Cosmology

Lecture 2: Dynamics of the Universe and Thermal History

CENTRAL EUROPEAN INSTITUTE FOR COSMOLOGY AND FUNDAMENTAL PHYSICS

Institute of Physics of the Czech Academy of Sciences

EUROPEAN UNION
European Structural and Investment Funds
Operational Programme Research, Development and Education

MINISTRY OF EDUCATION, YOUTH AND SPORTS

Constantinos Skordis — IDPASC, 19 Sep 2023

Friedman-Lemaître-Robertson-Walker (FLRW) spacetime: the geometry of a homogeneous-isotropic universe

Conventionally $a = 1$ Today

Scale factor – determined by matter content

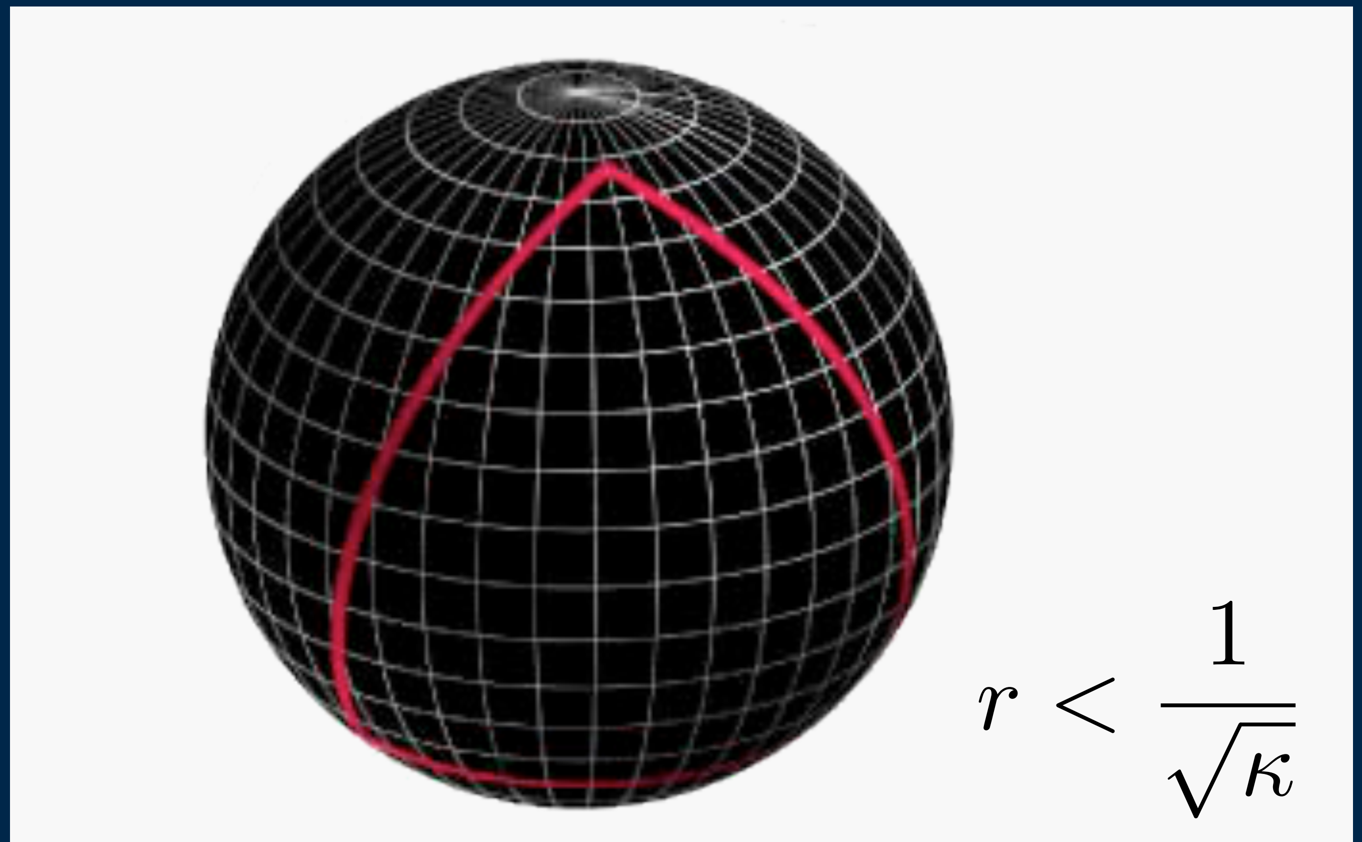
$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

(Spherical coordinates)

Spatial curvature

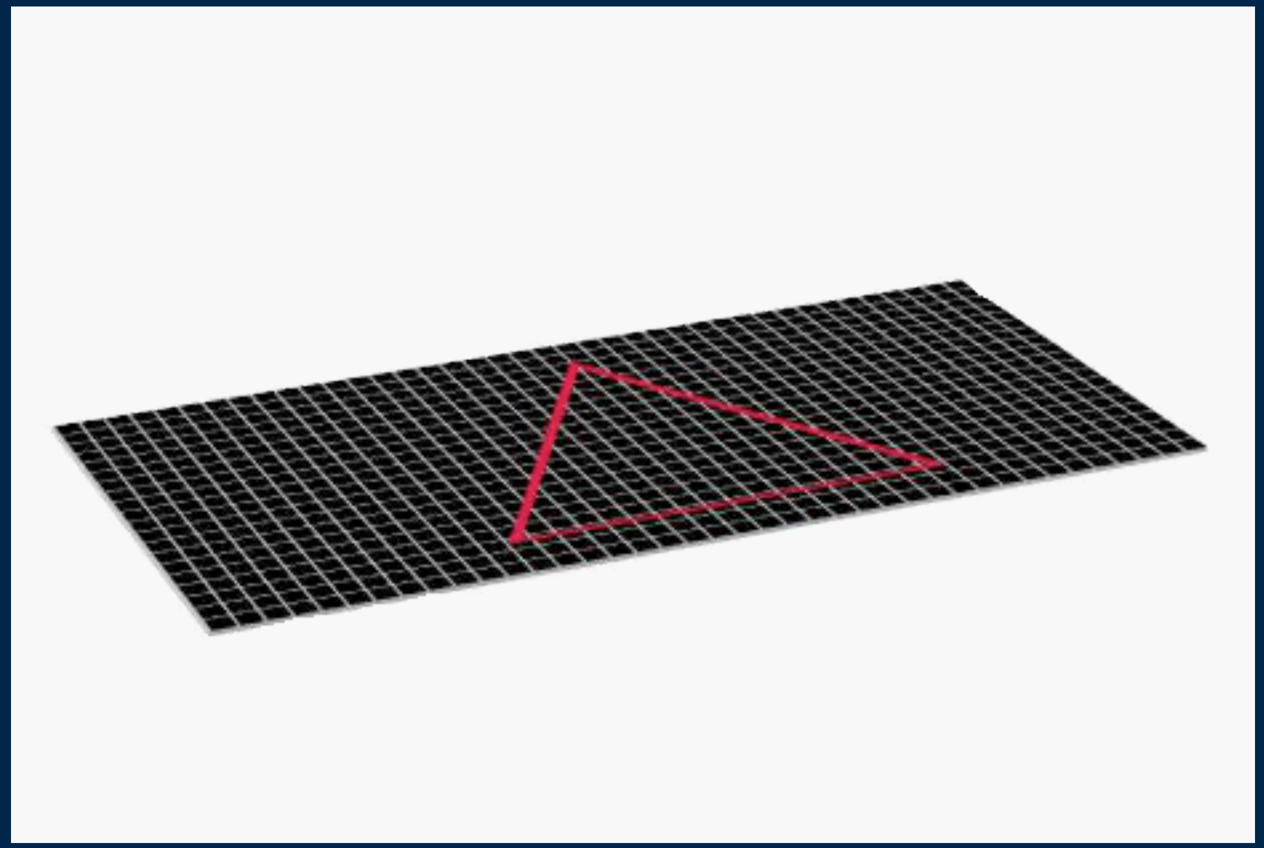
$\kappa > 0$

Positively curved (e.g. 3-sphere)



$\kappa = 0$

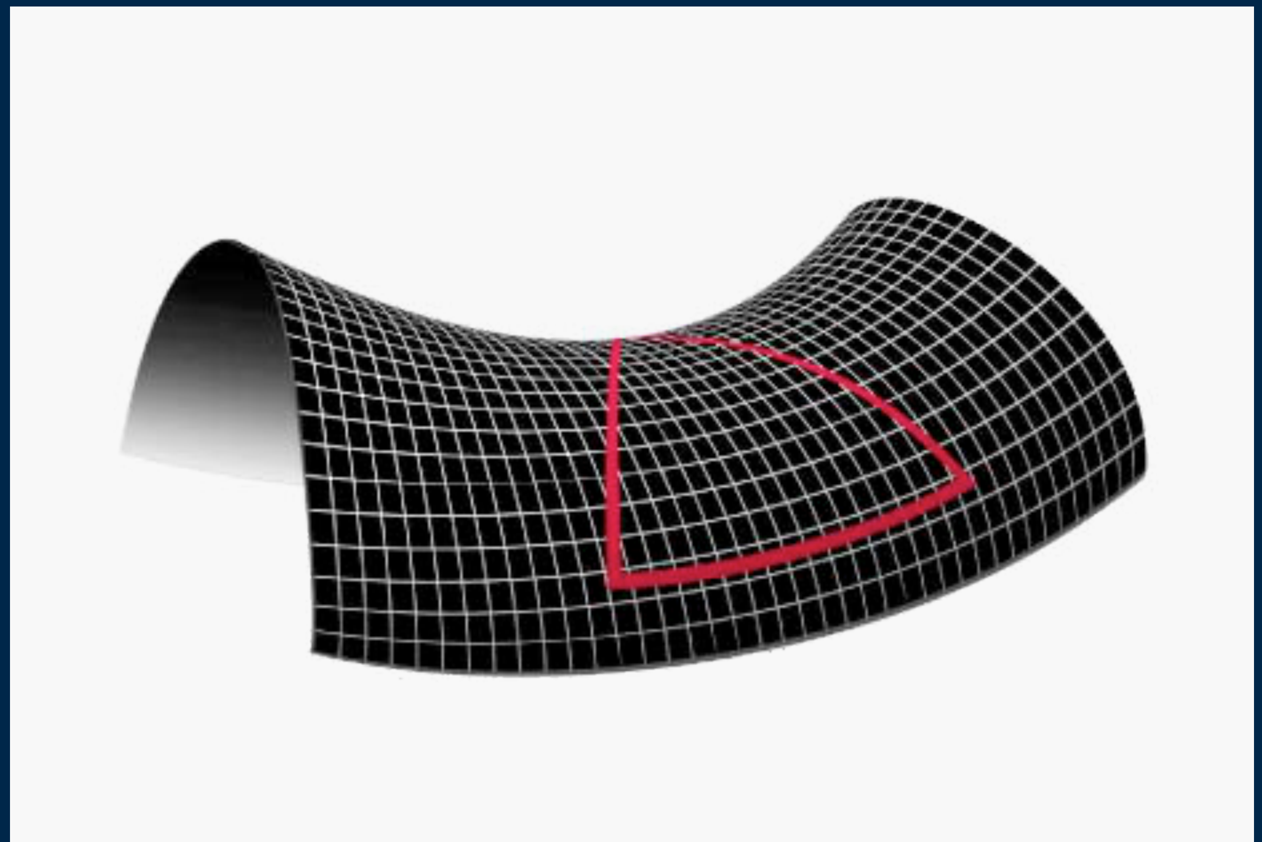
Spatially flat (Euclidean)



Preferred by the data

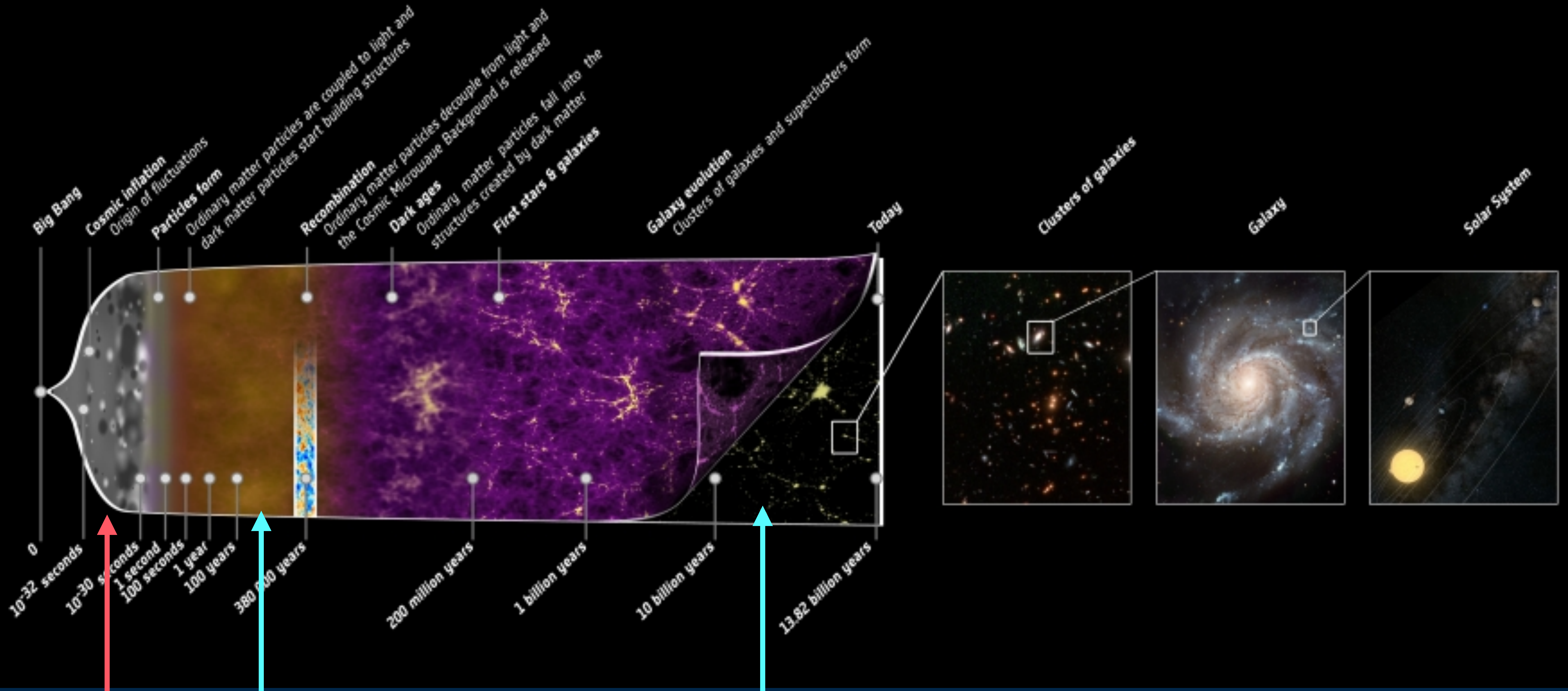
$\kappa < 0$

Negatively curved (e.g. hyperbolic 3-space)



Dynamics of homogeneous-isotropic Universe

How to determine $a(t)$



Radiation era

Matter era

Λ Era

$t \sim 41000yr$

$t \sim 9.94Gyr$

General Relativity — Gravity as geometry of spacetime

Einstein equations

Cosmological constant

Einstein tensor → $G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$

Matter stress-energy tensor

Schematically: $g_{\mu\nu} \xrightarrow{\partial} \Gamma^{\mu}_{\alpha\beta} \xrightarrow{\partial} R_{\mu\nu} \rightarrow R \rightarrow G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

For any metric

$\nabla_\mu G^{\mu}{}_{\nu} = 0$

$\nabla_\rho g_{\mu\nu} = 0$

→ Compatible with $\nabla_\mu T^{\mu}{}_{\nu} = 0$ → Stress-energy conservation

Spatially flat FLRW: $\kappa = 0$

(Cartesian coordinates)

Metric:

$$g_{\mu\nu}$$

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}$$

Christoffel symbols

$$\Gamma^{\alpha}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right)$$

FLRW

$$\Gamma^{i}_{0j} = H \delta^{i}{}_{j} \qquad \Gamma^{0}_{ij} = H g_{ij}$$

Hubble parameter

$$H(t) \equiv \frac{\dot{a}}{a}$$

Non-zero components

Ricci curvature

$$R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\alpha}_{\alpha\lambda} \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu} \Gamma^{\beta}_{\mu\alpha}$$

$$R_{00} = -3\dot{H} - 3H^2 \qquad R_{ij} = \left[\dot{H} + 3H^2 \right] g_{ij}$$

Scalar curvature

$$R \equiv g^{\mu\nu} R_{\mu\nu}$$

$$R = 6 \left[\dot{H} + 2H^2 \right]$$

Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ (Includes cosmological constant)

$$T_{00} = \bar{\rho}(t)$$

$$T_{ij} = \bar{P}(t) g_{ij} = \bar{P}(t) a^2 \gamma_{ij}$$

Friedman eq. $$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \sum_I \bar{\rho}_I - \frac{\kappa}{a^2}$$

Energy density = sum over all matter species

Spatial curvature

Raychaudhury eq. $$-2\frac{\ddot{a}}{a} - H^2 = -2\dot{H} - 3H^2 = 8\pi G \sum_I \bar{P}_I - \frac{\kappa}{a^2}$$

Pressure = sum over all matter species

Energy conservation $\nabla_\mu T^\mu{}_\nu = 0$ → $$\dot{\bar{\rho}}_I + 3H\left(\bar{\rho}_I + \bar{P}_I\right) = 0$$

I Labels species: e.g. baryons, dark matter, photons, neutrinos, cosmological constant, etc

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \sum_I \bar{\rho}_I - \frac{\kappa}{a^2}$$

Conventionally $a = 1$ Today

$$\bar{\rho}_I(a=1) = \bar{\rho}_{0,I}$$

Critical density: $\bar{\rho}_{\rm crit} \equiv \frac{3H^2}{8\pi G}$

Relative density: $\Omega_I \equiv \frac{\bar{\rho}_I}{\bar{\rho}_{\rm crit}} = \frac{8\pi G \bar{\rho}_I}{3H^2}$

Today $\Omega_0 \equiv \sum_I \Omega_{0,I}$ → $\kappa = (\Omega_0 - 1)\, H_0^2 \quad = \Omega_K H_0^2$

Negligible curvature $\Omega_K = -0.0106 \pm 0.0065$ CMB (Planck 2018)

Spatially flat

$$\sum_I \Omega_I = 1$$

Equation of state: a relation between density and pressure

$$\bar{P} = w\bar{\rho}$$

↓ Equation of state parameter

$$\dot{\bar{\rho}}_I + 3H\left(\bar{\rho}_I + \bar{P}_I\right) = 0$$

↓ Solve

$w = 1$	Stiff fluid, massless scalar field	$\rho_{\rm stiff} \propto a^{-6}$
$w = \frac{1}{3}$	Radiation (photons, massless neutrinos, gravitational waves, …)	$\rho_r \propto a^{-4}$
$w = 0$	Pressureless matter, also called dust (cool baryons, cold dark matter, galaxies, …)	$\rho_m \propto a^{-3}$
$w = -\frac{1}{3}$	Spatial Curvature, or cosmic strings	$\rho_{\rm strings} \propto a^{-2}$
$w = -\frac{2}{3}$	Domain walls	$\rho_{\rm walls} \propto a^{-1}$
$w = -1$	Cosmological constant	$\rho_\Lambda = \frac{\Lambda}{8\pi G} = \text{constant}$

Conventionally $a = 1$ Today

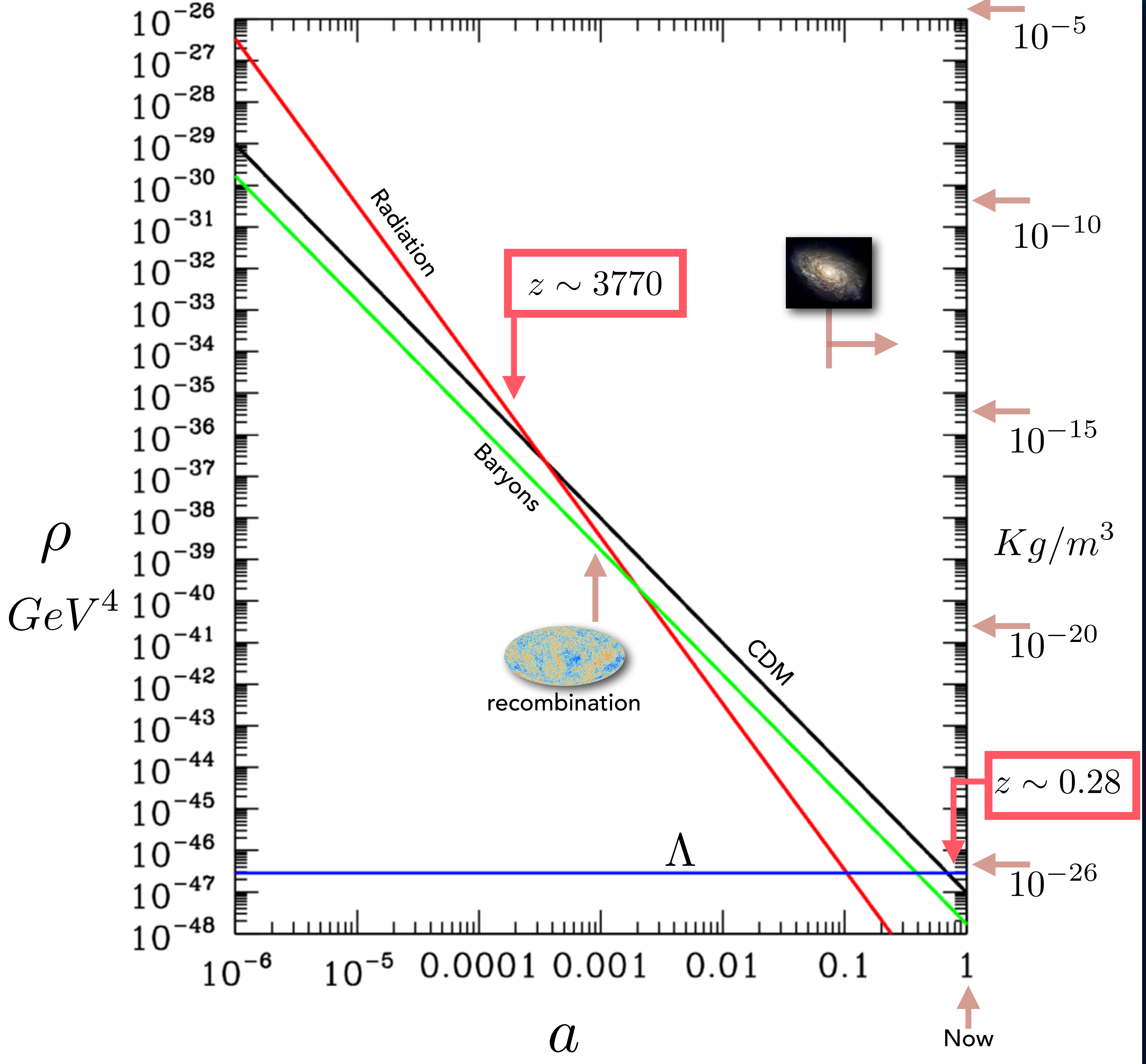
Simple solutions

Friedman eq. $$3\frac{\dot{a}^2}{a^2} = 8\pi G\bar{\rho}$$ → $a(t)$

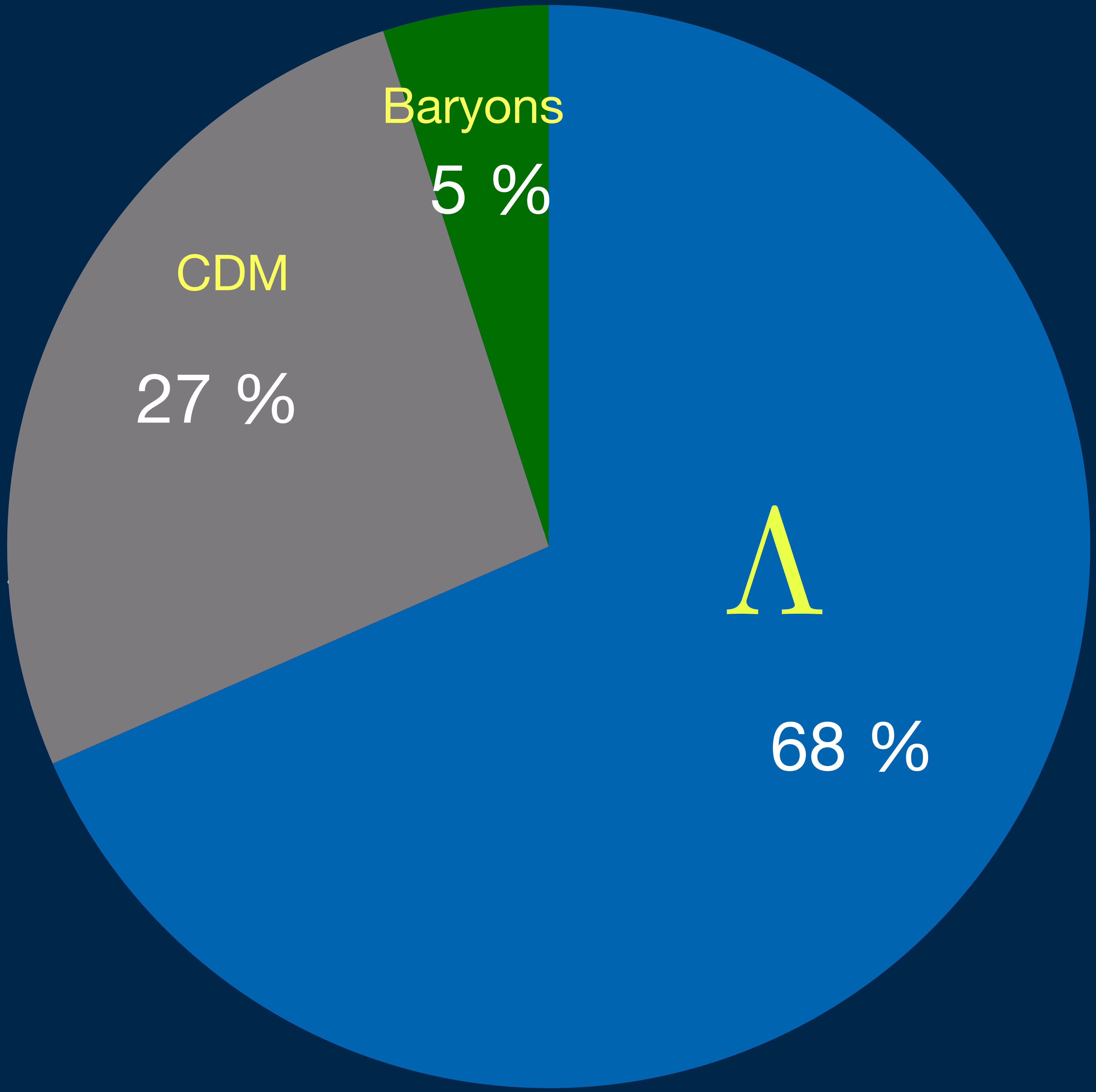
Today: $3H_0^2 = 8\pi G\bar{\rho}_0$

Stiff fluid	$w = 1$	$\bar{\rho}_{\rm stiff} = \bar{\rho}_0 a^{-6}$	$a = (3H_0 t)^{1/3}$
Radiation	$w = \frac{1}{3}$	$\bar{\rho}_r = \bar{\rho}_0 a^{-4}$	$a = \sqrt{2H_0 t}$
Pressureless matter	$w = 0$	$\bar{\rho}_m = \bar{\rho}_0 a^{-3}$	$a = \left(\frac{3}{2}H_0 t\right)^{2/3}$
Cosmological constant	$w = -1$	$\bar{\rho}_\Lambda = \bar{\rho}_0$	$a = e^{H_0 t}$

Expansion eras



Material content of the Universe



Hubble constant
$$\left\{ \begin{array}{ll} H_0 = 67.36 \pm 0.54 & \text{CMB (Planck 2018)} \\ H_0 = 73.04 \pm 1.04 & \text{Local (SH0ES 2021)} \end{array} \right\}$$
"Hubble tension"

Critical density $\bar{\rho}_{crit,0} = \frac{3H_0^2}{8\pi G} = 1.878 \times 10^{-26} h^2 \text{ kg m}^{-3}$

Negligible curvature $\Omega_K = -0.0106 \pm 0.0065$ CMB (Planck 2018)

Assuming flatness, CMB (Planck 2018)

Matter density	$\Omega_{0,m} = 0.3153 \pm 0.0073$
Cosmological constant	$\Omega_{0,\Lambda} = 0.6847 \pm 0.0073$
Cold dark matter	$\Omega_{0,c}h^2 = 0.1202 \pm 0.0014$ $\Omega_{0,c} \sim 0.2658$
Baryons	$\Omega_{0,b}h^2 = 0.02236 \pm 0.00015$ $\Omega_{0,b} \sim 0.0494$
Photons	$\Omega_{0\gamma} \sim 5.04 \times 10^{-5}$

Neutrinos $3.43 \times 10^{-5} \lesssim \Omega_{0\nu} \lesssim 0.01$

Expansion age of the Universe

$$a_{eq}^{r,m} = \frac{\Omega_{0r}}{\Omega_{0m}}$$

$$a_{eq}^{m,\Lambda} = \left(\frac{\Omega_{0m}}{\Omega_{0\Lambda}}\right)^{1/3}$$

$$a(t) = \sqrt{2H_0\sqrt{\Omega_{0r}}t}$$

$$a = \left(\frac{3}{2}H_0\sqrt{\Omega_{0m}}t\right)^{2/3}$$

$$a = e^{H_0\sqrt{\Omega_{0\Lambda}}t}$$

Radiation era

Matter era

Λ Era

$$a_{eq}^{r,m} \sim 2.6 \times 10^{-4}$$

$$z_{eq}^{r,m} \sim 3770$$

$$a_{eq}^{m,\Lambda} \sim 0.78$$

$$z_{eq}^{m,\Lambda} \sim 0.28$$

Beware: This is rather approximate

$$t = \frac{a^2}{2H_0\sqrt{\Omega_{0r}}}$$

$$t = t_{eq} + \frac{2a^{3/2}}{3H_0\sqrt{\Omega_{0m}}}$$

$$t = t_{eq,\Lambda} + \frac{\ln a}{H_0\sqrt{\Omega_{0\Lambda}}}$$

$t \sim 53155yr$

$t \sim 11.3\text{Gyr}$

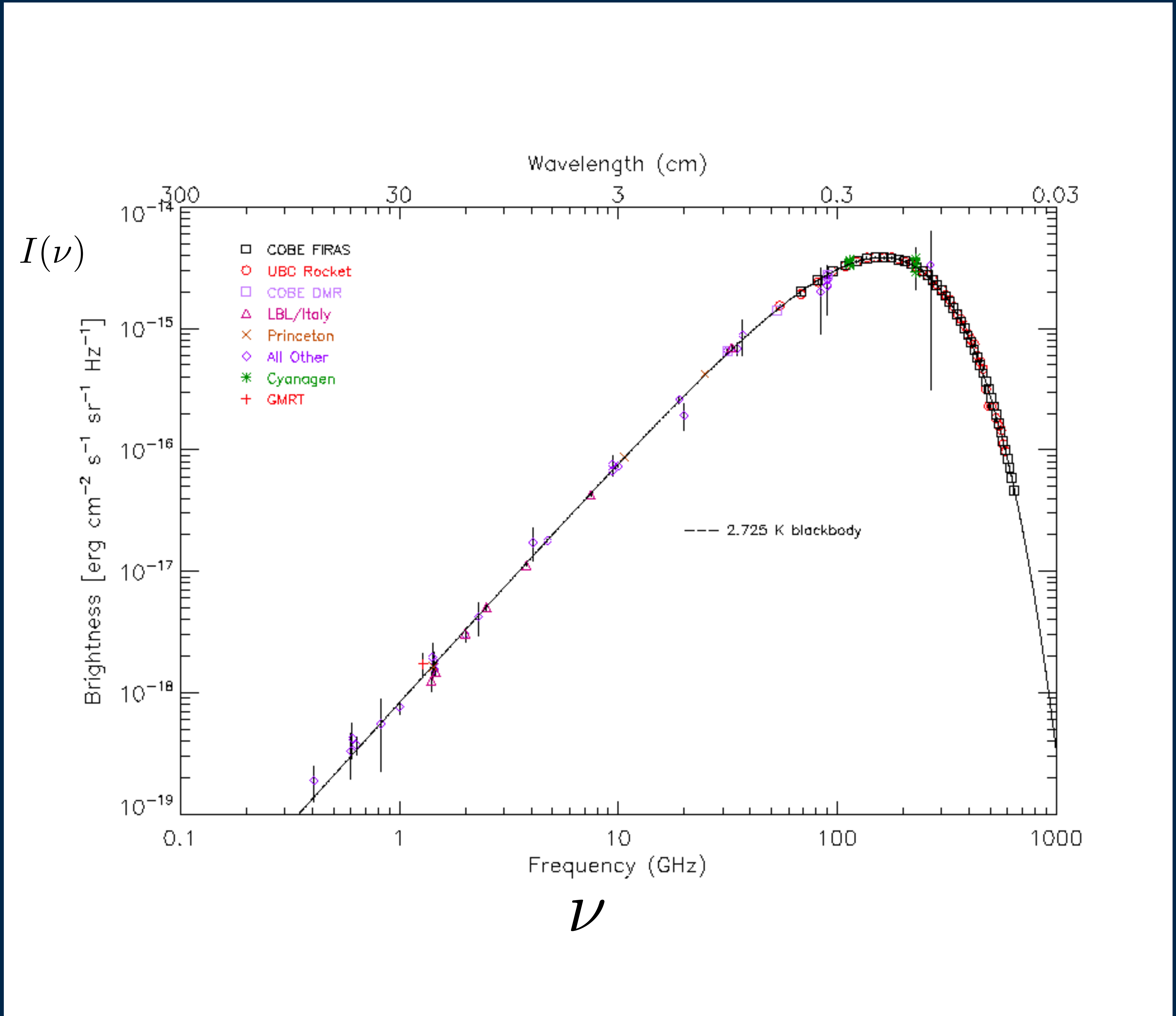
$t_0 \sim 15.5Gyr$

Accurate way:

$$t(a_{ref}) = \frac{1}{H_0}\int_0^{a_{ref}} \frac{da}{a\sqrt{\frac{\Omega_{0r}}{a^4} + \frac{\Omega_{0m}}{a^3} + \Omega_{0\Lambda}}}$$

→ $t \sim 9.94Gyr$ $t_0 \sim 13.205Gyr$

Thermal history of the Universe



Cosmic Microwave Background

$$\bar{T}_{CMB} = 2.7255K$$

Best Black body spectrum in the Universe

$$I(\nu) = \frac{4\pi\nu^3}{e^{2\pi\nu/T} - 1}$$

CMB at thermal equilibrium today

Expansions preserves Black Body spectrum

CMB at thermal equilibrium in the past

+ anything interacting with it.

Today	$z = 0$	$a = 1$	$t_0 \sim 13.7Gyr$	$T_{cmb} \sim 2.7255K$	Λ domination
Distant galaxies	$z = 5$	$a \sim 0.17$	$t \sim 1.15Gyr$	$T_{cmb} \sim 16K$	Matter domination
Photon decoupling	$z \sim 1100$	$a \sim 9 \times 10^{-4}$	$t \sim 350000yr$	$T_{cmb} \sim 3000K$	
		CMB anisotropies			
Matter-radiation equality	$z \sim 3700$	$a \sim 2.6 \times 10^{-4}$	$t \sim 41000yr$	$T_{cmb} \sim 10000K \sim eV$	
Nucleosynthesis ends	$z \sim 3 \times 10^8$	$a \sim 3 \times 10^{-9}$	$t \sim 200 - 300s$	$T \sim 0.05MeV$	
		Nuclear reactions efficient: formation of light elements			
e^+e^- annihilation	$z \sim 5 \times 10^9$	$a \sim 2 \times 10^{-10}$	$t \sim 1s$	$T \sim 0.5MeV$	Radiation domination
Neutrino decoupling n/p freeze-out	$z \sim 10^{10}$	$a \sim 10^{-10}$	$t \sim 0.2s$	$T \sim 1 - 2MeV$	
Quark-gluon phase transition	$z \sim 5 \times 10^{12}$	$a \sim 2 \times 10^{-13}$	$t \sim 10^{-5}$	$T \sim 200MeV$	
Electroweak unification		$a \sim 2 \times 10^{-16}$	$t \sim 10^{-12}s$	$T \sim 100GeV$	?
GUT??		$a \sim 10^{-28}$	$t \sim 10^{-36}s$	$T \sim 10^{15}GeV$	Inflation: Scalar field domination $V(\phi) \sim \Lambda_{inf}$
Quantum gravity??			$t \sim 10^{-43}s$ Graviton decoupling		

Particles at equilibrium

Distribution function: $f(t, \vec{x}, \vec{p}) \rightarrow f(t, p)$

Physical momentum

Describes collection of particles of given energy at time t

$$E(p) = \sqrt{p^2 + m^2}$$

Degrees of freedom

$n = n(T)$ Number density

$$n = \frac{g}{2\pi^2} \int dp \frac{p^2}{e^{E(p)/T} \pm 1}$$

$+$ Fermions
$-$ Bosons

$\rho = \rho(T)$ Energy density

$$\rho = \frac{g}{2\pi^2} \int \frac{dp p^2}{e^{E(p)/T} \pm 1} E(p)$$

$P = P(T)$ Pressure

$$P = \frac{g}{2\pi^2} \int \frac{dp p^2}{e^{E(p)/T} \pm 1} \frac{p^2}{3E(p)}$$

Remember

$$T = \frac{T_0}{a}$$

1st law of thermodynamics

$$TdS = dE + PdV$$

$S = S(T, V)$ Entropy

Energy $E = \rho V$ = energy density x volume

$\rho = \rho(T)$

Pressure $P = P(T)$

Volume $V = a^3$

$$\frac{\partial S}{\partial T} = \frac{V}{T}\frac{\partial \rho}{\partial T}$$

$$\frac{\partial S}{\partial V} = \frac{\rho + P}{T}$$

Consistency: $\frac{\partial}{\partial V}\frac{\partial S}{\partial T} = \frac{\partial}{\partial T}\frac{\partial S}{\partial V}$ → $dP = \frac{\rho + P}{T} dT$

$$dS = d\left[\frac{(\rho + P)V}{T}\right] = 0$$

Entropy is conserved for equilibrium processes

→ Entropy density $s = \frac{\rho + P}{T}$

Relativistic species $m \ll T$ $E(p) \approx p$

Number density → $n = \frac{gT^3}{2\pi^2} \int_0^\infty \frac{x^2}{e^x \pm 1}$

$n_B = \frac{g\zeta(3)}{\pi^2} T^3$ Bosons

$n_F = \frac{3}{4} n_B$ Fermions

Energy density → $\rho = \frac{gT^4}{2\pi^2} \int_0^\infty \frac{x^3 dx}{e^x \pm 1}$

$\rho_B = \frac{\pi^2 g}{30} T^4$ Bosons

$\rho_F = \frac{7}{8} \rho_B$ Fermions

Pressure → $P = \frac{1}{3} \rho$

Entropy density → $s = \frac{4}{3} \frac{\rho}{T} \propto n$

Relativistic degrees of freedom

Several relativistic species add together to the radiation energy density — Relevant only during the radiation era

— Neutrinos are then approximately massless

Effective massless degrees of freedom

$$\rho_r = \frac{\pi^2 g_*}{30} T_\gamma^4 \qquad g_* = \sum_{bosons,I} g_I \left(\frac{T_I}{T_\gamma}\right)^4 + \frac{7}{8} \sum_{fermions,I} g_I \left(\frac{T_I}{T_\gamma}\right)^4$$

$T \gtrsim 300 GeV$ All standard model particles are relativistic: $g_* \approx 106.75$

$1 MeV \leq T \leq 100 MeV$ Relativistic species are: $\gamma, 3 \times \nu, e^\pm$ ($m_{e^\pm} \sim 0.5 MeV$) $g_* = 2 + \frac{7}{8} \times 3 \times 2 + \frac{7}{8} \times 2 \times 2 = 10.75$

$T \ll 1 MeV$ Relativistic species are: $\gamma, 3 \times \nu$ $g_* = 2 + \frac{7}{8} \times 3 \times 2 \times \left(\frac{4}{11}\right)^{4/3} \approx 3.36$

$N_{eff,\nu} \approx 3.0395$ Due to non-instantaneous neutrino decoupling + QED effects at finite T, see Magnano et al., Phys.Lett. B534, 8 (2002)

$g_* \sim 3.38$

(Neutrinos become non-relativistic inside the matter era)

Neutrino decoupling

Neutrinos at equilibrium with other species $\bar{\nu}\nu \leftrightarrow e^+e^-$ $\nu e \leftrightarrow \nu e$

Interaction rate $\Gamma = n\langle\sigma v\rangle \sim T^3(G_F T^2) \sim G_F^2 T^5$

Neutrino decoupling when $\Gamma < H$

(This means that the expansion is faster than the interaction rate and prevents reactions above from occurring efficiently)

$$\frac{\Gamma}{H} \sim \frac{G_F^2 T^5}{\sqrt{8\pi G g_* T^4/3}} \sim \frac{G_F^2}{\sqrt{8\pi G g_*/3}} T^3 \sim \left(\frac{T}{1MeV}\right)^3$$

Neutrino decoupling when $T < 1MeV$

Neutrinos maintain the same temperature as the other species but otherwise mind their own business

e^+e^- annihilation — Neutrino temperature

Below $T \sim 0.5 MeV$ $e^+e^- \rightarrow \gamma$ Heat the photons

Use entropy conservation

$$S_{before} = S_{after}$$

$$\rightarrow s_b a_b^3 = s_a a_a^3$$

Before					
$s \rightarrow$	γ	ν	$\bar{\nu}$	e^+	e^-
	2	3	3	2	2

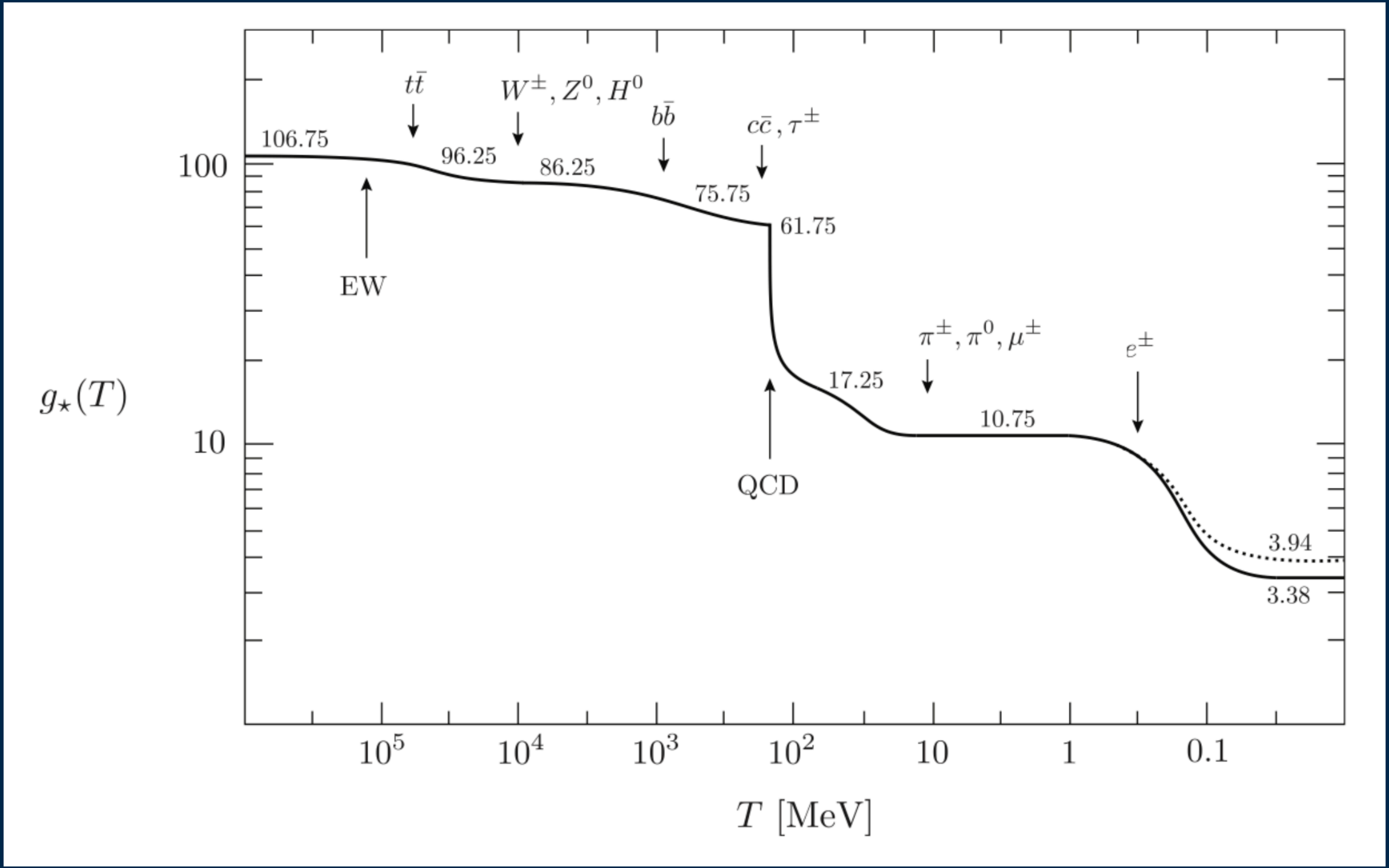
After			
$s \rightarrow$	γ	ν	$\bar{\nu}$
	2	3	3

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

$$T_{\nu,\text{today}} \sim 1.94537K$$

Relativistic degrees of freedom

From Julien Baur, PhD Thesis, see https://inspirehep.net/literature/1765226



Non-relativistic species $m \gg T$ $E(p) \approx m$

Number density → $n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}$

Energy density → $\rho = nm$

Pressure → $P = nT \ll \rho$ $(P \approx 0)$

e.g. massive neutrinos

$$\Omega_\nu = \frac{m_\nu}{94h^2 eV}$$

Gershtein, Zel'dovich (1966)

Marx & Szalay (1972)

Cowsik & McClelland (1972)

For $\Omega_\nu < 1$ $m_\nu \lesssim 45eV$

Summary

Friedman equation $H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\sum_I \bar{\rho}_I - \frac{\kappa}{a^2}$

Energy conservation $\dot{\bar{\rho}}_I + 3H\left(\bar{\rho}_I + \bar{P}_I\right) = 0$

Relative density: $\Omega_I \equiv \frac{\bar{\rho}_I}{\bar{\rho}_{\rm crit}} = \frac{8\pi G\bar{\rho}_I}{3H^2}$

Today: $\Omega_\Lambda \sim 0.68$ $\Omega_c \sim 0.27$ $\Omega_b \sim 0.05$

Expansion age $t(a_{ref}) = \frac{1}{H_0}\int_0^{a_{ref}} \frac{da}{a\sqrt{\frac{\Omega_{0r}}{a^4} + \frac{\Omega_{0m}}{a^3} + \Omega_{0\Lambda}}}$

Thermodynamics: entropy conserved

$m \ll T$ Relativistic species: $n \sim T^3$ $\rho \sim T^4$ $P = \frac{1}{3}\rho$ $s \sim T^3$

Fermions $\times\frac{3}{4}$ $\times\frac{7}{8}$ relative to bosons