# Cosmology exercise session - IDPASC 2023

**1.** By considering the trajectories of light taken between two galaxies receding from each other due to the expansion of the universe, show that the frequencies of the emitted and observed light, $\nu_{\text{emit}}$ and $\nu_{\text{obs}}$, are related to the scale factor of the universe at emission and observation $a_{\text{emit}}$ and $a_{\text{obs}}$ by

$$\frac{\nu_{\text{emit}}}{\nu_{\text{obs}}} \equiv 1 + z = \frac{a_{\text{obs}}}{a_{\text{emit}}}$$

This ability to use observed shifts in spectral lines to determine the size of the universe at the time the light was emitted, is probably the key result which enables the discipline of observational cosmology.

**2.** The spectrum of a distant object is determined to be due to Lyman-$\alpha$ line in Hydrogen with wavelength $\lambda = 1.21567 \times 10^7 m$. The spectrum was measured to have a wavelength of $\lambda_{obs} = 8.50969 \times 10^{-7} m$.

- **a.** What is the redshift of the object?
- **b.** In a spatially flat Universe, dominated by pressureless matter (Lecture 2), the Hubble parameter is given by $H = H_0/a^{3/2}$. Assuming a Hubble constant $H_0 = 70 km/s/Mpc$ what is
  - **i.** the co-moving distance of the object (in $Mpc$ and in $m$)?
  - **ii.** the angular-diameter distance of the object (in $Mpc$ and in $m$)?
  - **iii.** the luminosity distance of the object (in $Mpc$ and in $m$)?
- **c.** Repeat the above calculation for a spatially flat Universe with 30% pressureless matter and 70% cosmological constant (In that case $H = H_0\sqrt{0.3/a^3 + 0.7}$. See Lecture 2). You can use a mathematical software to perform the integral.

**3.** Objects of a given physical size $l$ are assumed to be perpendicular to our line of sight. If it subtends an angle $\Delta\theta$ (which is always small in astronomy because the distance scales are so large) then we define the Angular diameter distance $d_{\text{diam}}$ through

$$d_{\text{diam}} \equiv \frac{l}{\Delta\theta} \tag{1}$$

Recall the line element for the FRW universe

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right). \tag{2}$$

Start with an extended object of given transverse size $l$ situated at a comoving distance $r$ from us the observer. We are free to align the object as we want to, and so set $\phi = \text{const}$. Photons emitted from the object at time $t_{\text{em}}$ propagate along radial geodesics arriving today with an apparent angular separation $\Delta\theta$. The proper size of the object, $l$ is given by the interval between the emission events at the endpoints. In this case $dt = dr = d\phi = 0$ and so in the full metric we obtain

$$l = \sqrt{|ds^2|} = a(t_{\rm em})r\Delta\theta. \tag{3}$$

**3a.** Use the result in Eqn. (3) to show that the angular diameter distance and the luminosity distance can be related by the following equation (which holds in all curved spaces)

$$d_L = d_{\rm diam}(1+z)^2 \tag{4}$$

where $z$ corresponds to the redshift of the emitted photons.

**3b.** Recalling that

$$d_L = (1+z)\int_0^z \frac{dz'}{H(z')} \tag{5}$$

show for the case of a spatially flat matter dominated universe (i.e. no radiation or $\Lambda$ terms present) that

$$d_{\rm diam} = \frac{2H_0^{-1}}{(1+z)}\left(1 - \frac{1}{\sqrt{1+z}}\right) \tag{6}$$

**3c.** At what redshift is the angular diameter distance at its maximum value?
**3d.** Describe what happens to the angular diameter of objects as you go from low redshifts to large and explain why that is happening?

**4.** Derive the fluid equation (setting the speed of light $c = 1$ for convenience)

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{7}$$

from the acceleration and Friedmann equations

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a \tag{8}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{K}{a^2} \tag{9}$$

**5.** In this question (again setting $c = 1$) we will introduce the notion of conformal co-ordinates which can often make a problem easier to solve. In equations (8) and (9), change the variable from proper time $t$ to conformal time $\eta$ using $d\eta = \frac{dt}{a(t)}$ hence

$$\eta = \int^t \frac{dt}{a(t)}.$$

Show that the two equations become

$$a'' = \frac{4\pi G}{3}(\rho - 3p)a^3 - Ka \tag{10}$$

$$a'^2 = \frac{8\pi G}{3}\rho a^4 - Ka^2 \tag{11}$$

**6.** Show that $e^+e^-$ annihilation raises the photon temperature above that of neutrinos so that after the process is completed

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \tag{12}$$

**7.** Show that for massive neutrinos of mass $m_\nu$

$$\Omega_{0,\nu} = \frac{m_\nu}{94h^2 eV} \tag{13}$$

Hint: consider at which temperature neutrinos turn non-relativistic.

**8a.** Consider a universe with $K = \Lambda = 0$ and only one form of fluid with general equation of state $p = \omega\rho$ (where we have set the speed of light $c = 1$) and $\omega$ is a constant. By considering the fluid and Friedmann equations show that

$$a \propto t^{\frac{2}{3(1+\omega)}}.$$

**8b.** For what values of $\omega$ does the universe accelerate ?
**8c.** What happens to the solution as $\omega \to -1$?
**8d.** Now consider the case of $K \neq 0$ and $\Lambda = 0$. Show that for $3\omega \neq -1$ all the Friedmann models are solutions of the simple harmonic oscillator equation

$$\frac{d^2y}{d\eta^2} = -\frac{K(3\omega+1)^2 y}{4}$$

where $y = a^{(3\omega+1)/2}$ and $\eta$ is the conformal time defined by $d\eta = dt/a$.
**8e.** If $a(0) = 0$ sketch these solutions for $K = +1$.

**9.** This draws from tomorrow's lecture. But you can already have a go. The density contrast $\delta$ for matter evolves according to

$$\ddot{\delta} + 2H\dot{\delta} + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G\bar{\rho}\right)\delta = 0 \tag{14}$$

where $c_s^2$ is the sound speed (a constant) and $k$ is a (Fourier) wavenumber. In the lectures we found solutions to (14) assuming either matter or radiation domination. Find the solution to (14) on super-horizon scales during

- **a.** curvature domination assuming $\kappa < 0$.
- **b.** cosmological constant domination

and **c.** interpret your results

**10.** This question is for cosmological perturbation theory aficionados!
**(a)** Consider perturbed metric tensor in the conformal Newtonian gauge:

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(1+2\Psi) & 0 \\ 0 & (1-2\Phi)\gamma_{ij} + h_{ij}^{(T)} \end{pmatrix} \tag{15}$$

Find the inverse metric tensor $g^{\mu\nu}$ to linear order in the scalar modes $\Phi$, $\Psi$ and tensor mode $h_{ij}^{(T)}$.
Hint 1: You can raise the indices on $h_{ij}^{(T)}$ using the unperturbed spatial metric, i.e. $h^{(T)\,i}{}_{j} = \gamma^{ik} h_{kj}^{(T)}$.
Hint 2: Use the fact that scalar and tensor modes may be treated separately.

**(b)** Consider the 4-velocity of a fluid, $u_\mu$. The 4-velocity is assumed to be unit-timelike, i.e. $u_\mu u_\nu g^{\mu\nu} = -1$. Show that the background 4-velocity is $\bar{u}_\mu = a(1, \vec{0})$ and that the perturbed 0-component $\delta u_0$ is equal to $\delta u_0 = a\Psi$.

**(c)** Consider the equation for the tensor mode $h^{(T)}$ in the absence of shear:

$$h^{(T)\prime\prime} + 2\mathcal{H} h^{(T)\prime} + k^2 h^{(T)} = 0 \tag{16}$$

where $\mathcal{H} = \frac{a'}{a}$ and primes denote derivatives with respect to conformal time $\tau$ (related to $t$ by $dt = a d\tau$).

- Solve this equation on super-horizon scales in the matter and in the radiation eras.
- Explain briefly how you expect the solution to look like on sub-horizon scales and why (without actually solving the equation)