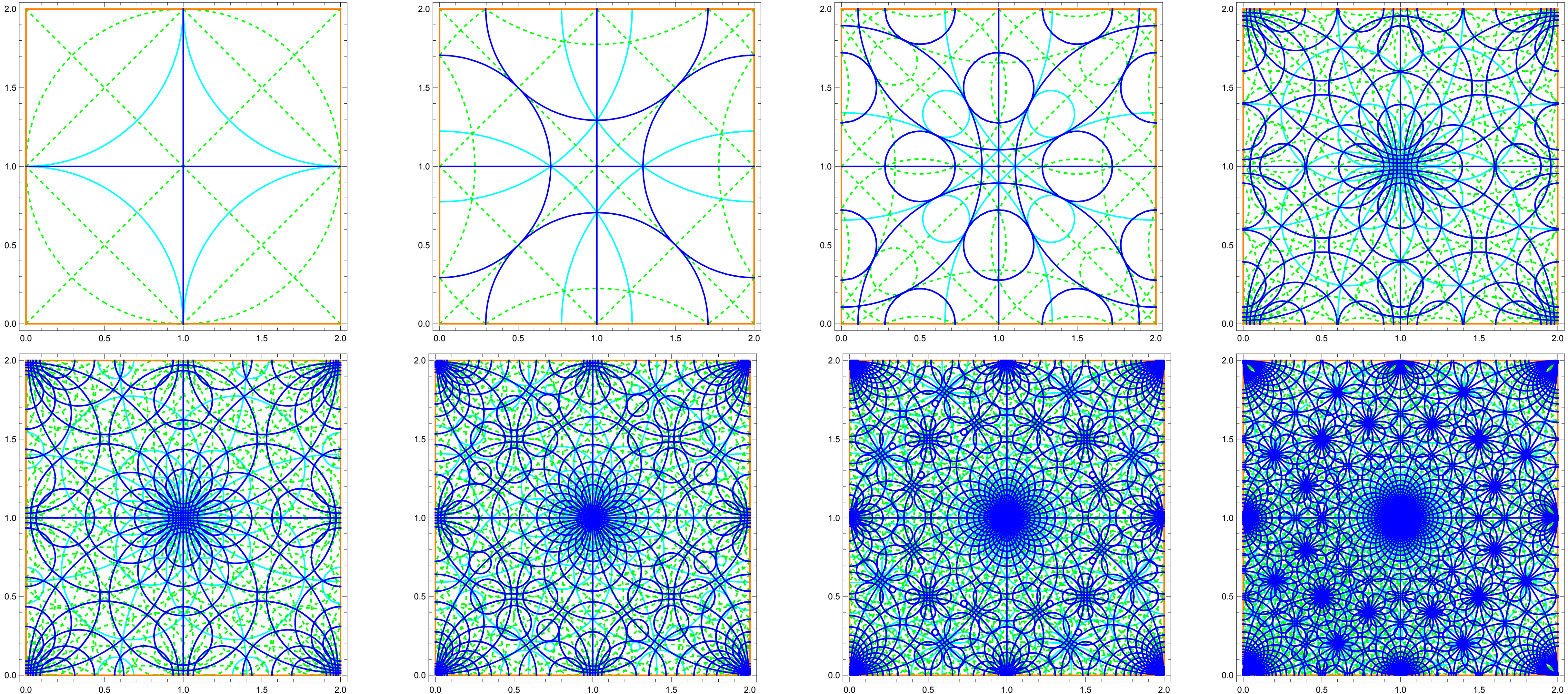

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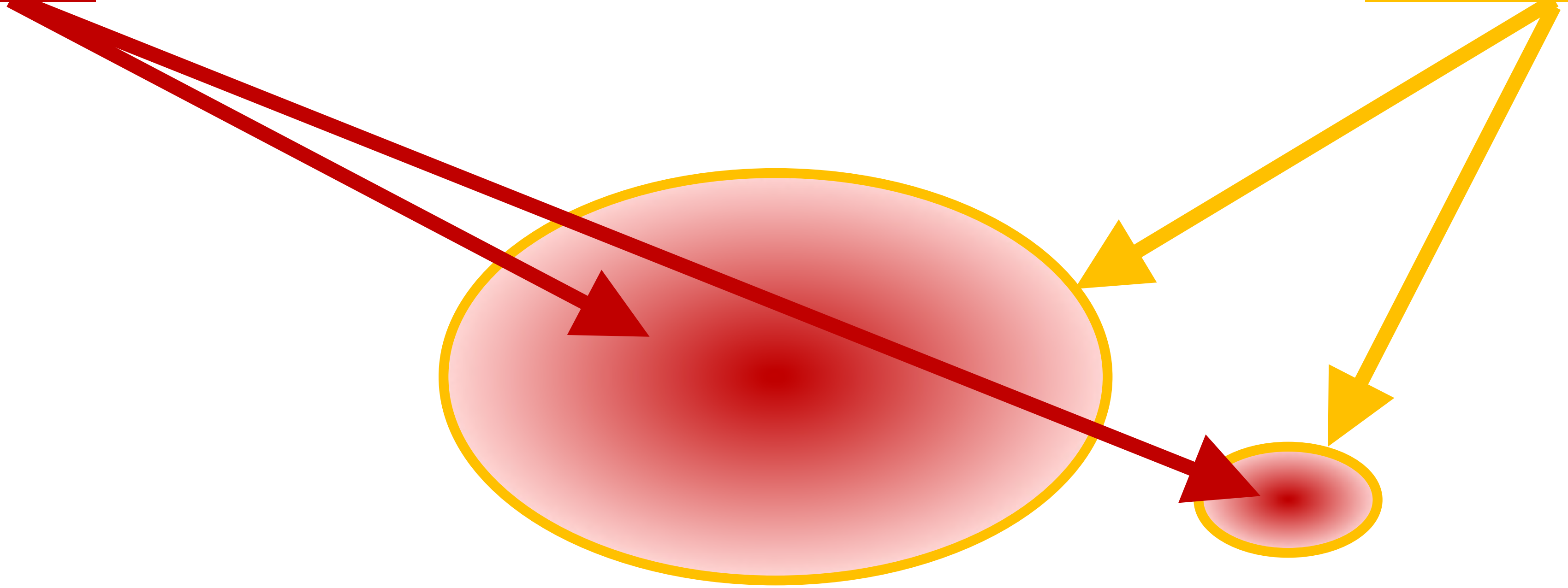
Tachyons appear inside hyper-ellipses defined by regions with **massless scalars**!



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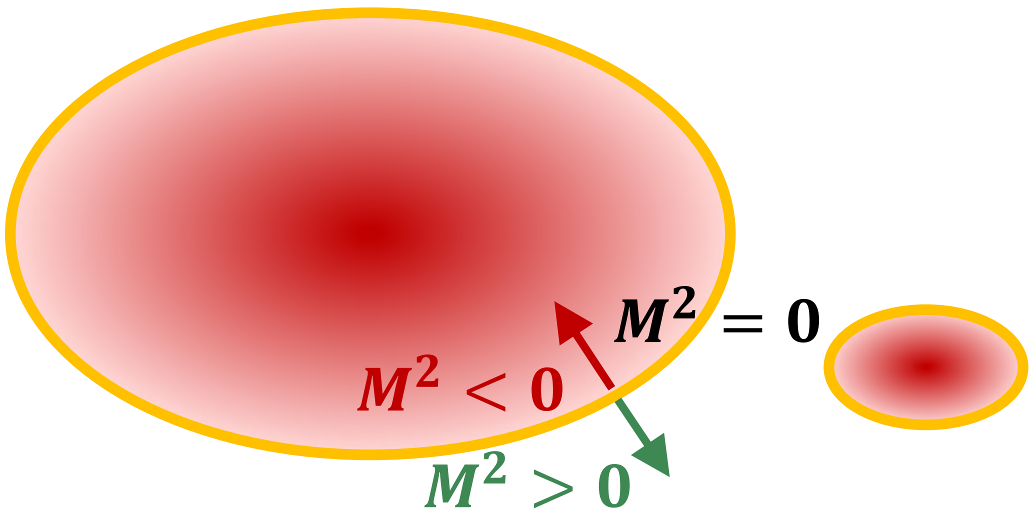

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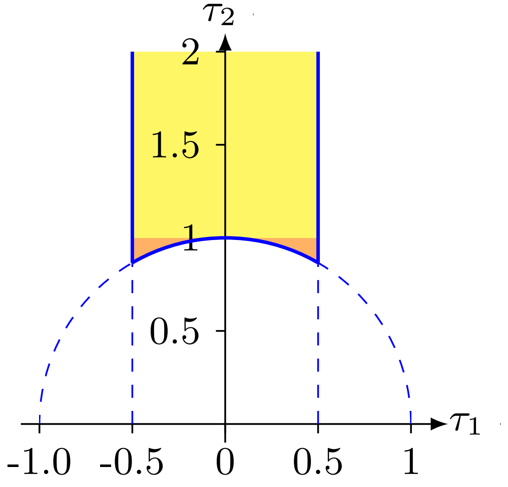
$$\boldsymbol{\Lambda}_{1-\text{loop}}(\mathrm{R},\mathrm{A}) = \int \frac{d^2\tau}{\tau_2^2}\boldsymbol{Z}(\tau,R,A)$$

$$\boldsymbol{Z} \sim \sum q^{p_L^2}\bar{q}^{p_R^2}$$



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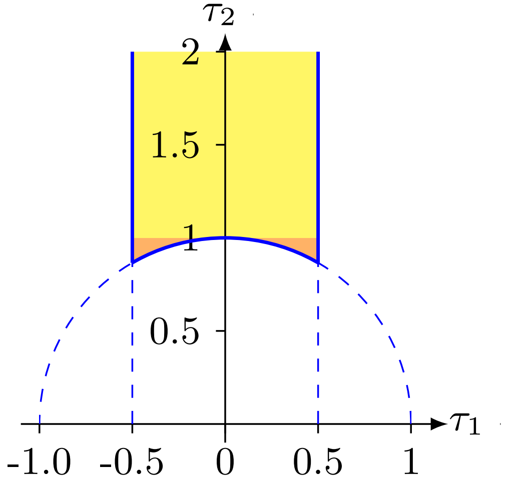

Massless gauge bosons and **fermions** have $p_L^2 + p_R^2 = 2$ → Finite contribution to Λ

Massless scalars have $p_L^2 + p_R^2 = 3$ → Finite contribution to Λ

Tachyons have $p_L^2 + p_R^2 < 3$ → **Infinite** contribution to Λ

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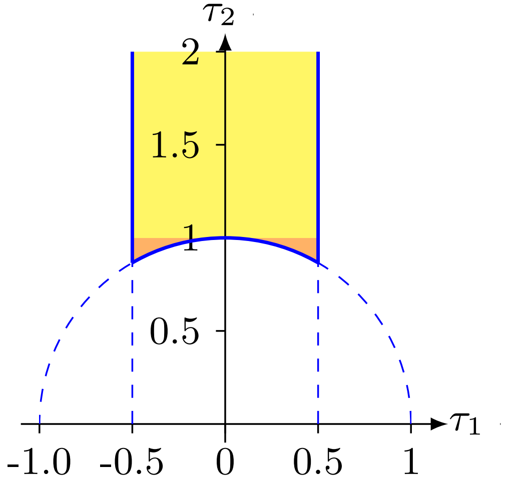


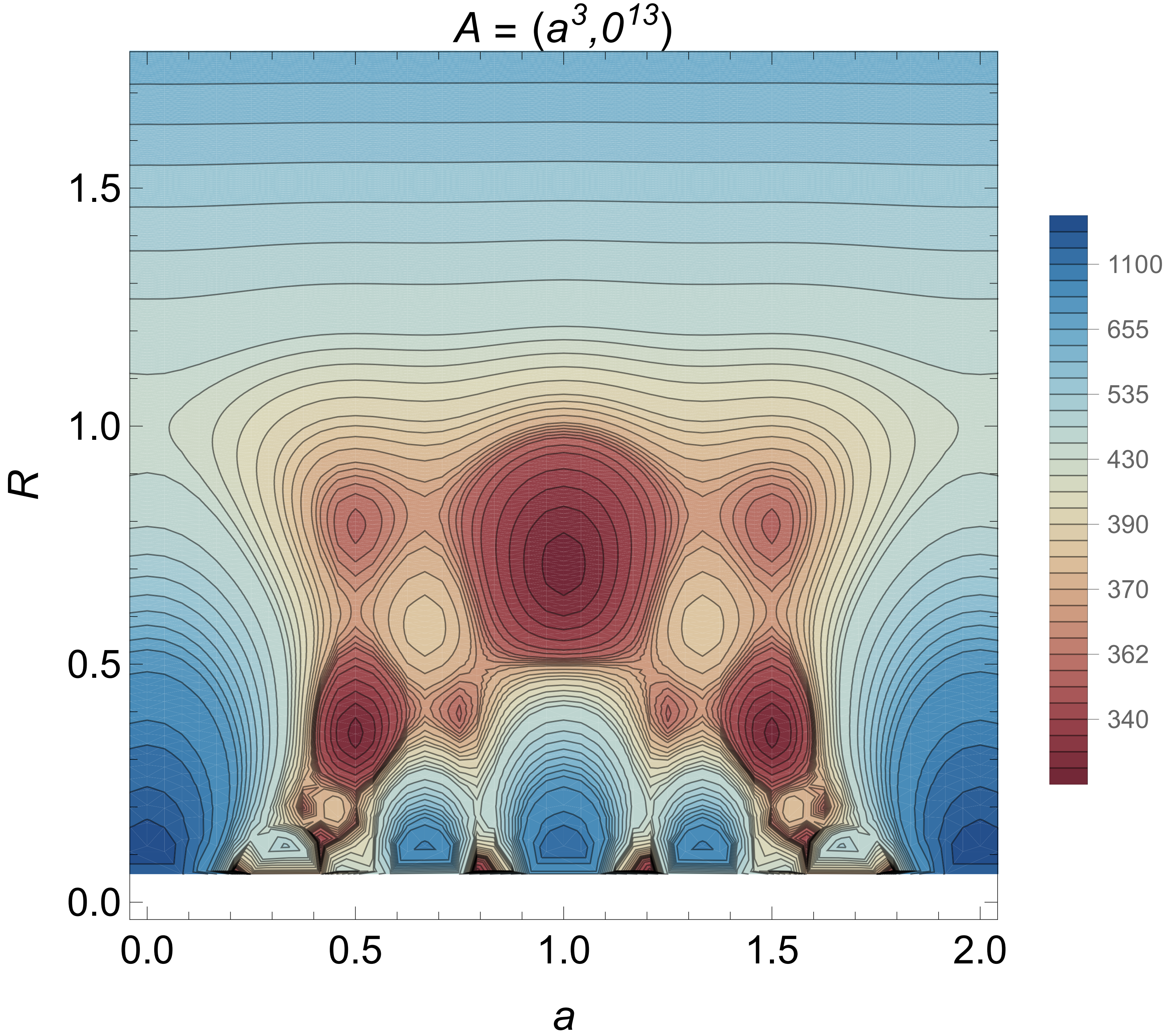
Massless gauge bosons and **fermions** have $p_L^2 + p_R^2 = 2$ → Finite contribution to $\boldsymbol{\Lambda}$

Massless scalars have $p_L^2 + p_R^2 = 3$ → Finite contribution to $\boldsymbol{\Lambda}$

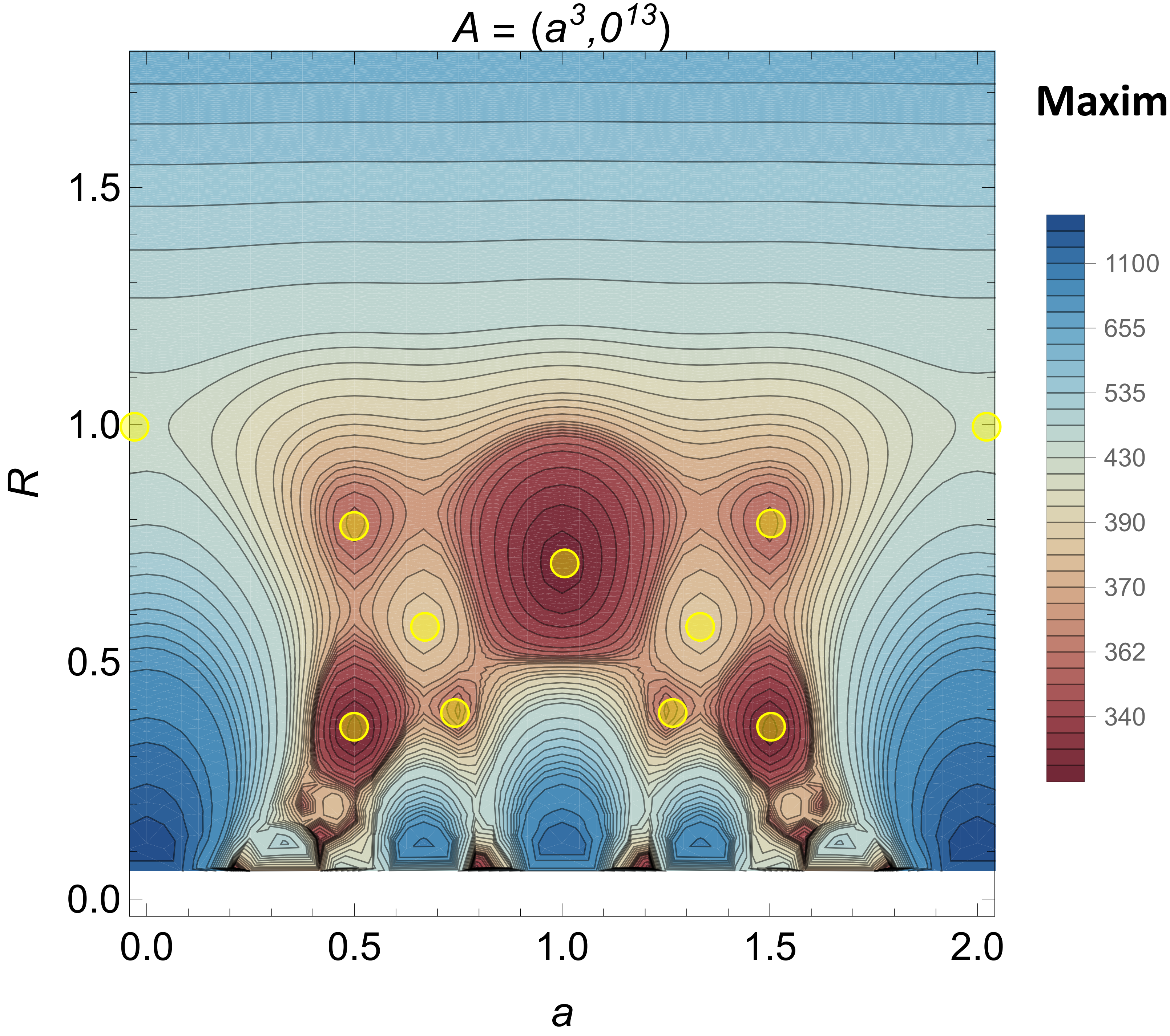
Tachyons have $p_L^2 + p_R^2 < 3$ → **Infinite** contribution to $\boldsymbol{\Lambda}$

Massive (and "off-shell") **states** may have a big contribution.

We must consider all states to compute $\boldsymbol{\Lambda}$



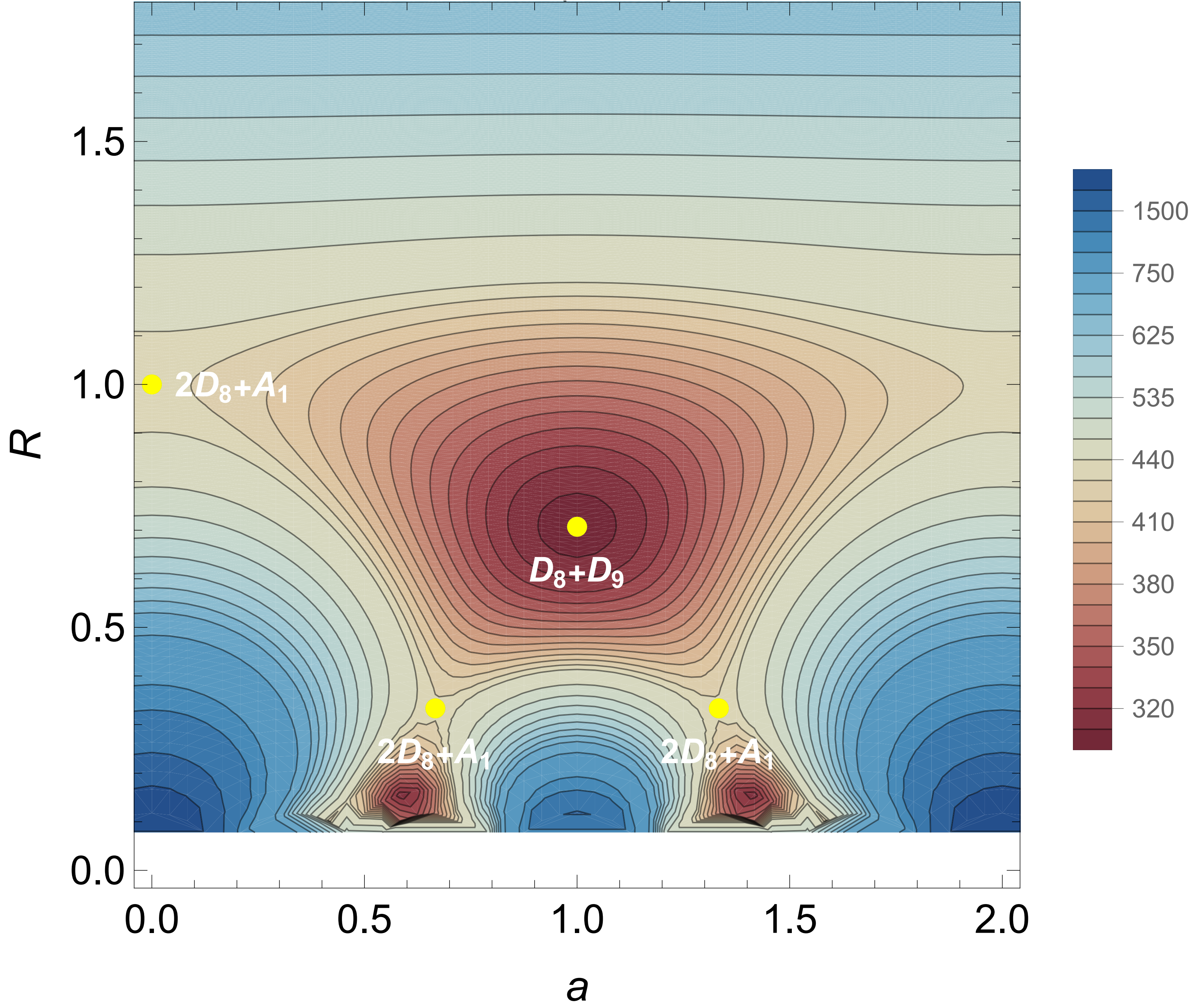
$A = (a^3, 0^{13})$
1.5
1.0
0.5
0.0
R
0.0
0.5
1.0
1.5
2.0
a
1100
655
535
430
390
370
362
340



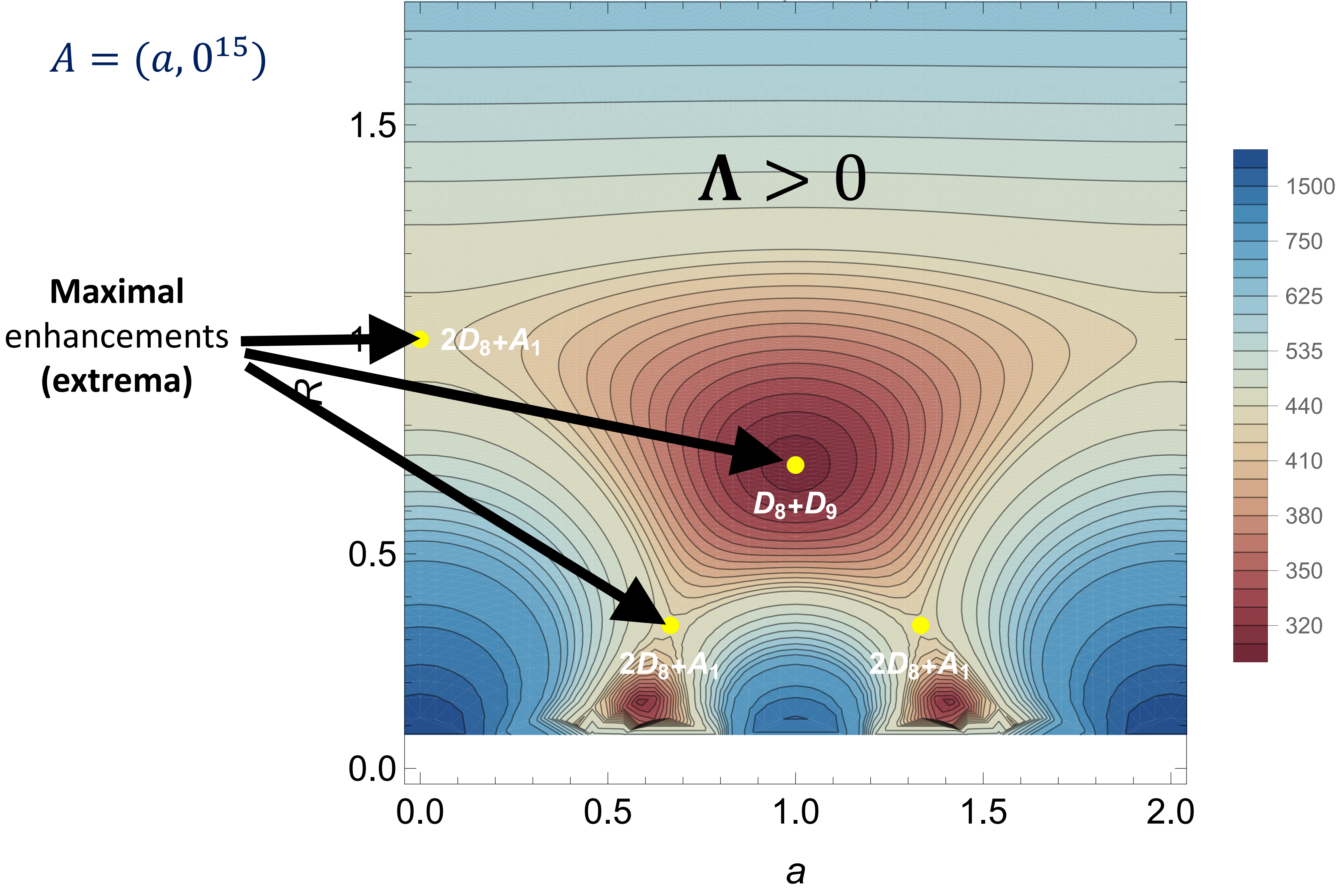
Maximal enhancement ⇒ **extremum** of $\boldsymbol{\Lambda}$

[Ginsparg, Vafa '87]

$A = (a, 0^{15})$

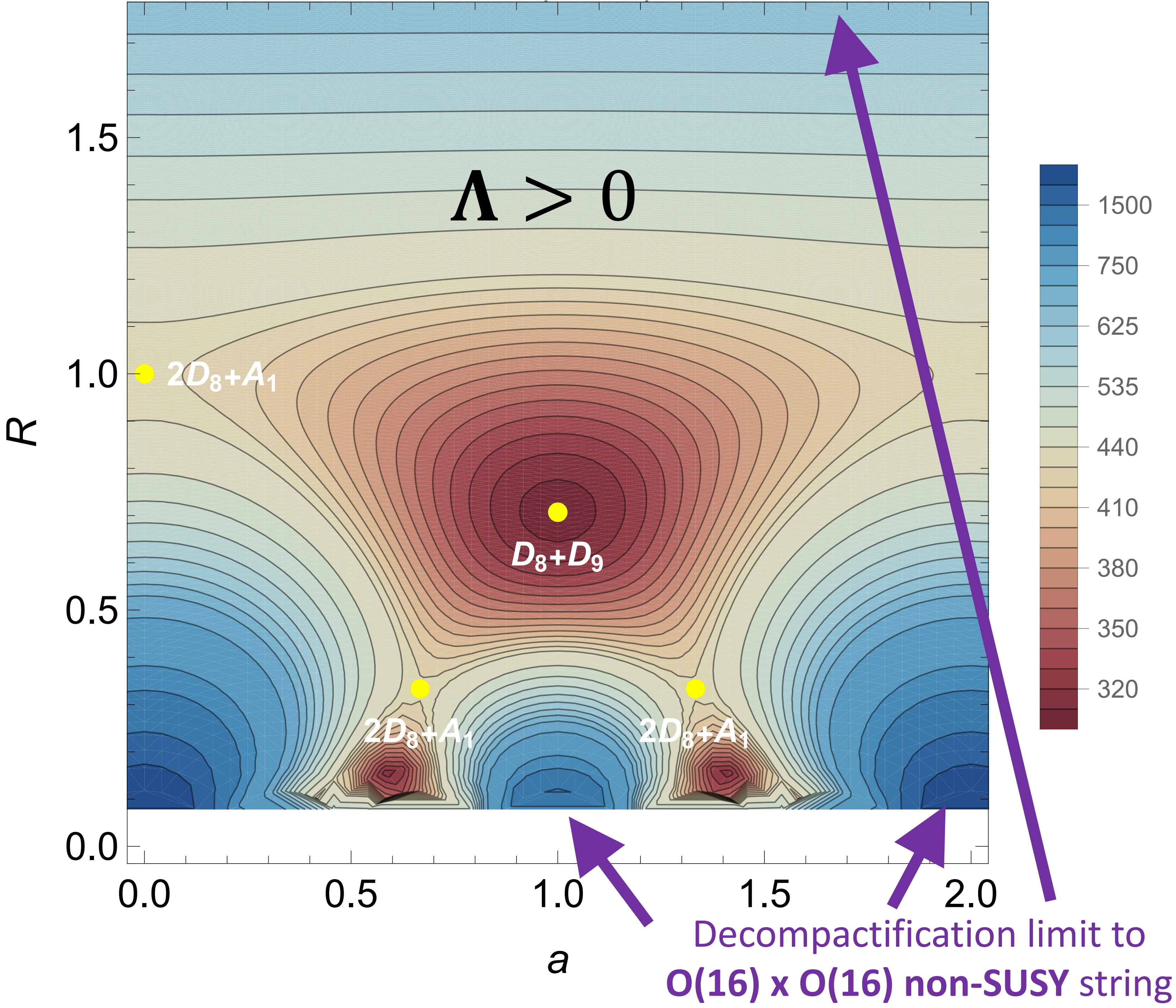


1.5
1.0
0.5
0.0
R
0.0
0.5
1.0
1.5
2.0
a
2D8+A1
D8+D9
2D8+A1
2D8+A1
1500
750
625
535
440
410
380
350
320



$A = (a, 0^{15})$
$\Lambda > 0$
Maximal enhancements (extrema)
$2D_8+A_1$
D_8+D_9
$2D_8+A_1$
$2D_8+A_1$
1.5
1
0.5
0.0
R
0.0
0.5
1.0
1.5
2.0
a
1500
750
625
535
440
410
380
350
320

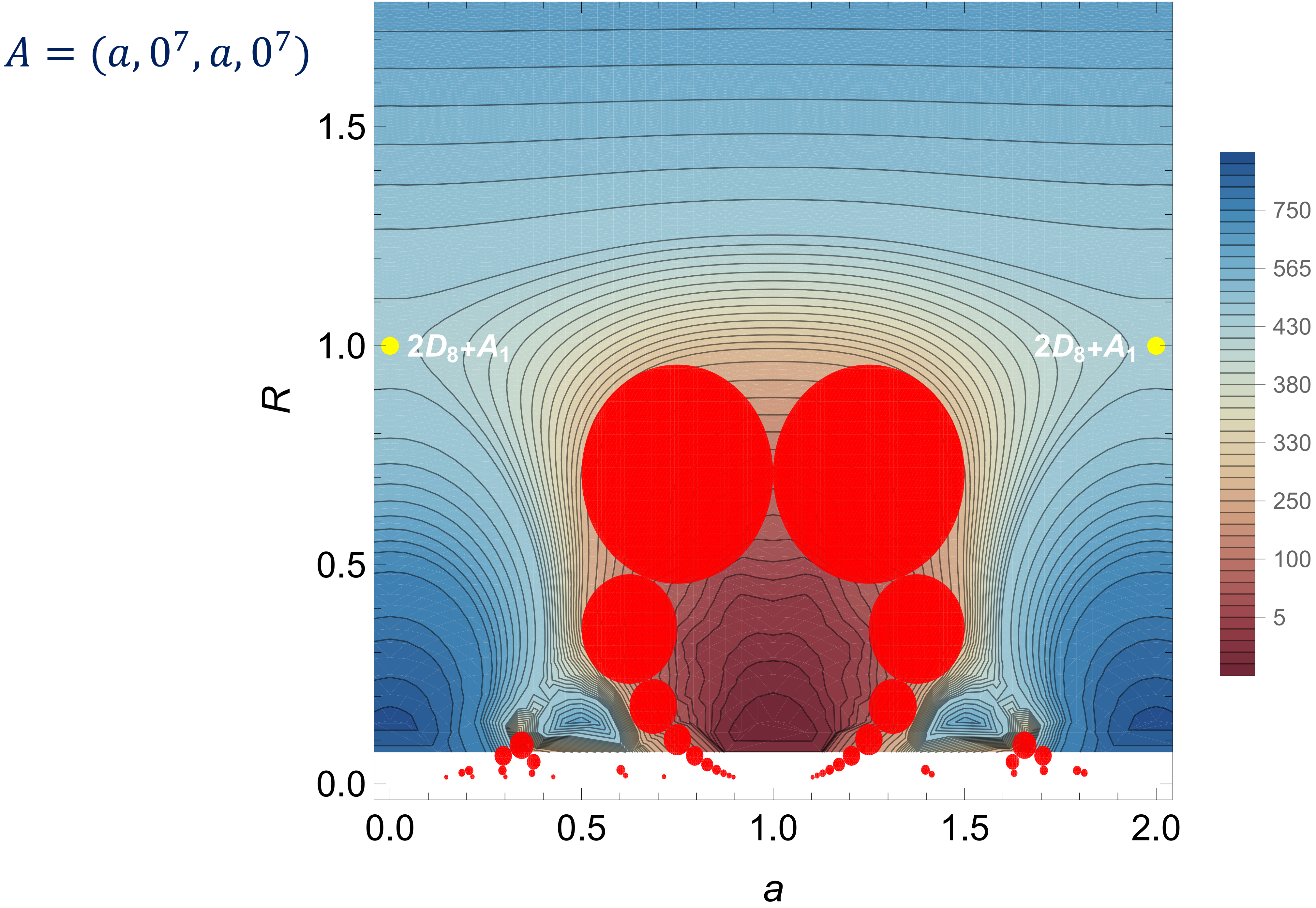
$A = (a, 0^{15})$



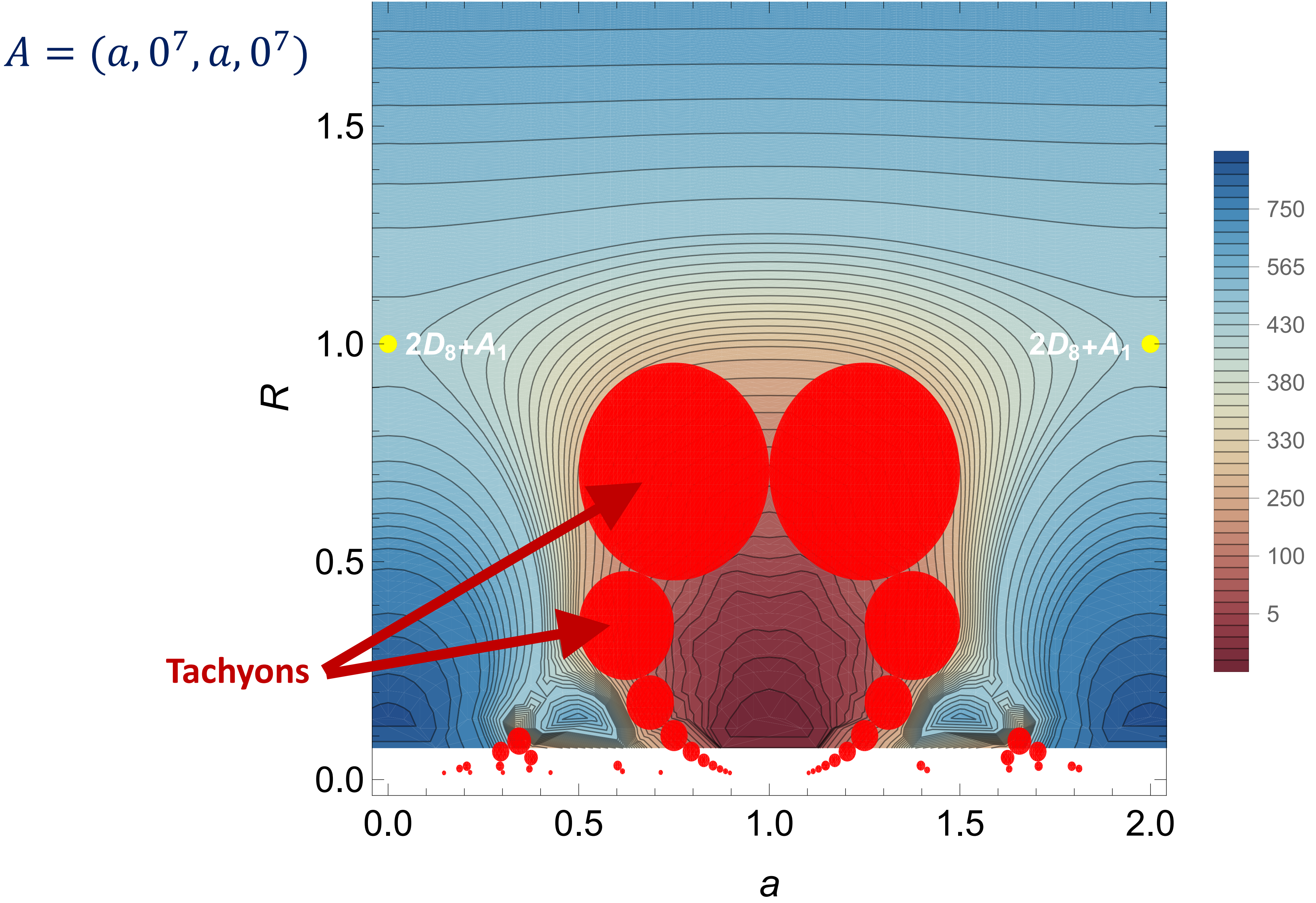
SUSY case:

[V. Collazuol, M. Graña, A. Herráez '22]

[see V. Collazuol's talk]

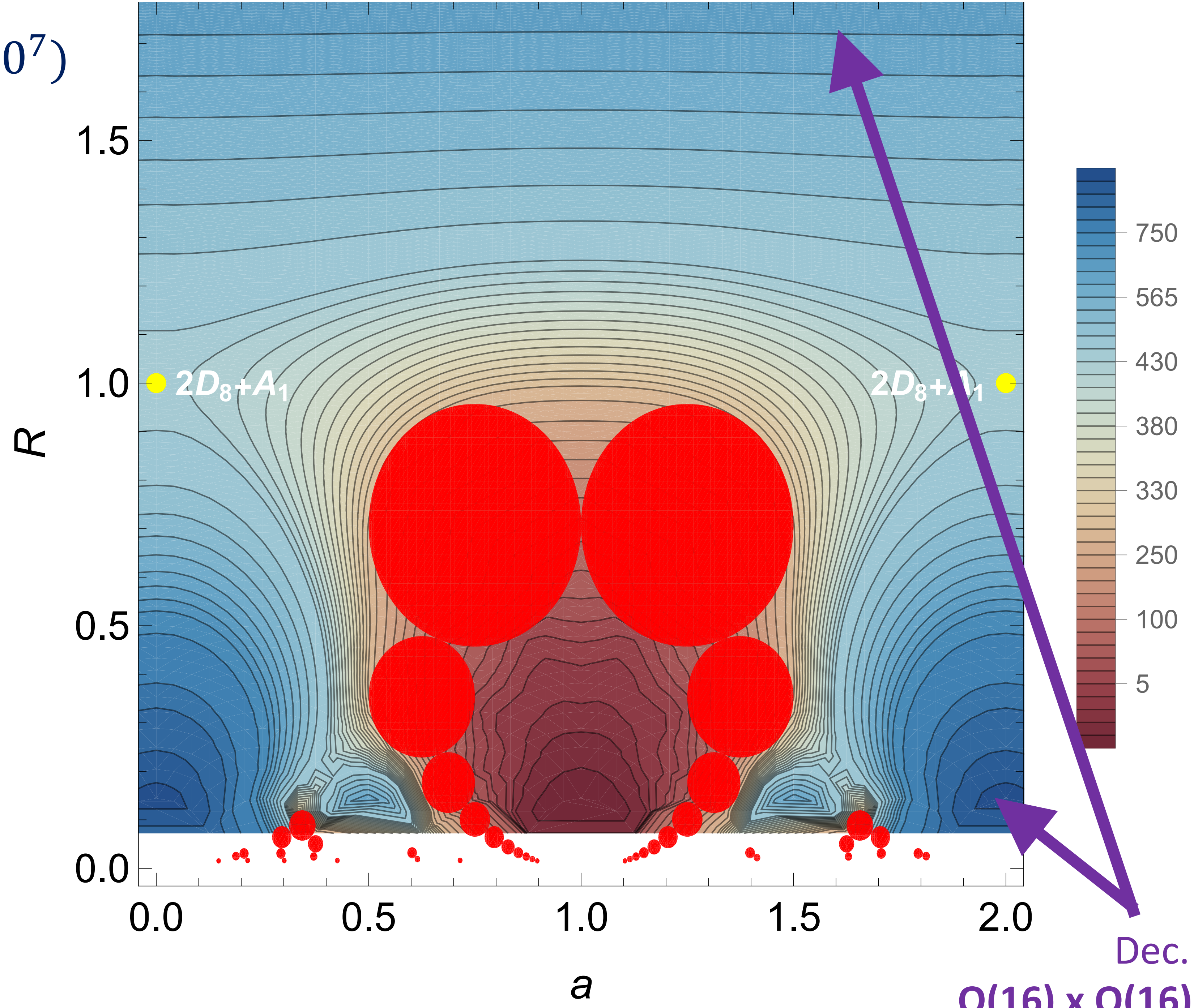


$A = (a, 0^7, a, 0^7)$
R
a
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
2D8+A1
2D8+A1
750
565
430
380
330
250
100
5



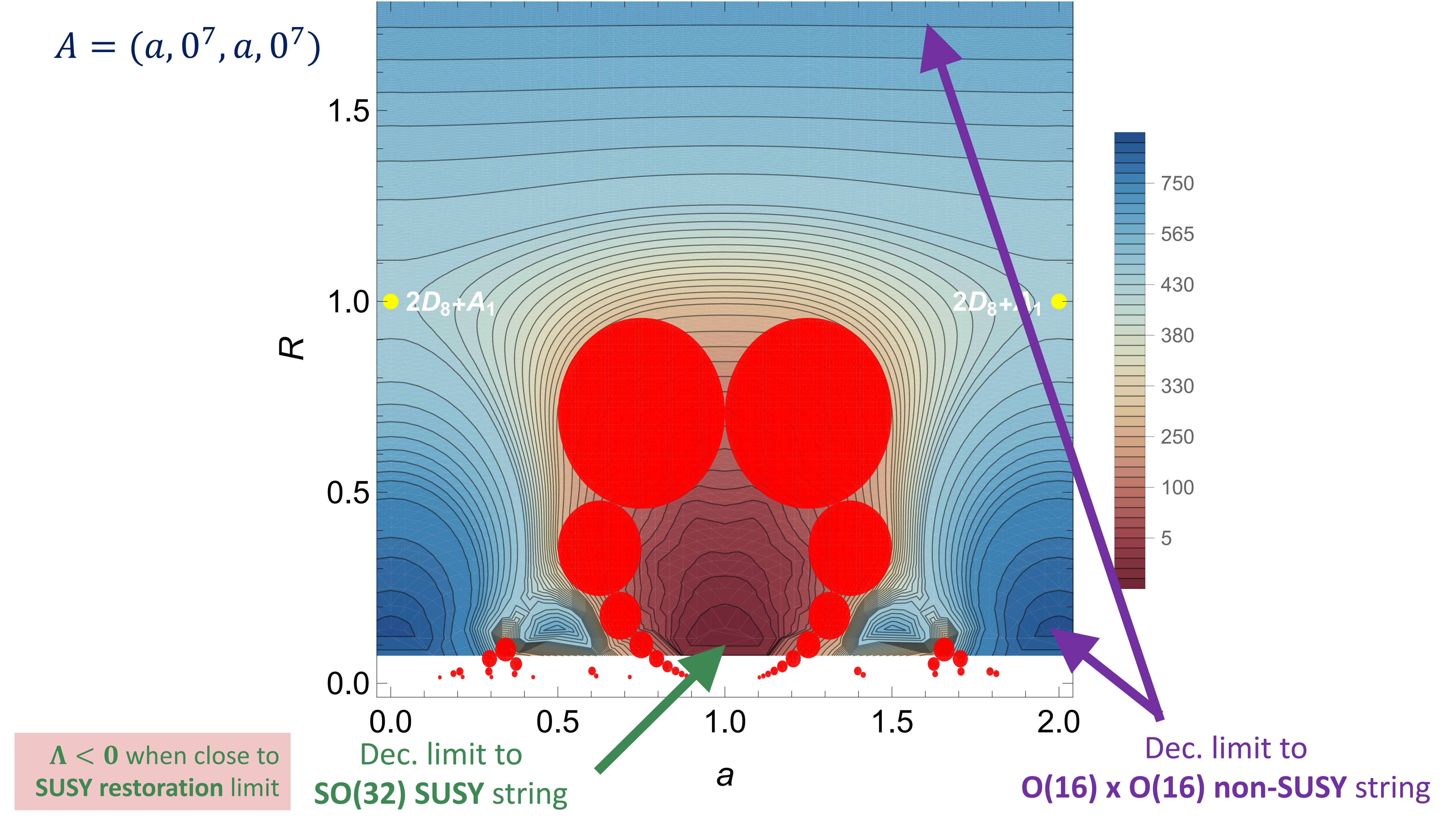
$A = (a, 0^7, a, 0^7)$
1.5
1.0
0.5
0.0
R
2D_8+A_1
2D_8+A_1
Tachyons
0.0
0.5
1.0
1.5
2.0
a
750
565
430
380
330
250
100
5

$A = (a, 0^7, a, 0^7)$



Dec. limit to

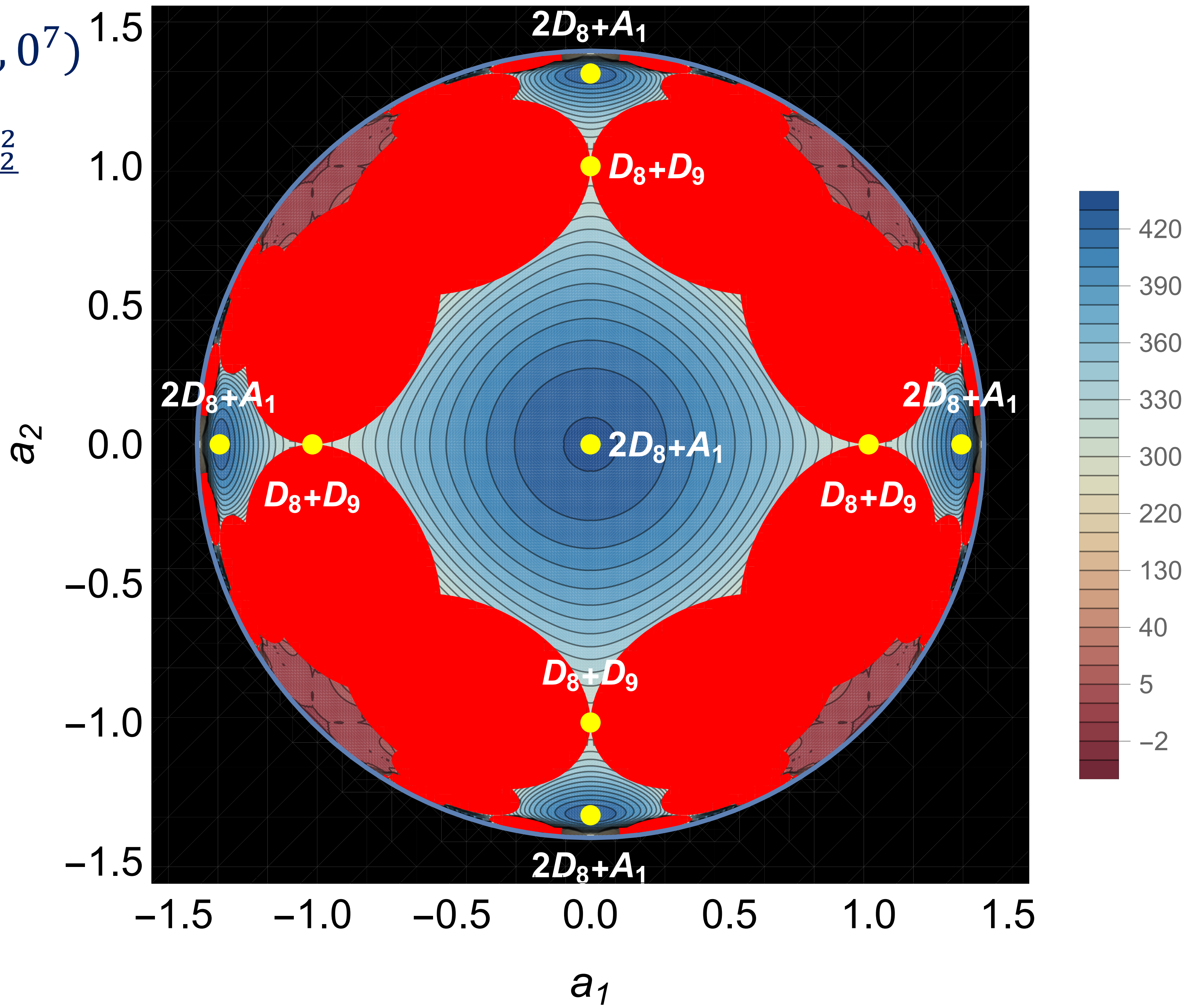
O(16) x O(16) non-SUSY string



$A = (a, 0^7, a, 0^7)$
R
a
1.5
1.0
0.5
0.0
0.0
0.5
1.0
1.5
2.0
750
565
430
380
330
250
100
5
$2D_8+A_1$
$2D_8+A_1$
$\Lambda < 0$ when close to
SUSY restoration limit
Dec. limit to
SO(32) SUSY string
Dec. limit to
O(16) x O(16) non-SUSY string

$$A = (a_1, 0^7, a_2, 0^7)$$

$$R^2 = 1 - \frac{a_1^2 + a_2^2}{2}$$

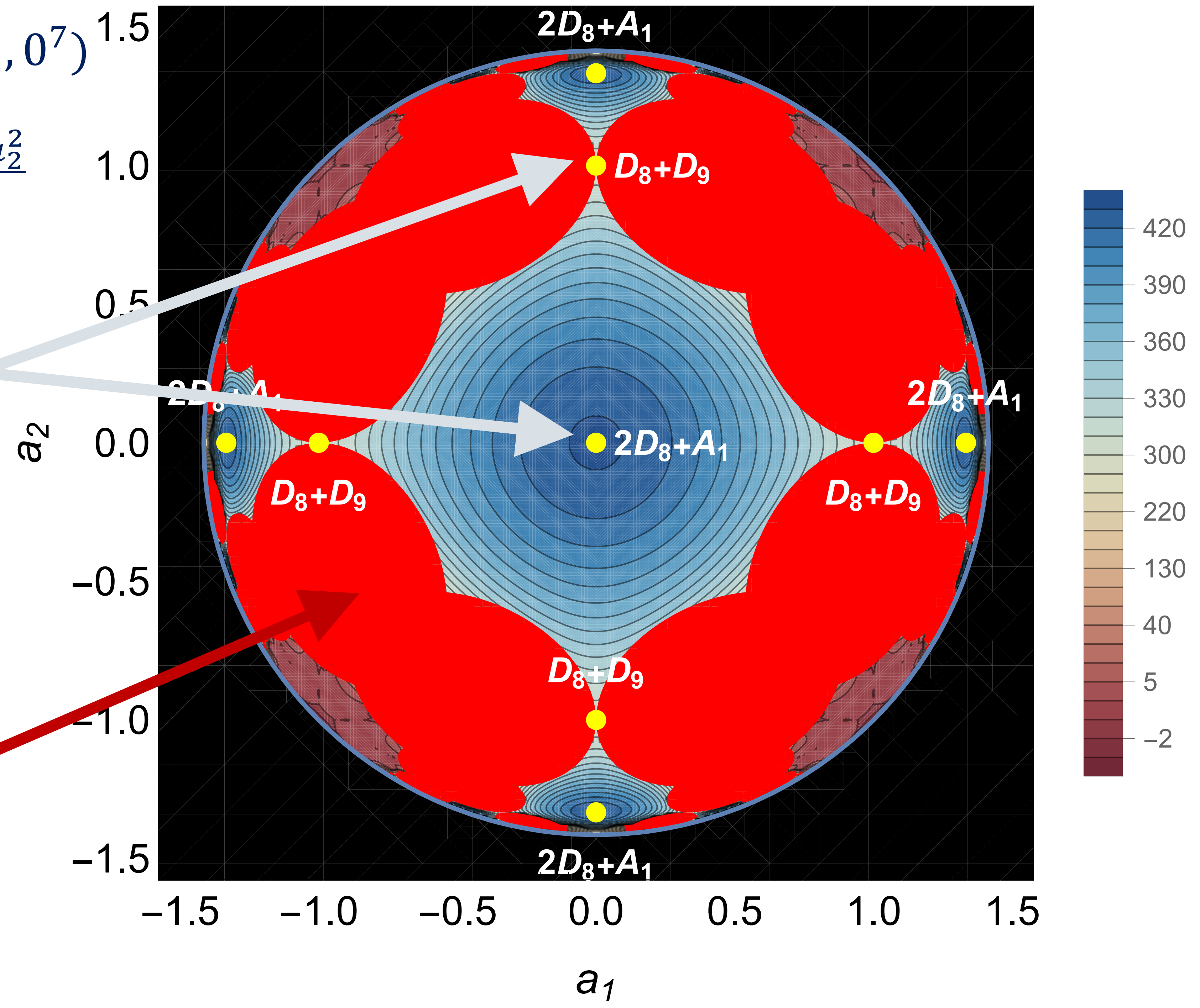


$A = (a_1, 0^7, a_2, 0^7)$

$R^2 = 1 - \frac{a_1^2 + a_2^2}{2}$

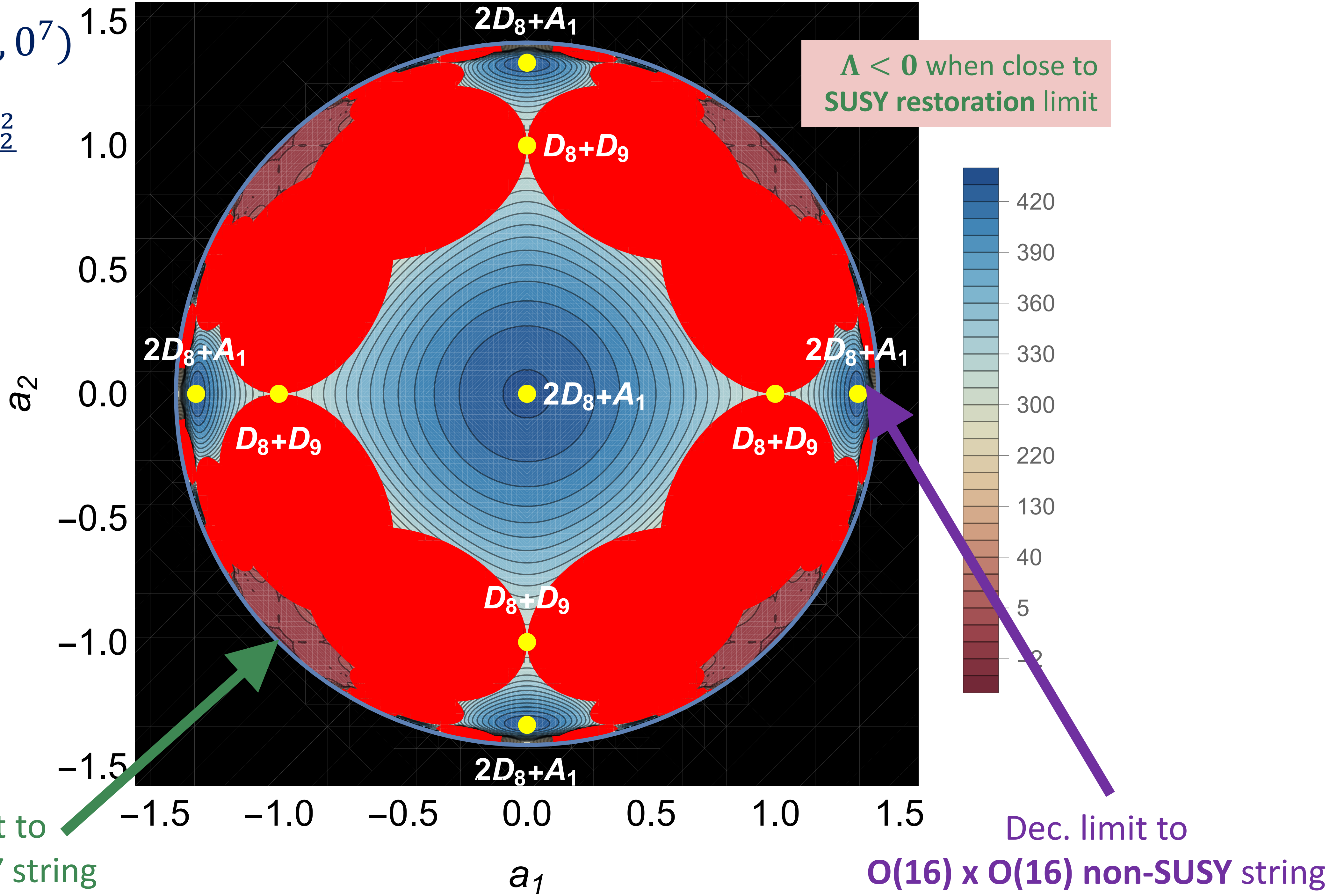
Maximal enhancements **(extrema)**

Tachyons

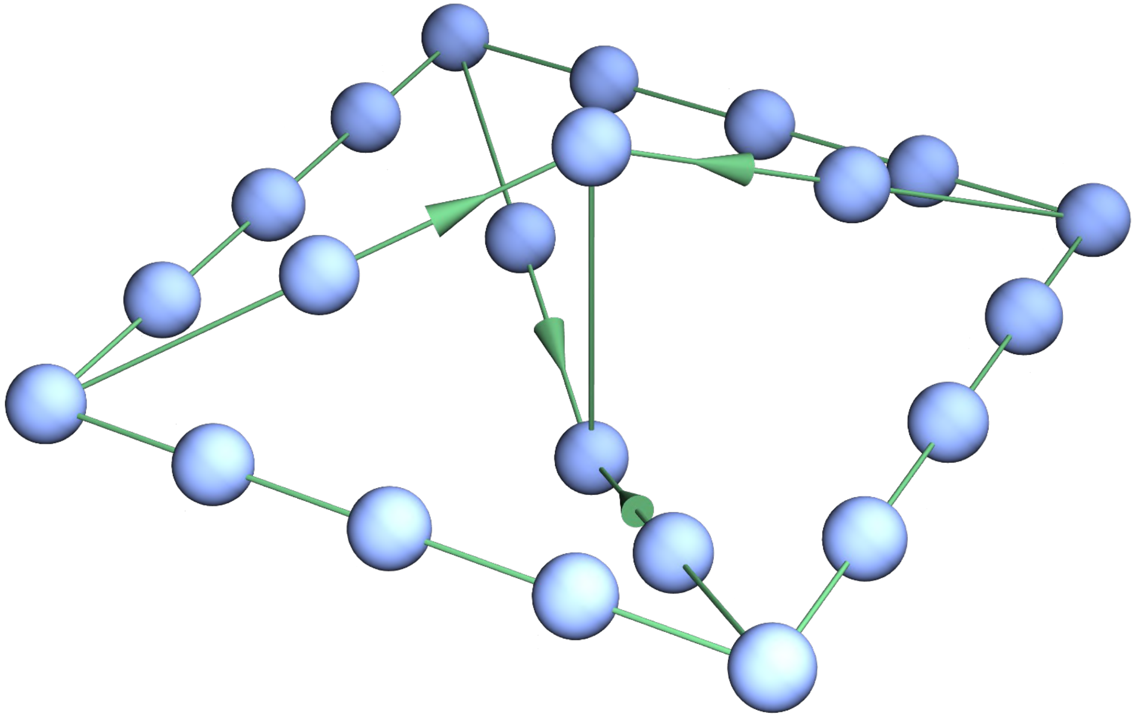


$A = (a_1, 0^7, a_2, 0^7)$

$R^2 = 1 - \frac{a_1^2 + a_2^2}{2}$



Extended Dynkin Diagram



107 maximal enhancements

[BF, M. Graña, H. P. de Freitas, S. Sethi WIP]

Most maximal enhancement points are **tachyonic**, but there are **8** free of tachyons

Are their Λ's **positive**, **negative**? **minima**? **maxima**? **saddle points**? Let's find out!

Extrema of the cosmological constant:

We compute $\boldsymbol{\Lambda}$ and its **Hessian** for the 8 non-tachyonic maximal enhancements:

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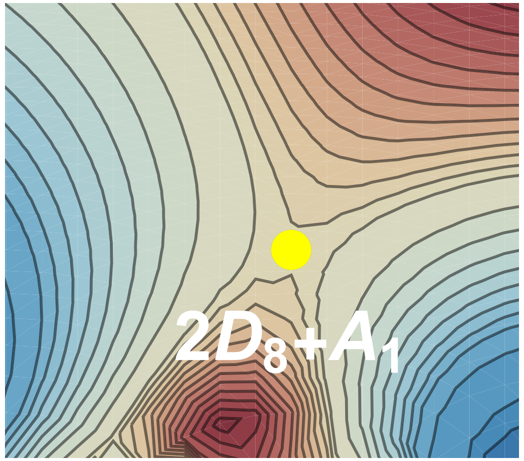
$$SO(16)\times SO(16)\times SU(2)\ \boldsymbol{\Lambda} = \mathbf{431.35}$$

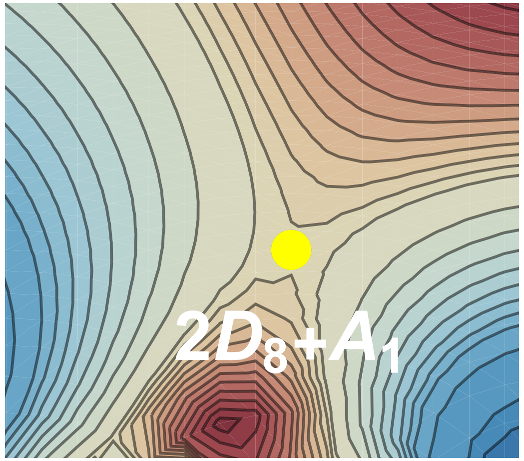
$$SO(16)\times SO(12)\times SU(3)\times SU(2)\ \boldsymbol{\Lambda} = \mathbf{384.51}$$

$$SO(16)\times SO(10)\times SU(5)\ \boldsymbol{\Lambda} = \mathbf{359.20}$$

Saddle points



Extrema of the cosmological constant:



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} **Saddle points**

$SO(10)\times SO(10)\times SU(8)\ \boldsymbol{\Lambda}=\mathbf{303.78}$ —> (local **maximum**)

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$SO(16)\times SO(10)\times SU(5)\ \boldsymbol{\Lambda} = \mathbf{359.20}$

→ **Saddle points**

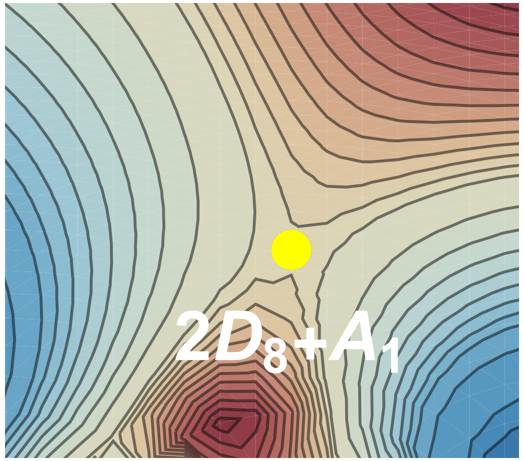
$SO(10)\times SO(10)\times SU(8)\ \boldsymbol{\Lambda} = \mathbf{303.78}$ —> (local **maximum**)

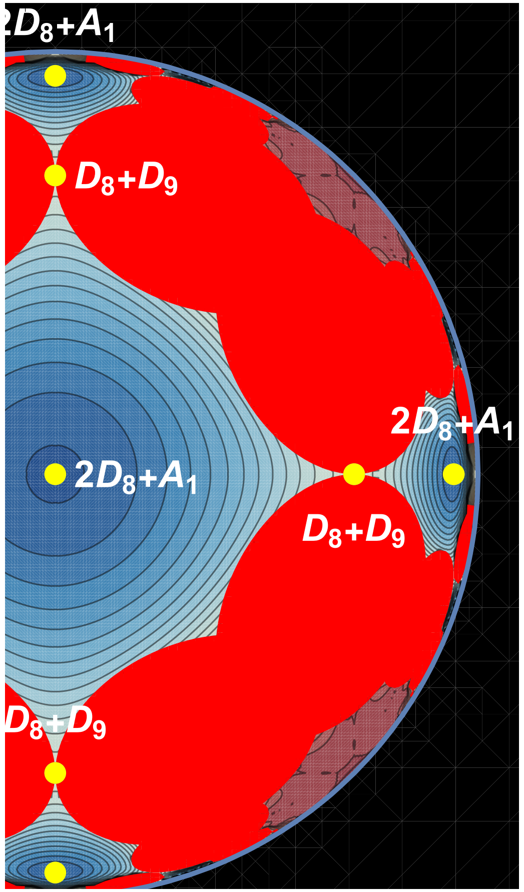
$SO(16)\times SO(18)\ \boldsymbol{\Lambda} = \mathbf{305.01}$

$SO(16)\times SO(10)\times SO(8)\ \boldsymbol{\Lambda} = \mathbf{305.01}$

$(SO(12)\times SU(2))^2\times SU(4)\ \ \boldsymbol{\Lambda} = \mathbf{305.01}$

→ **Knife-edges**





Extrema of the cosmological constant:

We compute $\boldsymbol{\Lambda}$ and its **Hessian** for the 8 non-tachyonic maximal enhancements:

$SO(16) \times SO(16) \times SU(2)\ \boldsymbol{\Lambda = 431.35}$

$SO(16) \times SO(12) \times SU(3) \times SU(2)\ \boldsymbol{\Lambda = 384.51}$

$SO(16) \times SO(10) \times SU(5)\ \boldsymbol{\Lambda = 359.20}$

→ **Saddle points**

$SO(10) \times SO(10) \times SU(8)\ \boldsymbol{\Lambda = 303.78}$ —> (local **maximum**)

$SO(16) \times SO(18)\ \boldsymbol{\Lambda = 305.01}$

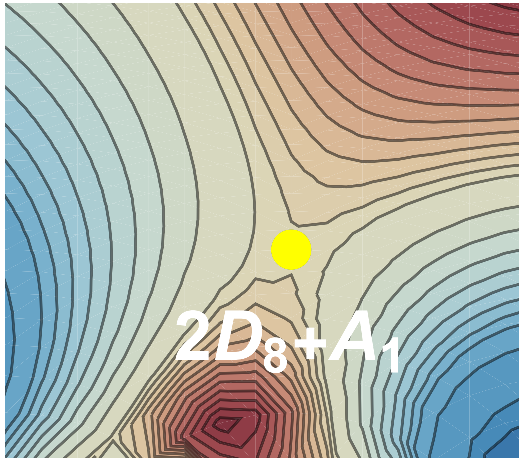
$SO(16) \times SO(10) \times SO(8)\ \boldsymbol{\Lambda = 305.01}$

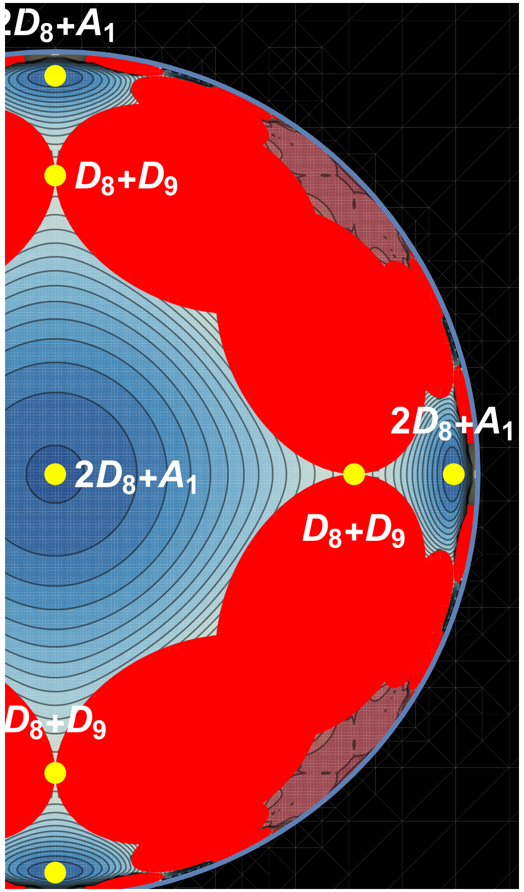
$(SO(12) \times SU(2))^2 \times SU(4)\ \ \boldsymbol{\Lambda = 305.01}$

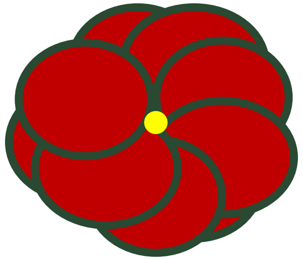
Completely surrounded by **tachyons**:

$E_6 \times SU(12) \times \boldsymbol{SU(2)_R}\ \ \boldsymbol{\Lambda = 180.43}$

→ **Knife-edges**







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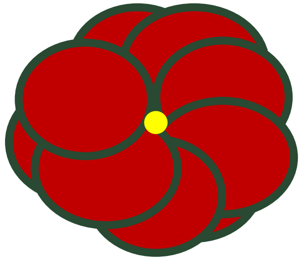
$(SO(12)\times SU(2))^2\times SU(4)$ $\boldsymbol{\Lambda = 305.01}$

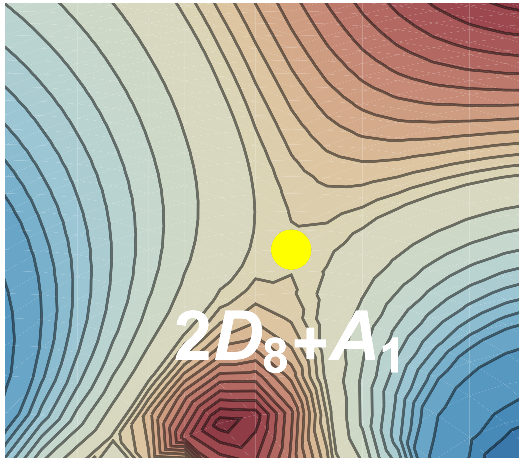
Completely surrounded by **tachyons**:

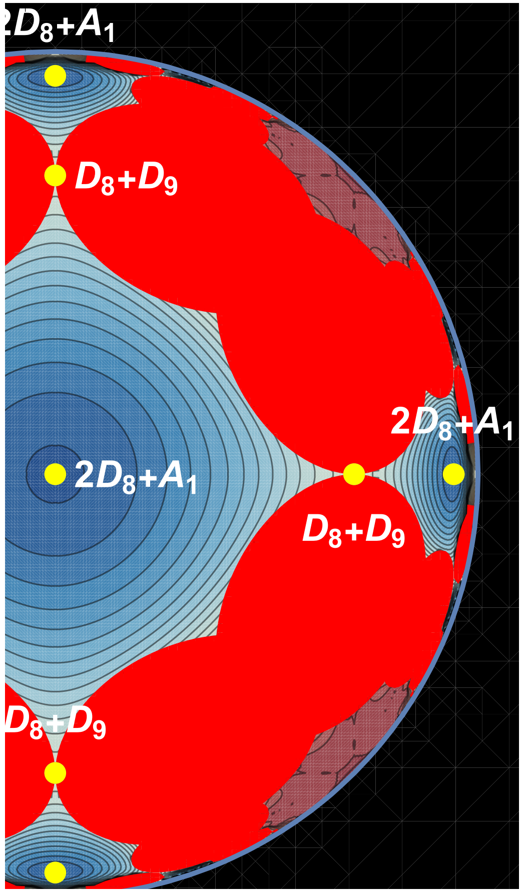
$E_6\times SU(12)\times \boldsymbol{SU(2)_R}$ $\boldsymbol{\Lambda = 180.43}$

} **Knife-edges**



No local **minima.**

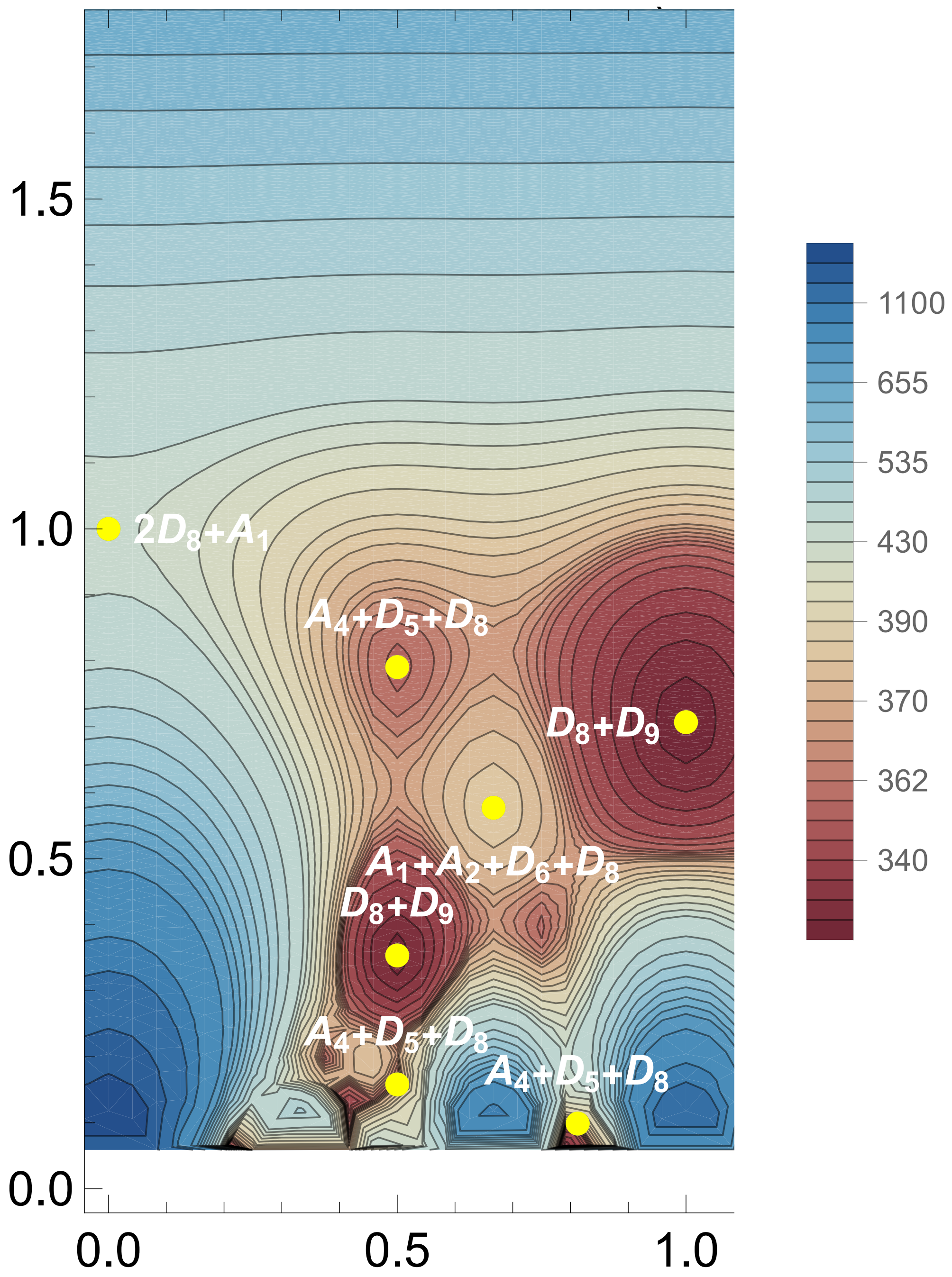




Maximal enhancement ⇒ **EXTREMA of Λ**

But there are also **extremal** points with **non-maximal** enhancement

$A = (a, a, a, 0^{13})$



Maximal enhancement ⇒ **EXTREMA of Λ**

But there are also **extremal** points with **non-maximal** enhancement

$SO(16)\times SO(10)\times SU(4)\times U(1)$ $\quad \boldsymbol{\Lambda = 367.15}$

$SO(16)\times SO(8)\times SU(5)\times U(1)$ $\quad \boldsymbol{\Lambda = 359.20}$

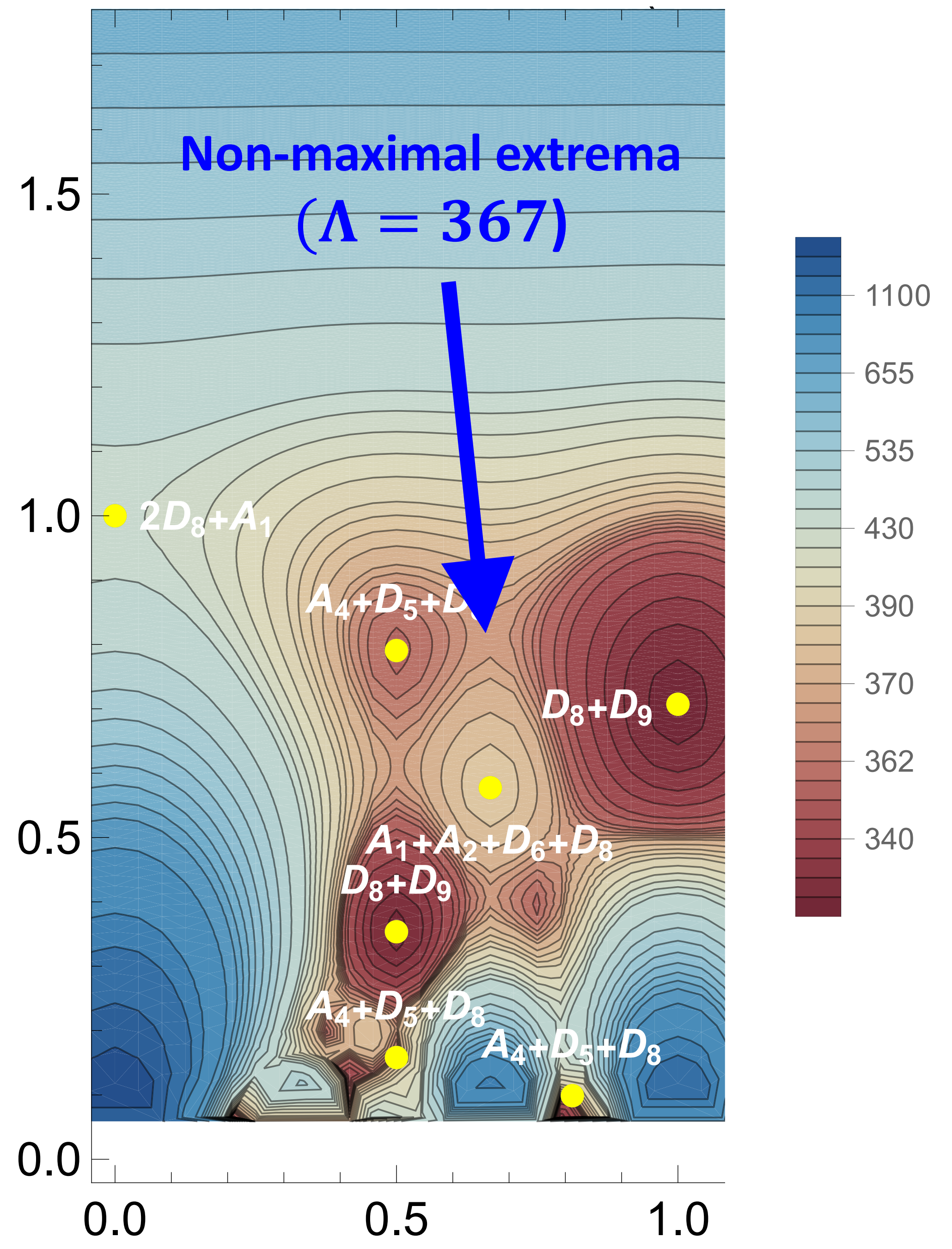
$SO(8)^4\times U(1)$ $\quad \boldsymbol{\Lambda = 305.01}$

$SU(8)^2\times SU(2)^2\times U(1)$ $\quad \boldsymbol{\Lambda = 305.01}$

(all saddle points or knife-edges)

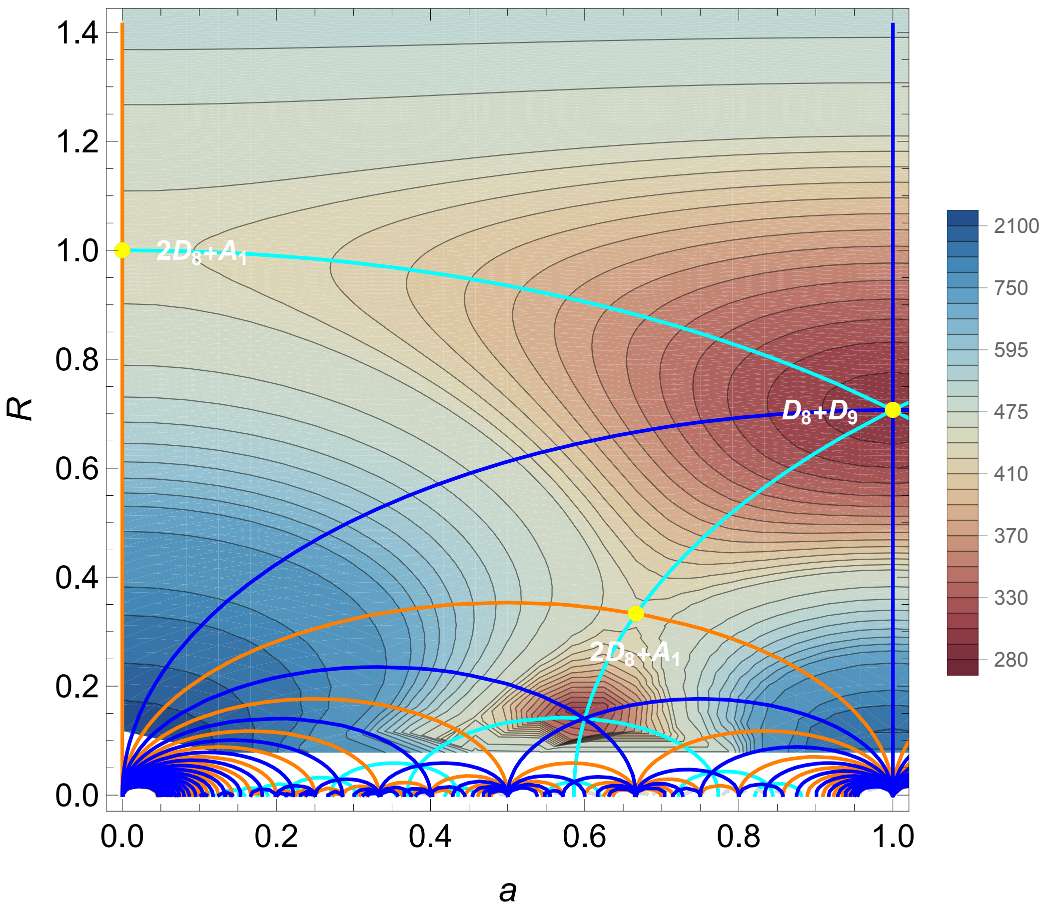
These are special points were extra **massless spinors** and/or **scalars** appear

$A = (a, a, a, 0^{13})$

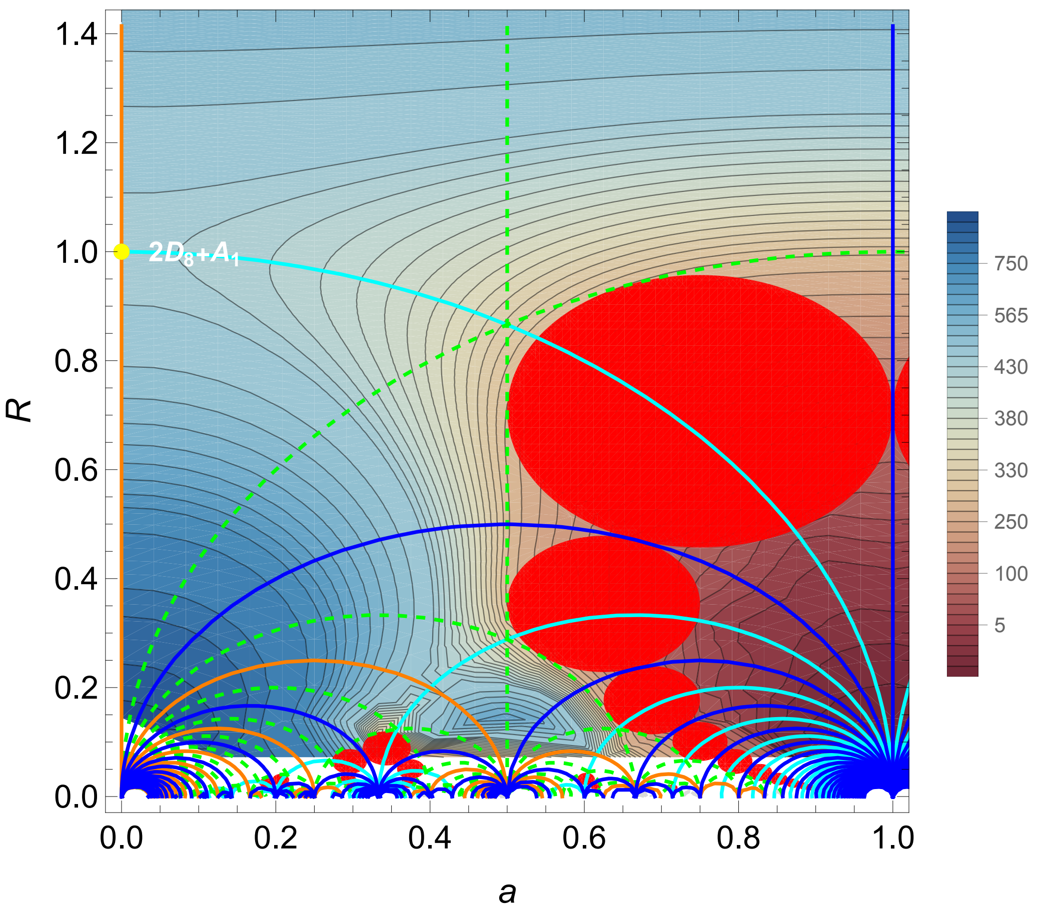


It is interesting to overlap the curves of **symmetry enhancement** and the **profile** of $\boldsymbol{\Lambda}$...

$A = (a, 0^{15})$



$A = (a, 0^7, a, 0^7)$



Massless bosons

Massless bosons + spinors

Massless spinors

Tachyon-free **maximal enhancements** connected trough these curves avoiding tachyons except for $E_6 \times SU(12) \times \boldsymbol{SU(2)_R}$

Conclusions

- No local **minimum** of Λ
- From the EDD we get the **fundamental region** for the **O(16)xO(16) circle compactification** moduli space.
- **107** maximal enhancements, **8** have finite extremal values of $\boldsymbol{\Lambda}$:
  - **1** local maximum.
  - **3** saddle points.
  - **4** knife-edges
- Seems that $\boldsymbol{\Lambda > 0}$ everywhere except close to **SUSY restoration** limits.
- **4** non-maximal points extremizing $\boldsymbol{\Lambda}$

Future work

- Some enhancement curves give **interpolations** between all **10D** theories at **infinite distance limits.**
  [see Koga's talk]

- Compactifying to **less** space-time **dimensions** → **more extrema!**

- These compactifications can be used to construct $\mathbf{AdS_3}$ **vacua!** [Baykara, Robbins, Sethi '22]

- Same analysis could be done for **reduced rank** non-SUSY theories.
  [Nakajima '23], [see H. P. de Freitas talk]

감사합니다!

Maximal enhancements

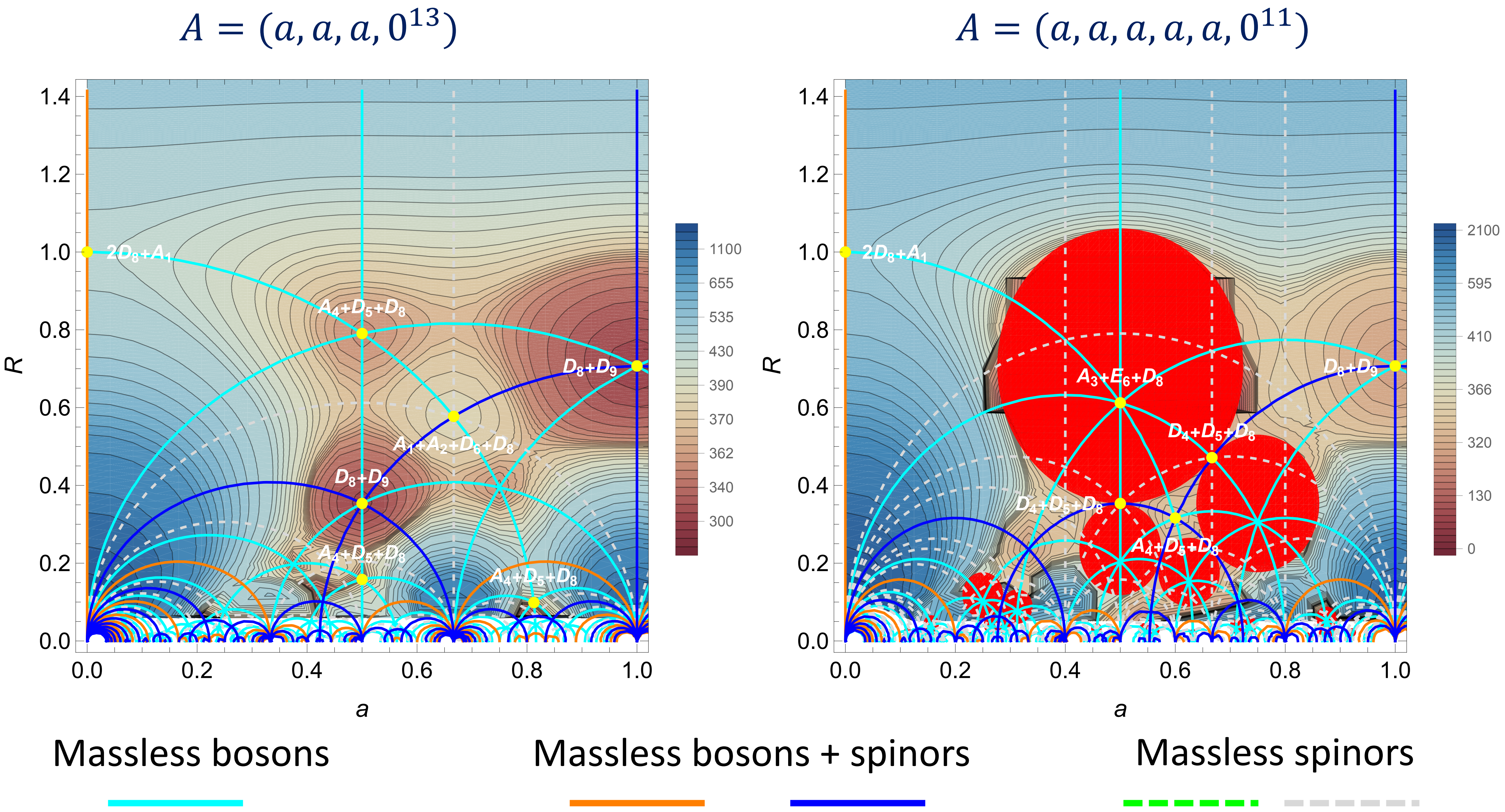
WL	v	s	c	0
(0^{16})	$[A_1+2D_8\,;\mathbb{Z}_2]$	$(1,128,1)$ $(1,1,128)$	$(1,16,16)$	none
$\left(\frac{1}{2}^2,0^{14}\right)$	$[A_1+A_2+D_6+D_8\,;\mathbb{Z}_2]$	$(2,1,32,1)$ $(1,1,1,128)$	(1,1,12,16)	none
$\left(\frac{1}{2}^3,0^{13}\right)$	$A_4+D_5+D_8$	(1,1,128)	(1,10,16)	none
$\left(\frac{1}{2}^3,0^5,\frac{1}{2}^3,0^5\right)$	$[A_7+2D_5\,;\mathbb{Z}_4]$	none	$(1,10,10)$ $(70,1,1)$	none
$(1,0^{15})$	D_8+D_9	(1,128)	(16,18)	$(128,1)\times 2$
$\left(\frac{1}{2}^4,0^{12}\right)$	$[D_4+D_5+D_8\,;\mathbb{Z}_2]$	$(1,1,128)$ $(8,10,1)$	(8,1,16)	$(8,16,1)\times 2$
$\left(\frac{1}{2}^2,0^6,\frac{1}{2}^2,0^6\right)$	$[2A_1+A_3+2D_6\,;\mathbb{Z}_2^2]$	$(2,1,1,32,1)$ $(1,2,1,1,32)$	$(1,1,1,12,12)$ $(2,2,6,1,1)$	$(2,1,1,32,1)\times 2$ $(1,2,1,1,32)\times 2$
$\left(\frac{1}{2}^5,0^3,\frac{1}{4}^7,\text{-}\frac{1}{4}\right)$	$[A_{11}+E_6\,;\mathbb{Z}_3]$	none	none	$(143,1)\times 2$ $(1,78)\times 2$

Maximal enhancements

Group	R^2	Wilson line	Λ	$\lambda(H_\Lambda)\times R^2$
$\left[Spin(16)^2\right]/\mathbb{Z}_2\times SU(2)$	1	0^{16}	431.354	-306^{16}, 831
$\left[Spin(16)\times Spin(12)\times SU(2)\right]/\mathbb{Z}_2\times SU(3)$	$\frac{3}{4}$	$0^{14}, \frac{1}{2}^2$	383.516	-307^{15}, 544^2
$Spin(16)\times Spin(10)\times SU(5)$	$\frac{5}{8}$	$0^{13}, \frac{1}{2}^3$	359.196	-569^5, -256^8, 355^4
$\left[Spin(10)^2\times SU(8)\right]/\mathbb{Z}_4$	$\frac{1}{4}$	$0^4, \frac{1}{2}^4, \frac{1}{4}^8$	303.778	-195^{17}
$Spin(18)\times Spin(16)$	$\frac{1}{2}$	$0^{15}, 1$	305.013	-1283^8, 588^9
$\left[Spin(16)\times Spin(10)\times Spin(8)\right]/\mathbb{Z}_2$	$\frac{1}{2}$	$0^{12}, \frac{1}{2}^4$	305.013	-1283^4, -347^8, 588^5
$\left[Spin(12)^2\times SU(4)\times SU(2)^2\right]/\mathbb{Z}_2^2$	$\frac{1}{2}$	$0^6, \frac{1}{2}^2, 0^6, \frac{1}{2}^2$	305.013	-1283^2, -347^{12}, 588^3
$\left[E_6\times SU(12)\right]/\mathbb{Z}_3$	$\frac{1}{8}$	$0^3, \frac{1}{2}^5, -\frac{1}{4}, \frac{1}{4}^7$	180.426	-72^{17}

Non-maximal extrema

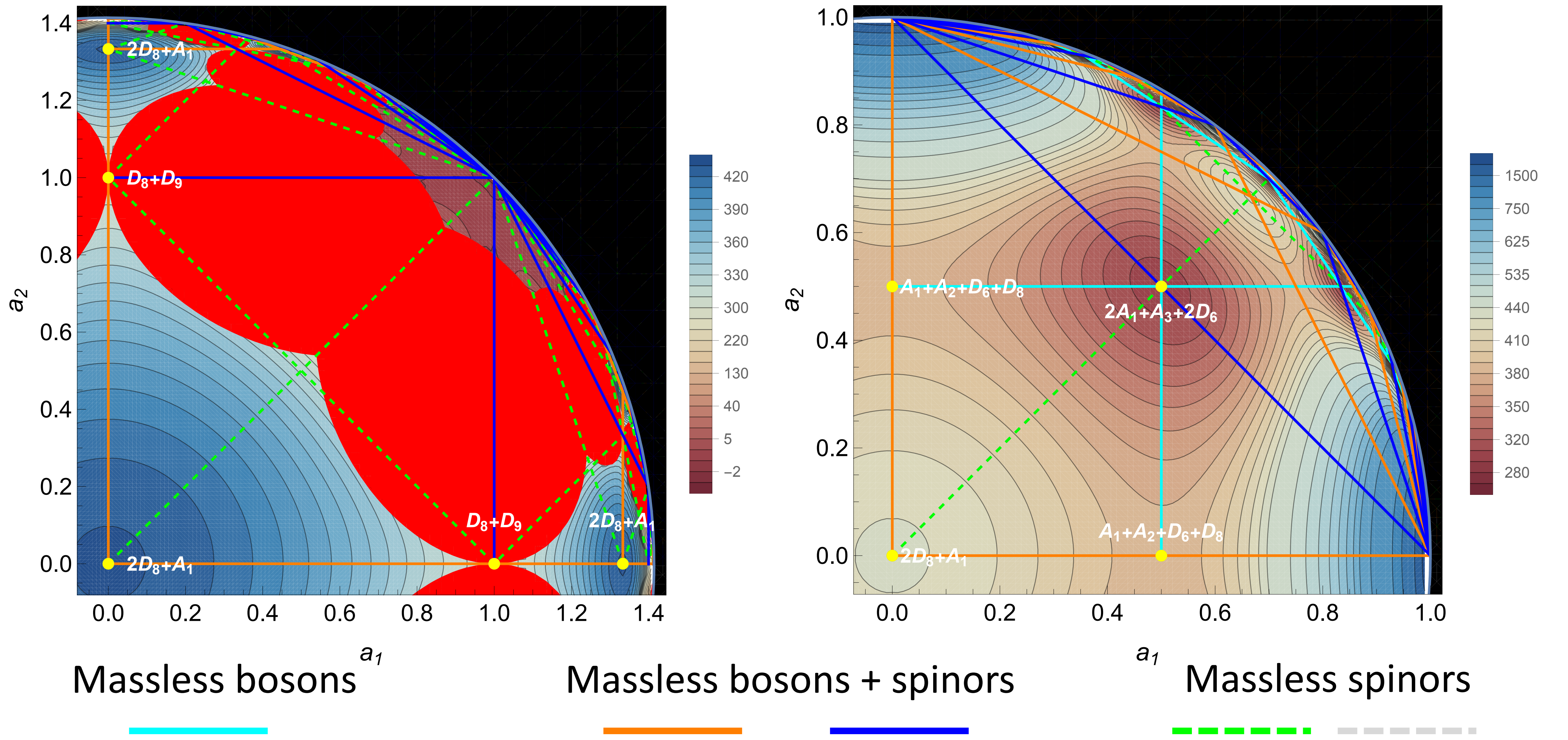
Algebra	R^2	Wilson line	Λ	$\lambda(H_\Lambda) \times R^2$
$D_8 + D_5 + A_3$	$3/8$	$\left(0^5, \frac{1}{2}^3, 0^8\right)$	367.146	$-338^{14}, 424^3$
$D_8 + D_4 + A_4$	$2/5$	$\left(0^3, \frac{4}{5}^5, 0^8\right)$	359.196	$-569, -412^8, -256^4, 355^4$
$4D_4$	$1/2$	$\left(0^4, \frac{1}{2}^4, 0^4, \frac{1}{2}^4\right)$	305.013	$-347^{16}, 588$
$2A_7 + 2A_1$	$1/2$	$\left(\frac{1}{4}^{16}\right)$	305.013	$-1283, -347^{14}, 588^2$



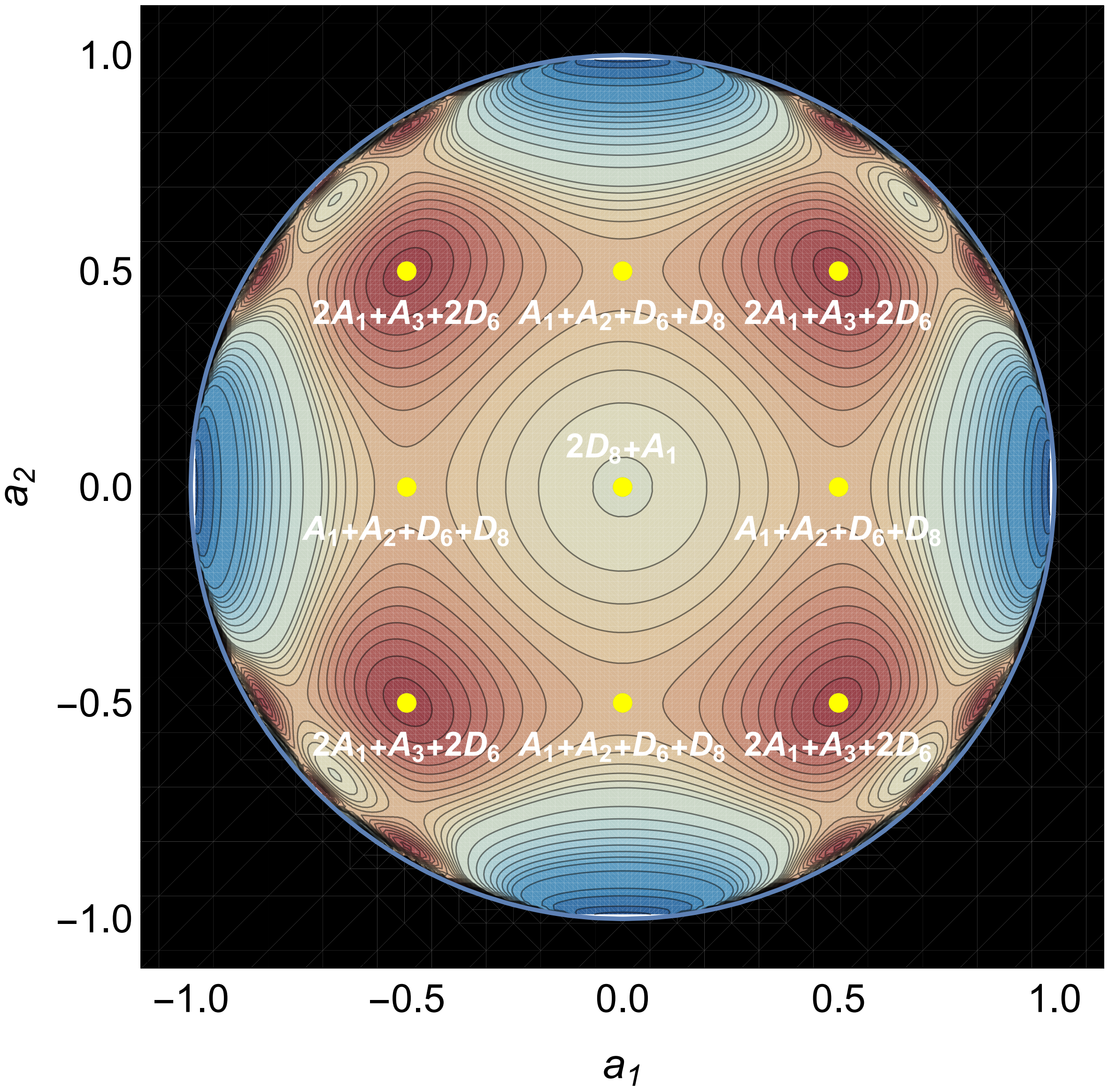
$A = (a, a, a, 0^{13})$
$A = (a, a, a, a, a, 0^{11})$
$2D_8+A_1$
$A_4+D_5+D_8$
D_8+D_9
$A_1+A_2+D_6+D_8$
$A_3+E_6+D_8$
$D_4+D_5+D_8$
R
a
1100
655
535
430
390
370
362
340
300
2100
595
410
366
320
130
0
Massless bosons
Massless bosons + spinors
Massless spinors

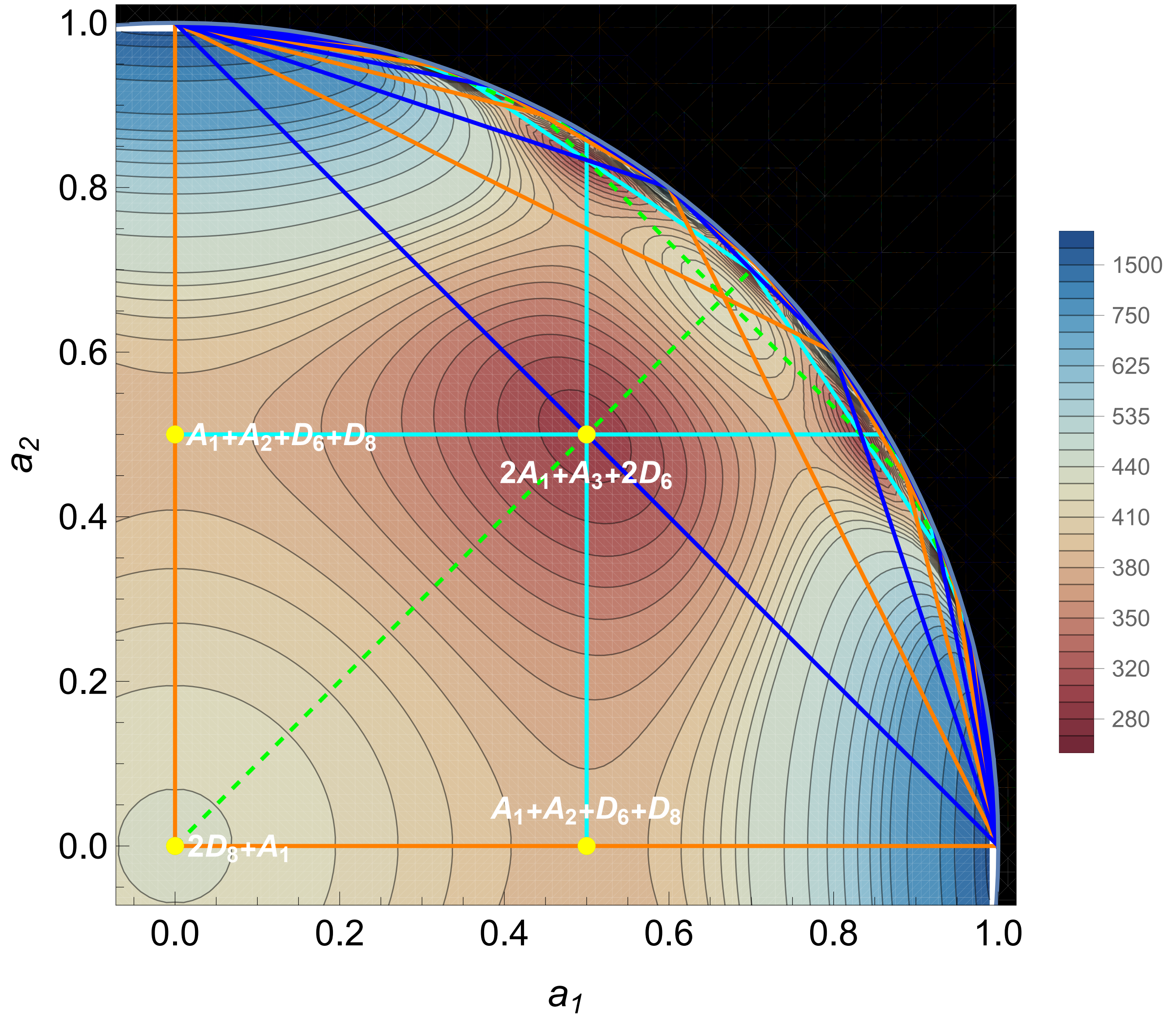
$$A = (a_1, 0^7, a_2, 0^7) \qquad R^2 = 1 - \frac{a_1^2 + a_2^2}{2}$$

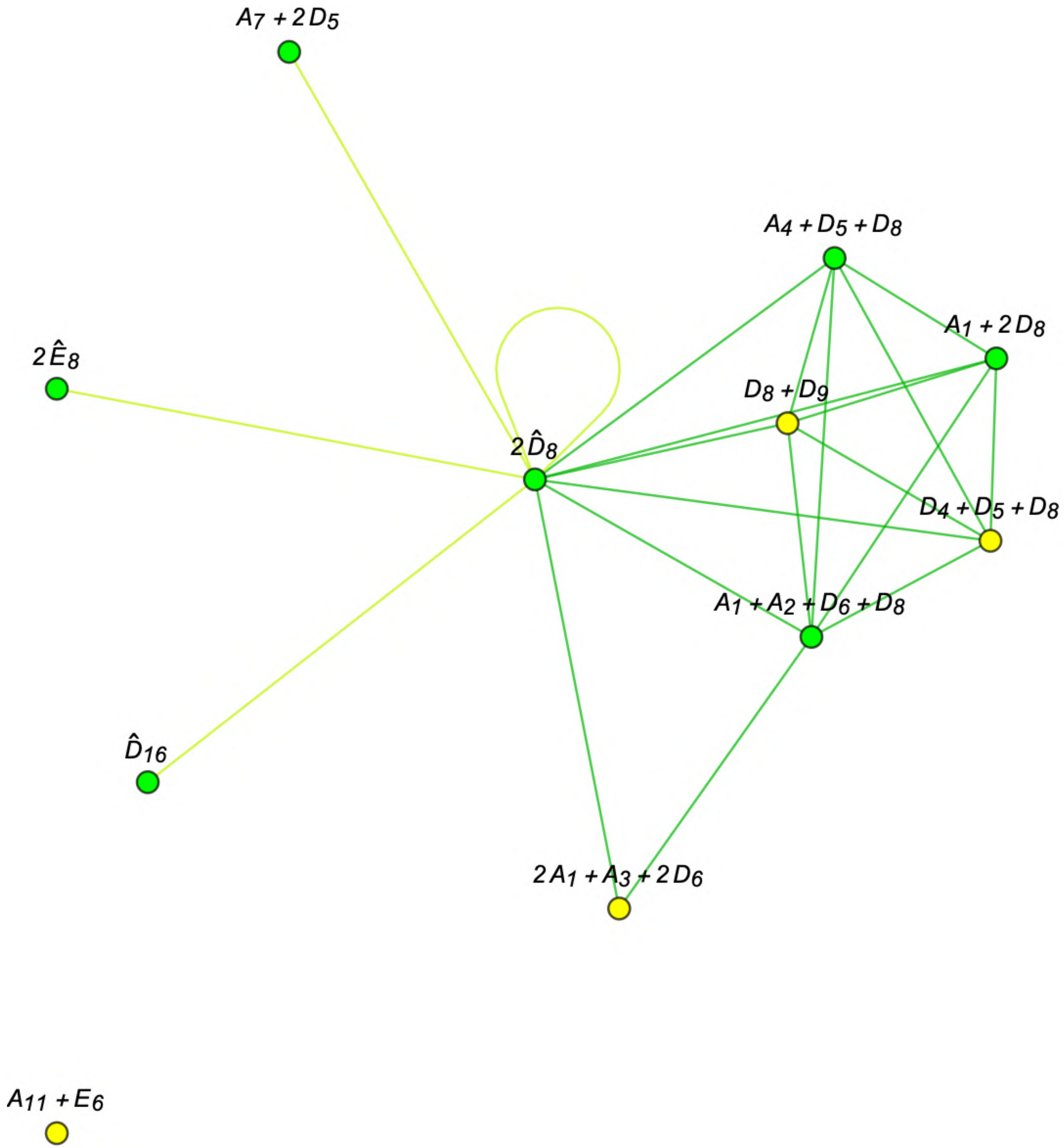
$$A = (a_1, a_1, 0^6, a_2, a_2, 0^6) \qquad R^2 = 1 - (a_1^2 + a_2^2)$$

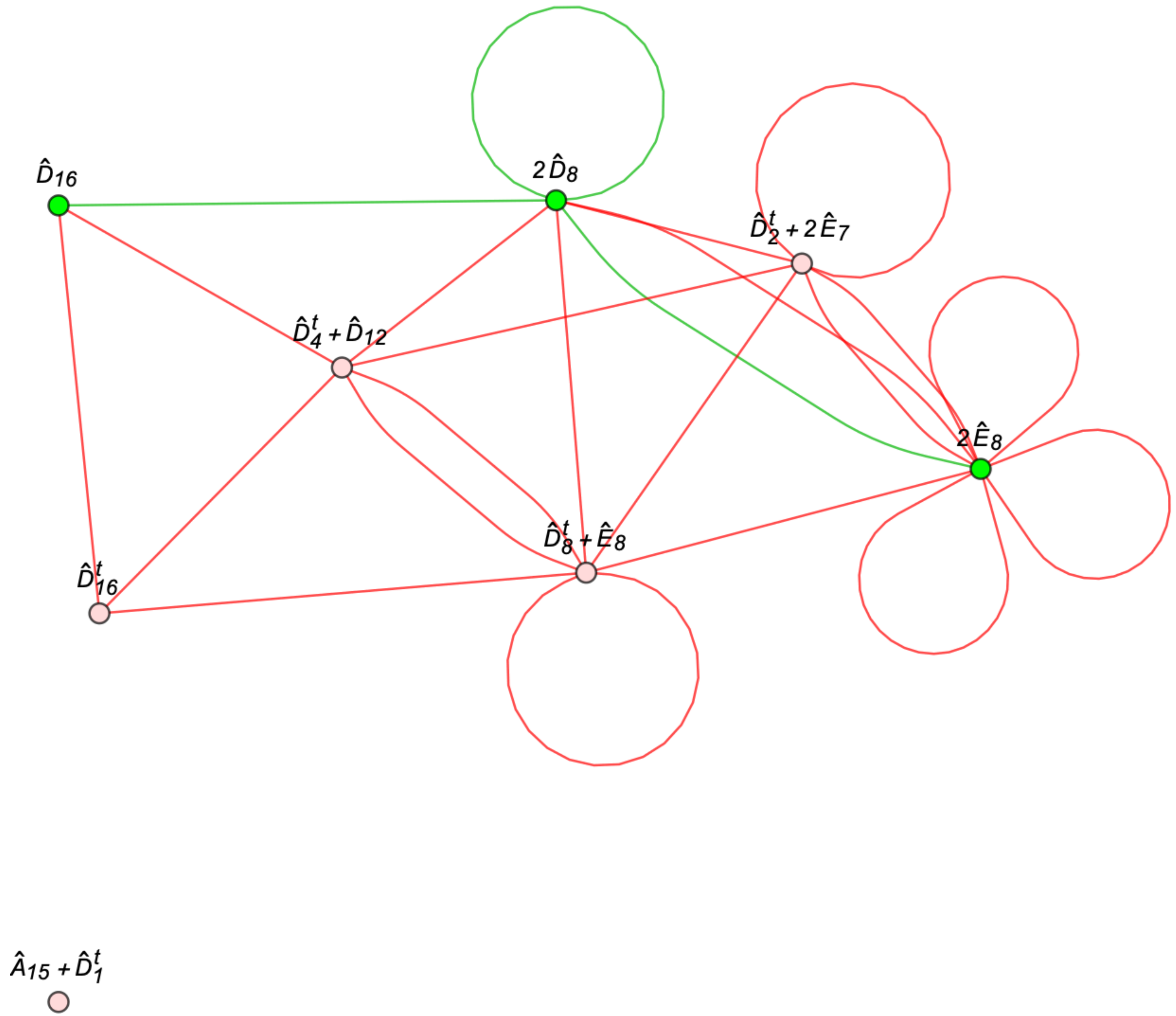


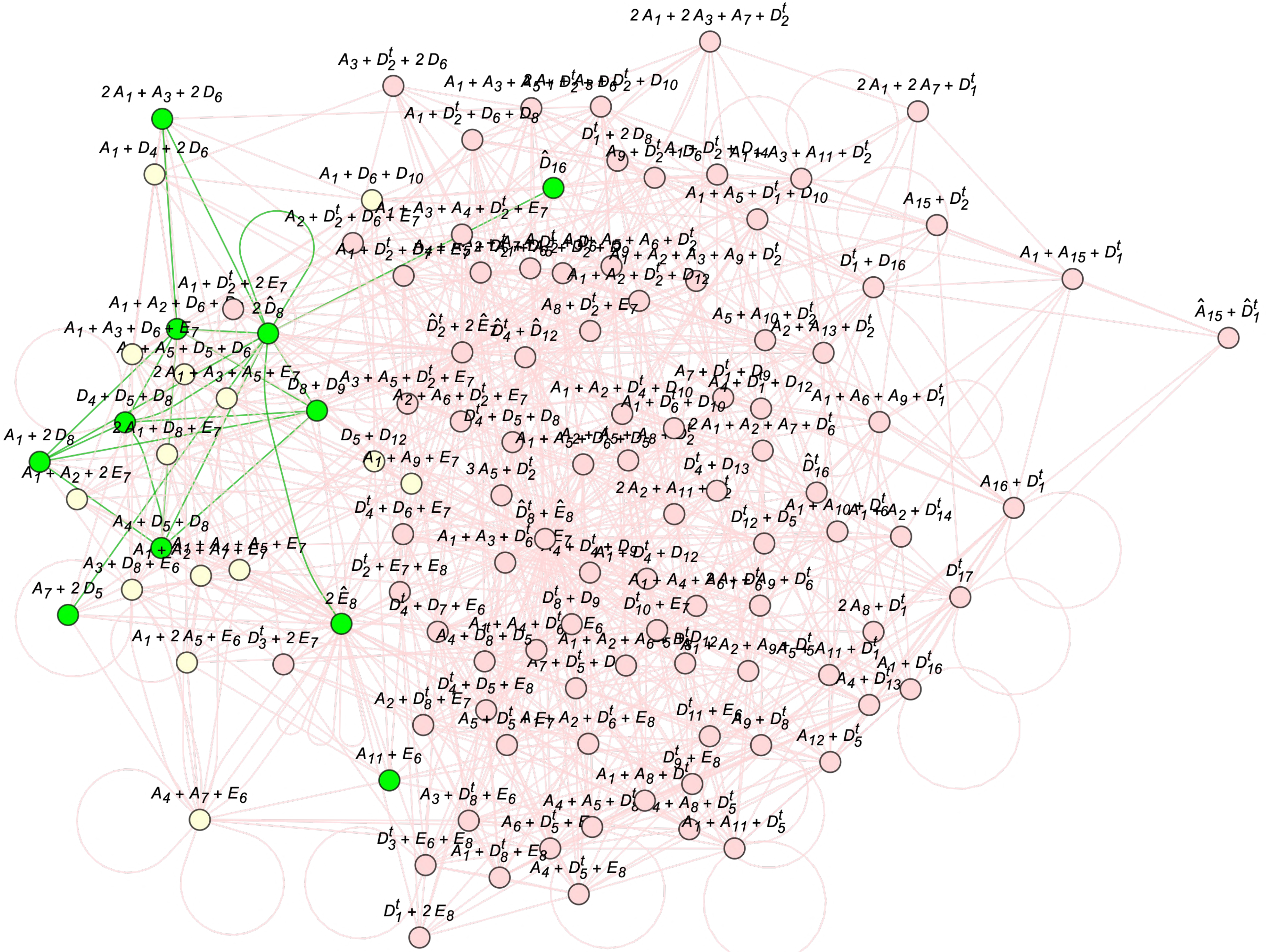
$$A = (a_1, a_1, 0^6, a_2, a_2, 0^6) \qquad R^2 = 1 - (a_1^2 + a_2^2)$$





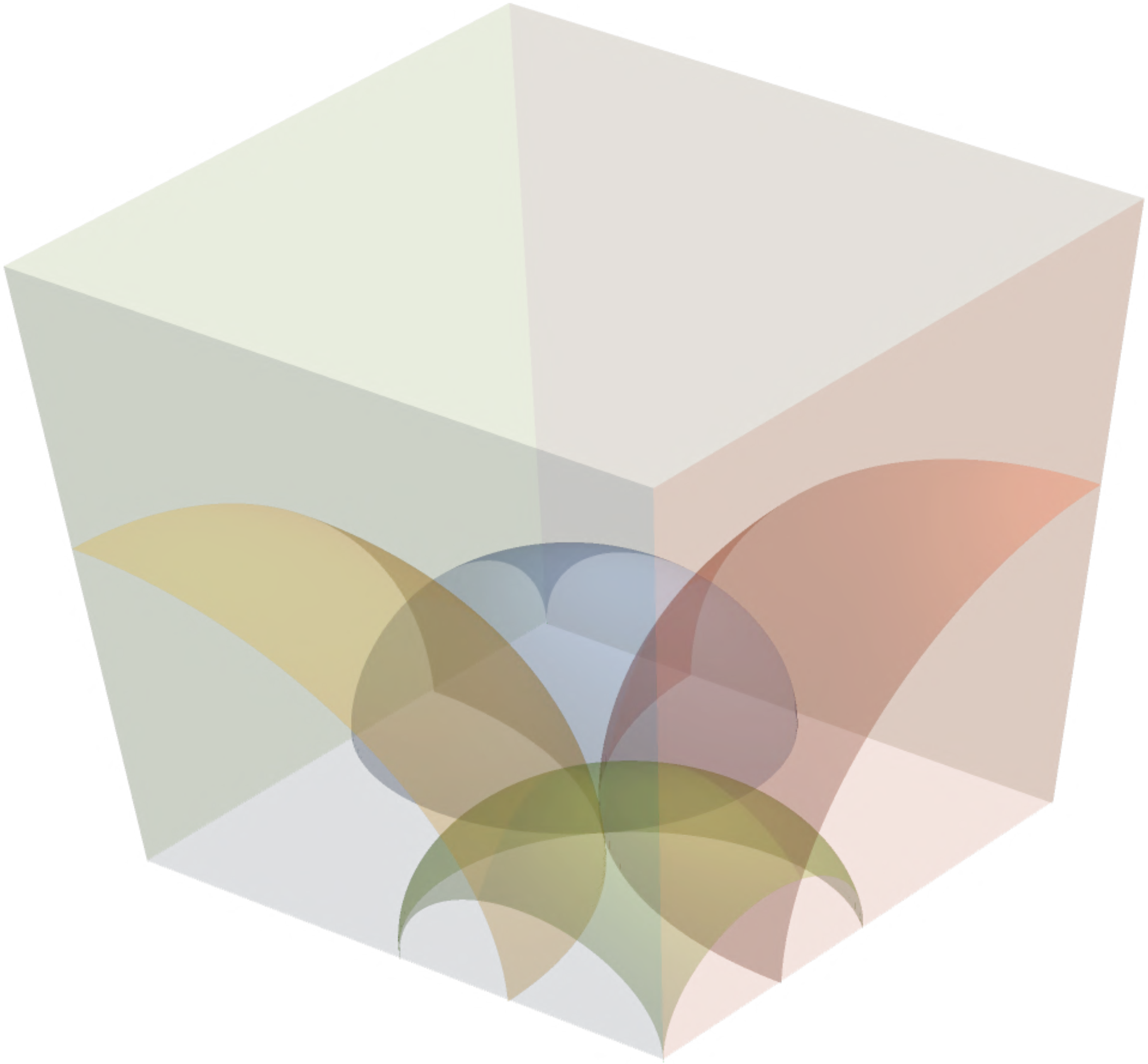

$A_7 + 2D_5$
$A_4 + D_5 + D_8$
$A_1 + 2D_8$
$2\hat{E}_8$
$D_8 + D_9$
$2\hat{D}_8$
$D_4 + D_5 + D_8$
$A_1 + A_2 + D_6 + D_8$
$\hat{D}_{16}$
$2A_1 + A_3 + 2D_6$
$A_{11} + E_6$

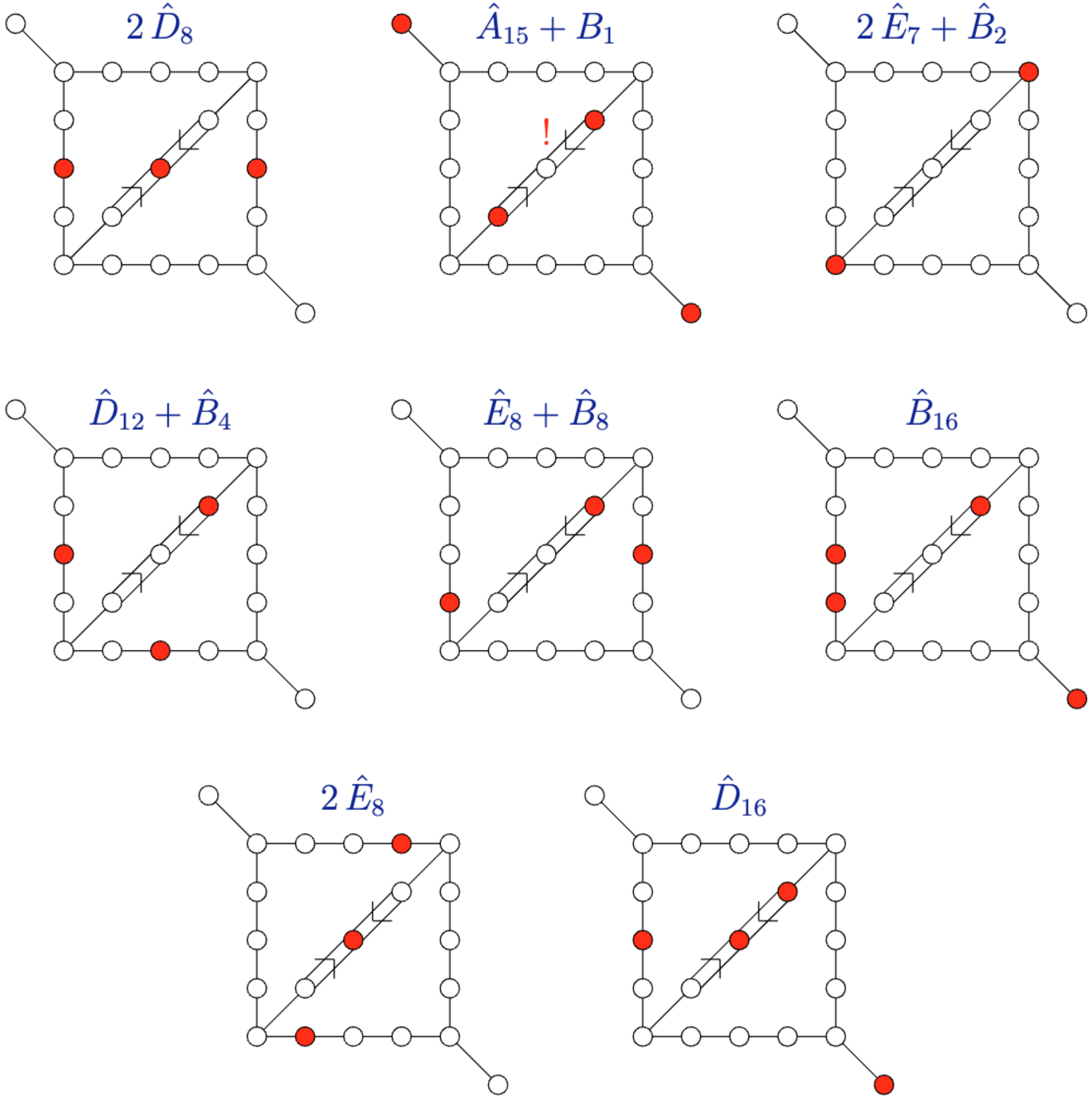

$\hat{D}_{16}$
$2\hat{D}_8$
$\hat{D}_2^t + 2\hat{E}_7$
$\hat{D}_4^t + \hat{D}_{12}$
$2\hat{E}_8$
$\hat{D}_8^t + \hat{E}_8$
$\hat{D}_{16}^t$
$\hat{A}_{15} + \hat{D}_1^t$



$2A_1 + 2A_3 + A_7 + D_2^t$
$A_3 + D_2^t + 2D_6$
$2A_1 + 2A_7 + D_1^t$
$2A_1 + A_3 + 2D_6$
$A_1 + D_2^t + D_6 + D_8$
$A_1 + D_4 + 2D_6$
$A_1 + D_6 + D_{10}$
$\hat{D}_{16}$
$A_15 + D_2^t$
$A_1 + A_5 + D_1^t + D_{10}$
$A_1 + A_{15} + D_1^t$
$A_1 + A_3 + A_4 + D_2^t + E_7$
$A_2 + D_2^t + D_6 + E_7$
$A_1 + A_2 + A_3 + A_9 + D_2^t$
$A_1 + D_2^t + 2E_7$
$D_1^t + D_{16}$
$\hat{A}_{15} + \hat{D}_1^t$
$A_1 + A_2 + D_2^t + D_{12}$
$A_8 + D_2^t + E_7$
$2\hat{D}_8$
$A_5 + A_{10} + D_2^t$
$A_2 + A_{13} + D_2^t$
$A_1 + A_3 + D_6 + E_7$
$2A_1 + A_3 + A_5 + E_7$
$A_3 + A_5 + D_2^t + E_7$
$A_1 + A_2 + D_4^t + D_{10}$
$A_1 + A_6 + A_9 + D_1^t$
$D_8 + D_9$
$A_2 + A_6 + D_2^t + E_7$
$D_4 + D_5 + D_8$
$2A_1 + A_2 + A_7 + D_6^t$
$A_1 + 2D_8$
$2A_1 + D_8 + E_7$
$D_5 + D_{12}$
$A_1 + A_9 + E_7$
$3A_5 + D_2^t$
$D_4^t + D_{13}$
$A_1 + A_2 + 2E_7$
$\hat{D}_{16}^t$
$A_{16} + D_1^t$
$A_4 + D_5 + D_8$
$D_4^t + D_6 + E_7$
$\hat{D}_8^t + \hat{E}_8$
$D_{12}^t + D_5$
$A_3 + D_8 + E_6$
$A_7 + 2D_5$
$D_2^t + E_7 + E_8$
$2\hat{E}_8$
$D_4^t + D_7 + E_6$
$D_8^t + D_9$
$D_{10}^t + E_7$
D_{17}^t
$2A_8 + D_1^t$
$A_1 + 2A_5 + E_6$
$D_3^t + 2E_7$
$A_1 + D_{16}^t$
$D_4^t + D_5 + E_8$
$A_2 + D_8^t + E_7$
$D_{11}^t + E_6$
$A_9 + D_8^t$
$A_{12} + D_5^t$
$A_{11} + E_6$
$D_9^t + E_8$
$A_4 + A_7 + E_6$
$A_3 + D_8^t + E_6$
$A_1 + A_{11} + D_5^t$
$D_3^t + E_6 + E_8$
$A_1 + D_8^t + E_8$
$A_4 + D_5^t + E_8$
$D_1^t + 2E_8$

V	
V+S	
V+C	
V+S+C	
S	
C	
S+C	




$2\,\hat{D}_8$
$\hat{A}_{15} + B_1$
!
$2\,\hat{E}_7 + \hat{B}_2$
$\hat{D}_{12} + \hat{B}_4$
$\hat{E}_8 + \hat{B}_8$
$\hat{B}_{16}$
$2\,\hat{E}_8$
$\hat{D}_{16}$

#	L	H	k	m	N_b	N_f	Wilson line
1	$2A_1+3A_5$	$\mathbb{Z}_6$	11141	$-1,-\frac{1}{3}$	94	160	$0,0,0,\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{4}{5},0,0,\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5}$
2	$A_1+2A_1+A_3+A_5+A_6$	$\mathbb{Z}_2$	111230	$-1,-\frac{5}{12}$	90	128	$0,0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6},0,0,\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18}$
3	$2A_1+2A_1+2A_3+A_7$	$\mathbb{Z}_2\mathbb{Z}_4$	1111004 0011116	$-1,-\frac{1}{2}$	88	96	$0,0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6},0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2}$
4	$2A_1+A_2+A_5+A_8$	$\mathbb{Z}_3$	00146	$-1,-\frac{1}{2}$	112	80	$0,0,\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{7}{8}$
5	$A_1+2A_1+A_2+A_3+A_9$	$\mathbb{Z}_2$	111005	$-1,-\frac{7}{12}$	114	48	$0,0,\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{3}{16},\frac{1}{2},\frac{1}{2},-\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8}$
6	$2A_1+A_5+A_{10}$	1		$-1,-\frac{17}{66}$	144	80	$0,0,\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18},\frac{7}{18},-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6}$
7	$2A_1+2A_2+A_{11}$	$\mathbb{Z}_6$	111110	$-1,-\frac{2}{3}$	148	0	$0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},-\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8}$
8	$A_1+2A_1+A_3+A_{11}$	$\mathbb{Z}_4$	01119	$-1,-\frac{1}{3}$	150	48	$0,0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6},0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}$
9	$2A_1+A_2+A_{13}$	1		$-1,-\frac{17}{42}$	192	0	$0,0,\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{11}{16},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{7}{8}$
10	$2A_1+A_{15}+A_1^{(R)}$	$\mathbb{Z}_4$	114	-1	244	0	$0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6}$
11	$2A_1+D_1+2A_7$	$\mathbb{Z}_2\mathbb{Z}_4$	11104 00126	$-1,0$	118	0	$0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}$
12	D_1+2A_8	$\mathbb{Z}_3$	036	$-1,0$	146	0	$-\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{4}{5}$
13	$A_1+D_1+A_6+A_9$	$\mathbb{Z}_2$	1105	$-1,-\frac{2}{7}$	136	0	$0,0,\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{8}{11},-\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22}$
14	$D_1+A_5+A_{11}$	$\mathbb{Z}_6$	1210	$-1,-\frac{1}{2}$	164	0	$0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$
15	$A_1+D_1+A_{15}$	$\mathbb{Z}_4$	014	-1	244	0	$0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4}$
16	D_1+A_{16}	1		-1	274	0	$-\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},\frac{7}{22},-\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{8}{11}$
17	A_7+2D_5	$\mathbb{Z}_4$	213		136	170	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$
18	$A_4+A_8+D_5$	1		$-1,-\frac{4}{9}$	132	0	$0,0,0,\frac{2}{19},\frac{2}{19},\frac{2}{19},\frac{2}{19},\frac{17}{19},\frac{9}{38},\frac{9}{38},\frac{9}{38},\frac{9}{38},\frac{9}{38},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
19	$A_1+A_2+A_9+D_5$	$\mathbb{Z}_2$	1052	$-1,-\frac{2}{5}$	138	0	$0,0,0,\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},-\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{1}{2},\frac{1}{2}$
20	$A_1+A_{11}+D_5$	$\mathbb{Z}_2$	062	$-1,-\frac{1}{3}$	174	0	$0,0,0,\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},-\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7}$
21	$A_{12}+D_5$	1		$-1,-\frac{4}{13}$	196	0	$0,0,0,\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},\frac{3}{11},-\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{9}{22},\frac{13}{22}$
22	$A_7+D_5+D_5$	$\mathbb{Z}_2$	422	$-1,-\frac{1}{2}$	136	0	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8}$
23	$A_1+2A_1+A_3+A_5+D_6$	$\mathbb{Z}_2^2$	0002310 1110311	$-1,-\frac{1}{4}$	108	176	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2}$
24	$2A_1+A_4+A_5+D_6$	$\mathbb{Z}_2$	110301	$-1,-\frac{1}{6}$	114	208	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
25	$2A_1+A_2+A_7+D_6$	$\mathbb{Z}_2$	110411	$-1,-\frac{1}{3}$	126	128	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8}$
26	$2A_1+A_9+D_6$	$\mathbb{Z}_2$	00510	$-1,-\frac{1}{10}$	154	128	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}$
27	$A_1+A_5+D_5+D_6$	$\mathbb{Z}_2$	13211	$-\frac{1}{4}$	132	224	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$
28	$2A_1+A_3+2D_6$	$\mathbb{Z}_2^2$	0021010 1100101	0	136	296	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,0,0,\frac{1}{2},\frac{1}{2}$
29	$2A_1+A_3+2D_6$	$\mathbb{Z}_2^2$	0021001 1100110	$-1,0$	136	256	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
30	$A_1+D_4+2D_6$	$\mathbb{Z}_2^2$	0011001 1101110	$-\frac{1}{2}$	146	320	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
31	$A_1+A_4+A_6+D_6$	1		$-1,-\frac{9}{14},-\frac{3}{10}$	124	128	$0,0,0,\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{11}{12},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{1}{2},\frac{1}{2}$
32	$2A_1+A_2+A_7+D_6$	$\mathbb{Z}_2$	110411	$-1,-\frac{5}{8},-\frac{1}{3}$	126	128	$0,0,\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{13}{14},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{1}{2},\frac{1}{2}$
33	$2A_1+A_9+D_6$	$\mathbb{Z}_2$	10511	$-1,-\frac{3}{5},-\frac{1}{10}$	154	128	$0,0,\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},-\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9}$
34	$A_1+A_{10}+D_6$	1		$-1,-\frac{13}{22}$	172	128	$0,0,\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},-\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{4}{7}$
35	$A_1+A_5+D_5+D_6$	$\mathbb{Z}_2$	13211	$-1,-\frac{2}{3},-\frac{1}{4}$	132	128	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2}$
36	$A_4+D_5+D_8$	1			172	288	$0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$
37	$D_4+D_5+D_8$	$\mathbb{Z}_2$	11210	0	176	336	$0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
38	$D_4+D_5+D_8$	$\mathbb{Z}_2$	11201	$-1,0$	176	256	$0,0,0,0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
39	$A_1+2A_1+D_6+D_8$	$\mathbb{Z}_2^2$	0111101 1001010	$-1,-\frac{1}{2}$	178	256	$0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},0,0,\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
40	$A_1+A_2+D_6+D_8$	$\mathbb{Z}_2$	100110		180	384	$0,0,0,0,0,0,0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2}$
41	A_1+2D_8	$\mathbb{Z}_2$	01010		226	512	$0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0$
42	$D_1+2D_8+A_1^{(R)}$	$\mathbb{Z}_2^2$	00110 11011	$-1,0$	226	0	$0,0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3},-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
43	$A_4+A_5+D_8$	1		$-1,-\frac{1}{6}$	162	256	$0,0,0,\frac{1}{17},\frac{1}{17},\frac{1}{17},\frac{1}{17},\frac{16}{17},\frac{6}{17},\frac{6}{17},\frac{6}{17},\frac{6}{17},\frac{6}{17},\frac{6}{17},\frac{6}{17},\frac{6}{17}$
44	$A_1+A_2+A_6+D_8$	1		$-1,-\frac{1}{7}$	162	256	$0,0,\frac{1}{20},\frac{1}{20},\frac{1}{20},\frac{1}{20},\frac{1}{20},\frac{19}{20},\frac{7}{20},\frac{7}{20},\frac{7}{20},\frac{7}{20},\frac{7}{20},\frac{7}{20},\frac{7}{20},\frac{7}{20}$
45	$A_1+A_8+D_8$	1		$-1,-\frac{1}{9}$	186	256	$-\frac{6}{13},\frac{6}{13},\frac{6}{13},\frac{6}{13},\frac{6}{13},\frac{6}{13},\frac{6}{13},\frac{6}{13},\frac{2}{13},\frac{2}{13},\frac{2}{13},\frac{2}{13},\frac{2}{13},\frac{2}{13},\frac{2}{13},\frac{2}{13}$
46	A_9+D_8	1		$-1,-\frac{1}{10}$	202	256	$0,0,0,\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{4}{5},0,0,0,\frac{7}{15},\frac{7}{15},\frac{7}{15},\frac{7}{15},\frac{8}{15}$
47	$A_4+D_5+D_8$	1		$-1,-\frac{1}{5}$	172	256	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
48	$A_1+2A_1+A_5+D_9$	$\mathbb{Z}_2$	11132	$-1,-\frac{2}{3}$	180	224	$0,0,0,0,0,0,0,1,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$
49	$D_1+A_7+D_9$	$\mathbb{Z}_2$	142	$-1,-\frac{1}{2}$	202	0	$0,0,0,0,0,0,0,1,-\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$
50	$A_4+D_4+D_9$	1		$-1,-\frac{1}{5}$	188	288	$0,0,0,0,0,0,0,1,0,0,0,\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5}$
51	D_8+D_9	1		0	256	416	$0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1$
52	D_8+D_9	1		$-1,0$	256	256	$0,0,0,0,0,0,0,1,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$
53	$2A_1+2A_1+A_3+D_{10}$	$\mathbb{Z}_2^2$	0100210 1011001	$-1,-\frac{3}{4},0$	200	208	$0,0,0,0,0,0,0,1,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}$
54	$A_1+D_1+A_5+D_{10}$	$\mathbb{Z}_2^2$	00310 11001	$-1,-\frac{2}{3}$	214	0	$0,0,0,0,0,0,0,1,-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2}$
55	$A_1+A_2+D_4+D_{10}$	$\mathbb{Z}_2$	101101	$-1,-\frac{1}{3}$	212	320	$0,0,0,0,0,0,0,1,0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2}$
56	$A_1+D_6+D_{10}$	$\mathbb{Z}_2$	01010	$-\frac{1}{2}$	242	304	$0,0,0,0,0,0,0,1,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2}$
57	$A_1+D_6+D_{10}$	$\mathbb{Z}_2$	11101	$-1,-\frac{1}{2}$	242	128	$0,0,0,0,0,0,0,1,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}$
58	$A_1+2A_1+A_2+D_{12}$	$\mathbb{Z}_2$	011010	$-1,-\frac{5}{6},-\frac{1}{3}$	276	192	$0,0,0,0,0,0,0,1,0,0,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
59	$D_1+A_4+D_{12}$	$\mathbb{Z}_2$	1010	$-1,-\frac{4}{5}$	286	0	$0,0,0,0,0,0,0,1,-\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
60	$A_1+D_4+D_{12}$	$\mathbb{Z}_2$	01110	$-1,-\frac{1}{2}$	290	384	$0,0,0,0,0,0,0,1,0,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
61	D_5+D_{12}	1		$-\frac{3}{4}$	304	240	$0,0,0,0,0,0,0,1,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2}$
62	$D_5+D_{12}+A_1^{(R)}$	$\mathbb{Z}_2$	210	$-1,0$	304	0	$0,0,0,0,0,0,0,1,\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
63	D_5+D_{12}	1		$-1,-\frac{3}{4},0$	304	0	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}$
64	D_4+D_{13}	1		$-1,0$	336	416	$0,0,0,0,0,0,0,1,0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}$
65	A_4+D_{13}	1		$-1,-\frac{4}{5},-\frac{1}{5}$	332	0	$0,0,0,\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{1}{12},\frac{11}{12},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{7}{24},\frac{17}{24}$
66	$A_1+2A_1+D_{14}$	$\mathbb{Z}_2$	10010	$-1,-\frac{1}{2}$	370	224	$0,0,0,0,0,0,0,1,0,0,\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4}$
67	$A_1+A_2+D_{14}$	1		$-1,-\frac{5}{6},-\frac{1}{3}$	372	0	$0,0,\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{1}{14},\frac{13}{14},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{2}{7},\frac{5}{7}$
68	$D_1+D_{16}+A_1^{(R)}$	$\mathbb{Z}_2$	001	$-1,0$	482	0	$0,0,0,0,0,0,0,1,-\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{5}{6}$
69	A_1+D_{16}	1		$-1,-\frac{1}{2}$	482	0	$-\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},\frac{4}{9},-\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9}$
70	$D_{17}+A_1^{(R)}$	1		$-1,0$	544	0	$-\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{4}{7},-\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14},\frac{3}{14}$
71	$A_1+2A_5+E_6$	$\mathbb{Z}_3$	0241	$-\frac{1}{3}$	134	80	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}$
72	$A_4+A_7+E_6$	1		$-\frac{13}{40}$	148	70	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
73	$A_{11}+E_6+A_1^{(R)}$	$\mathbb{Z}_3$	41	0	204	0	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4}$
74	$A_6+D_5+E_6$	1		$-1,-\frac{4}{7}$	154	0	$0,0,0,0,0,\frac{2}{13},\frac{2}{13},\frac{11}{13},\frac{7}{26},\frac{7}{26},\frac{7}{26},\frac{7}{26},\frac{7}{26},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
75	$A_1+A_4+D_6+E_6$	1		$-1,-\frac{7}{10},-\frac{1}{6}$	154	128	$0,0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{7}{8},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{1}{2},\frac{1}{2}$
76	$D_4+D_7+E_6$	1		$-1,-\frac{1}{4}$	180	224	$0,0,0,0,0,0,0,\frac{3}{4},0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
77	$A_3+D_8+E_6$	1		$-\frac{1}{4}$	196	224	$0,0,0,0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
78	$A_3+D_8+E_6$	1		$-1,-\frac{1}{4}$	196	256	$0,0,0,0,0,\frac{1}{11},\frac{1}{11},\frac{10}{11},\frac{4}{11},\frac{4}{11},\frac{4}{11},\frac{4}{11},\frac{4}{11},\frac{4}{11},\frac{4}{11},\frac{4}{11}$
79	$D_{11}+E_6$	1		$-1,-\frac{2}{3}$	292	0	$0,0,0,0,0,\frac{1}{8},\frac{1}{8},\frac{7}{8},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{5}{16},\frac{11}{16}$
80	$A_1+2A_1+A_3+A_4+E_7$	$\mathbb{Z}_2$	111201	$-1,-\frac{1}{5}$	164	272	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{1}{2},\frac{1}{2}$
81	$2A_1+A_3+A_5+E_7$	$\mathbb{Z}_2$	11031	$-\frac{2}{3},-\frac{1}{4}$	172	176	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2}$
82	$2A_1+A_3+A_5+E_7$	$\mathbb{Z}_2$	11031	$-1,-\frac{1}{4}$	172	304	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
83	$A_1+A_4+A_5+E_7$	1		$-\frac{7}{10},-\frac{1}{6}$	178	152	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
84	$2A_1+A_2+A_6+E_7$	1		$-1,-\frac{1}{7}$	178	224	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7},\frac{3}{7}$
85	$A_1+A_2+A_7+E_7$	$\mathbb{Z}_2$	1041	$-\frac{5}{8}$	190	182	$0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
86	$2A_1+A_8+E_7$	1		$-1,-\frac{1}{9}$	202	224	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5}$
87	$A_1+A_9+E_7$	1		$-\frac{3}{5}$	218	112	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}$
88	$A_5+D_5+E_7$	$\mathbb{Z}_2$	321	$-1,-\frac{2}{3}$	196	0	$0,0,0,0,0,0,\frac{1}{5},\frac{4}{5},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
89	$2A_1+A_2+D_6+E_7$	$\mathbb{Z}_2$	110101	$-1,-\frac{1}{3}$	196	352	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
90	$A_1+A_3+D_6+E_7$	$\mathbb{Z}_2$	12111	$-\frac{3}{4},0$	200	248	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
91	$D_4+D_6+E_7$	$\mathbb{Z}_2$	11011	$-1,-\frac{1}{2}$	210	192	$0,0,0,0,0,0,\frac{1}{4},\frac{3}{4},0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
92	$A_1+A_3+D_6+E_7$	$\mathbb{Z}_2$	12111	$-1,-\frac{3}{4},0$	200	128	$0,0,0,0,0,0,\frac{1}{6},\frac{5}{6},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{1}{2}$
93	$A_1+2A_1+D_7+E_7$	$\mathbb{Z}_2$	11121	$-1,0$	216	336	$0,0,0,0,0,0,0,\frac{1}{2},0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
94	$2A_1+D_8+E_7$	$\mathbb{Z}_2$	10101	$-\frac{1}{2}$	242	304	$0,0,0,0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
95	$A_2+D_8+E_7$	1		$-1,-\frac{1}{3}$	244	256	$0,0,0,0,0,0,\frac{1}{8},\frac{7}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8}$
96	$D_{10}+E_7$	1		$-1,-\frac{1}{2}$	306	0	$0,0,0,0,0,0,\frac{1}{6},\frac{5}{6},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}$
97	$A_1+2A_1+2E_7$	$\mathbb{Z}_2$	01111	$-1,-\frac{1}{2}$	258	448	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},0,\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
98	$A_1+A_2+2E_7$	1		$-\frac{5}{6}$	260	224	$0,0,0,0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
99	$A_3+2E_7+A_1^{(R)}$	$\mathbb{Z}_2$	211	$-1,0$	264	0	$0,0,0,0,0,0,\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
100	$D_4+D_5+E_8$	1		$-1,-\frac{3}{4},0$	304	160	$0,0,0,0,0,\frac{1}{4},\frac{1}{4},\frac{3}{4},0,0,0,\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
101	$A_4+D_5+E_8$	1		$-1,-\frac{4}{5},-\frac{1}{5}$	300	0	$0,0,0,0,0,0,0,\frac{5}{7},\frac{5}{14},\frac{5}{14},\frac{5}{14},\frac{5}{14},\frac{5}{14},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
102	$A_1+A_2+D_6+E_8$	1		$-1,-\frac{5}{6},-\frac{1}{3}$	308	128	$0,0,0,0,0,0,0,\frac{3}{4},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2}$
103	$A_1+D_8+E_8$	1		$-1,-\frac{1}{2}$	354	256	$0,0,0,0,0,0,0,\frac{4}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{2}{5}$
104	$D_9+E_8+A_1^{(R)}$	1		$-1,0$	384	0	$0,0,0,0,0,0,0,\frac{3}{4},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{5}{8}$
105	$A_3+E_6+E_8$	1		$-1,-\frac{2}{3}$	324	0	$0,0,0,0,0,0,0,\frac{3}{5},\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
106	$2A_1+E_7+E_8$	1		$-1,-\frac{1}{2}$	370	224	$0,0,0,0,0,0,0,\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$
107	$D_1+2E_8+A_1^{(R)}$	1		$-1,0$	482	0	$0,0,0,0,0,0,0,\frac{1}{3},\frac{1}{6},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}$