Cosmology exercise session – Solutions - IDPASC 2023

1. Suppose we have two observers A and B separated by coordinate distance r_{AB} (i.e. the comoving distance). A emits a light pulse at time t_{emit} which reaches B at time t_{obs}. Now we know light rays are null, hence they satisfy $ds^2 = 0$ along the path of a photon. It follows that

$$\int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{dt}{a(t)} = \int_0^{r_{AB}} \frac{dr}{\sqrt{1-Kr^2}}.$$

Now imagine A emits a second pulse at time $t_{\text{emit}} + \delta t_{\text{emit}}$ which arrives at B at time $t_{\text{obs}} + \delta t_{\text{obs}}$, It travels the same comoving distance hence we have

$$\int_{t_{\text{emit}}+\delta t_{\text{emit}}}^{t_{\text{obs}}+\delta t_{\text{obs}}} \frac{dt}{a(t)} = \int_0^{r_{AB}} \frac{dr}{\sqrt{1-Kr^2}} = \int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{dt}{a(t)}$$

Hence it follows that for the case where the scale factor does not evolve much during the period between the first and second pulse being emitted (such as a wavelength of light) then

$$\frac{\delta t_{\text{obs}}}{a(t_{\text{obs}})} = \frac{\delta t_{\text{emit}}}{a(t_{\text{emit}})}$$

In particular considering the case where the time difference corresponds to say one wavelength, we can say that light of freq ν_{emit} at A will be detected with frequency ν_{obs} at B where

$$\frac{\nu_{\text{obs}}}{\nu_{\text{emit}}} = \frac{\delta t_{\text{emit}}}{\delta t_{\text{obs}}} = \frac{a(t_{\text{emit}})}{a(t_{\text{obs}})}$$

Thus the light received by B is redshifted

$$1+z = \frac{a(t_{\text{obs}})}{a(t_{\text{emit}})}, \qquad z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{\nu_{\text{emit}} - \nu_{\text{obs}}}{\nu_{\text{obs}}}.$$

2.

- **a.** We have that redshift is equal to

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{8.50969 - 1.21567}{1.21567} = 6 \tag{1}$$

  corresponding to scale factor $a = 1/7 \approx 0.142857$.

- **b.** We have that $H = H_0/a^{3/2} = H_0(1+z)^{3/2}$ and $H_0 = 70km/s/Mpc = 3.336 \times 10^{-4} Mpc^{-1}$.
  - **i.** The comoving distance is

$$\begin{aligned} d_{com} &= \int_0^z \frac{dz}{H} = \frac{1}{H_0}\int_0^6 (1+z)^{-3/2} dz \approx \frac{1.24407}{H_0} \\ &\approx \frac{1.24407}{3.336 \times 0.7 \times 10^{-4}} Mpc \approx 5.327 Gpc \approx 1.6434 \times 10^{26} m \end{aligned} \tag{2}$$

– **ii.** The angular-diameter distance of the object is

$$d_A = \frac{d_{com}}{1+z} = \frac{1}{7} d_{com} \approx 761 Mpc \approx 2.3 \times 10^{25} m \tag{3}$$

– **iii.** The luminosity distance of the object is

$$d_L = \frac{d_{com}}{1+z} = (1+z)^2 d_A = 7 d_{com} \approx 37.2 Gpc \approx 1.15 \times 10^{27} m \tag{4}$$

- **c.** Here, $H = H_0 \sqrt{0.3(1+z)^3 + 0.7}$.

– **i.** The comoving distance is

$$d_{com} = \int_0^z \frac{dz}{H} = \frac{1}{H_0} \int_0^6 \frac{1}{\sqrt{0.3(1+z)^3 + 0.7}} dz \approx \frac{1.92561}{H_0} = 1.17172 d_{com}^{\Omega_\Lambda = 0}$$
$$\approx 1.17172 \times 5.327 Gpc \approx 8.25 Gpc \approx 2.5 \times 10^{26} m \tag{5}$$

– **ii.** The angular-diameter distance of the object is

$$d_A = 1.17172 d_A^{\Omega_\Lambda = 0} \sim 891 Mpc = 2.69 \times 10^{25} m \tag{6}$$

– **iii.** The luminosity distance of the object is

$$d_L = 1.17172 d_L^{\Omega_\Lambda = 0} = 43.5879 Gpc \approx 2.3 \times 1.3410^{27} m \tag{7}$$

3a. We find the angular diameter distance as

$$d_A = a(t_{em}) r \tag{8}$$

where $a_{em} = a(t_{em})$ is the scale factor at emission. Meanwhile the luminosity distance for a source of intrinsic luminosity L_s is

$$d_L = \sqrt{\frac{L_s}{4\pi \mathcal{F}}} \tag{9}$$

where $\mathcal{F}$ is the observed flux. What we are after is the luminosity distance in an expanding Universe compared to the one in a static Universe. The latter, would be equal to r_{com}. Now, luminosity is Energy per unit time, that is $\delta E / \delta t$. We saw in the lectures that energy scales as $E_{em} = E_{obs}/a(t_{em})$ and we also saw from exercise 1 that $\delta t_{em} = a(t_{em}) \delta t_{obs}$ (assuming $a = 1$ today). So by the time light comes to us as observers, the luminosity of the source would appear to be

$$L_{obs} = L_s a^2(t_{em}) \tag{10}$$

which we may associate with the luminosity distance in case of a static Universe. So

$$d_L = \sqrt{\frac{L_s}{4\pi \mathcal{F}}} = \frac{1}{a} \sqrt{\frac{L_{obs}}{4\pi \mathcal{F}}} \tag{11}$$
$$= \frac{r_{com}}{a} \tag{12}$$

Eliminating r_{com} we then find the desired relation

$$d_L = \frac{d_A}{a^2} = d_A (1+z)^2 \tag{13}$$

3b. For matter domination $H = H_0(1+z)^{3/2}$. Hence,

$$d_A = \frac{1}{H_0}\frac{1}{1+z}\int_0^z (1+z')^{3/2} dz' \tag{14}$$

$$= \frac{2}{H_0(1+z)}\left(1 - \frac{1}{\sqrt{1+z}}\right) \tag{15}$$

3c. Setting to zero the derivative of the equation in 3b gives that the angular diameter distance in a purely matter dominated universe reaches a maximum at redshift $z = 5/4$.

3d. The Universe is expanding.

4. In the following three questions we set $c = 1$ for convenience. To obtain the fluid equation, first differentiate the Friedmann equation to yield

$$\frac{2\dot{a}\ddot{a}}{a^2} - \frac{2\dot{a}^3}{a^3} = \frac{8\pi G}{3}\dot{\rho} + \frac{2\kappa\dot{a}}{a^3}.$$

Rearrange to obtain

$$\frac{2\dot{a}}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\kappa}{a^2}\right) = \frac{8\pi G}{3}\dot{\rho} \tag{16}$$

Recall the acceleration (17) and Friedmann (18) equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{17}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \tag{18}$$

Sub into (16) to obtain

$$\frac{2\dot{a}}{a}\left(-\frac{4\pi G}{3}(\rho + 3p) - \frac{8\pi G}{3}\rho\right) = \frac{8\pi G}{3}\dot{\rho}$$

Finally simplify to obtain the required result

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{19}$$

5. In terms of conformal time η, we have

$$\frac{da}{dt} = \frac{da}{d\eta}\frac{d\eta}{dt} = \frac{a'}{a} \tag{20}$$

where $a' \equiv \frac{da}{d\eta}$ and $\frac{d\eta}{dt} = \frac{1}{a}$. Similarly we have

$$\ddot{a} = \frac{d(\dot{a})}{dt} = \frac{d(\dot{a})}{d\eta}\frac{d\eta}{dt} = \frac{a''}{a^2} - \frac{a'^2}{a^3}. \tag{21}$$

For the acceleration equation, sub (21) into (17) and use (20) and (18) to eliminate the $\frac{a'^2}{a^4}$ term. The answer then follows:

$$a'' = \frac{4\pi G}{3}(\rho - 3p)a^3 - \kappa a \tag{22}$$

For the Friedmann equation, simply sub (20) into (18) and the answer pops out:

$$a'^2 = \frac{8\pi G}{3}\rho a^4 - \kappa a^2 \tag{23}$$

6. Before e^+e^- annihilation, the relevant radiation species are photons (2 dof), 3 neutrino flavous (1 dof each), 3 anti-neutrino flavous (1 dof each), positrons (2 dof each) and electrons (2 dof each), for a total of 10 fermionic and 2 bosonic dof. Hence, the entropy density before s_b is

$$s_b = \frac{4}{3}\sum_I \frac{\rho_I}{T_I} = \frac{4}{3}\frac{\pi^2}{30}\left(2 + \frac{7}{8} * 10\right) T_\nu^3 \tag{24}$$

$$= \frac{43\pi^2}{90} T_\nu^3 \tag{25}$$

where I runs over all the species above and where all species have the same temperature, which we choose to be the neutrino temperature T_ν.

After the annihilation, he relevant radiation species are photons (2 dof), 3 neutrino flavous (1 dof each) and 3 anti-neutrino flavous (1 dof each), for a total of 6 fermionic and 2 bosonic dof. But, now while the neutrino temperature is unchanged by the annihilation, and remains at T_ν, the photon thermal bath is heated by the annihilation so that the new photon temperature is T_γ.

Hence, the entropy density after s_a is

$$s_b = \frac{4}{3}\sum_I \frac{\rho_I}{T_I} = \frac{4}{3}\frac{\pi^2}{30}\left(2T_\gamma^3 + \frac{7}{8} * 6\, T_\nu^3\right) \tag{26}$$

$$= \frac{4\pi^2}{90}\left[2\left(\frac{T_\gamma}{T_\nu}\right)^3 + \frac{21}{4}\right] T_\nu^3 \tag{27}$$

Since, the entropy is concerved, and since the annihilation is assumed to be instantaneous in redshift, then $S_b = S_a$, leading to

$$\frac{43}{4} = 2\left(\frac{T_\gamma}{T_\nu}\right)^3 + \frac{21}{4} \tag{28}$$

hence,

$$\left(\frac{T_\gamma}{T_\nu}\right)^3 = \frac{11}{4} \tag{29}$$

hence

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \tag{30}$$

7. At the beginning massive neutrinos were relativistic ($T_\nu \gg m$), hence, for one neutrino/anti-neutrino flavour

$$n_\nu = \frac{3}{4}\frac{2\zeta(3)}{\pi^2}T_\nu^3 \tag{31}$$

where the factor of 3/4 is because neutrinos are fermions, and the 2 dof count one helicity of one neutrino and one helicity of the antineutrino. Now we have seen that $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$, hence,

$$n_\nu = \frac{3}{11}\frac{2\zeta(3)}{\pi^2}T_\gamma^3 \tag{32}$$

$$= \frac{3}{11}n_\gamma \tag{33}$$

When $T \sim m$ and hence forth, neutrinos are non-relativistic so that

$$\rho_\nu = m_\nu n_\nu \tag{34}$$

At that point $n_\nu = \frac{3}{11}n_\gamma$ is still valid so that

$$\rho_\nu = \frac{3}{11}m_\nu\frac{2\zeta(3)}{\pi^2}T_\gamma^3 \tag{35}$$

$$= m_\nu\frac{6\zeta(3)}{11\pi^2}T_{0\gamma}^3\frac{1}{a^3} \tag{36}$$

$$= \frac{\rho_{0\nu}}{a^3} \tag{37}$$

where

$$\rho_{0\nu} = m_\nu\frac{6\zeta(3)}{11\pi^2}T_{0\gamma}^3 \tag{38}$$

is the density of one flavour of massive neutrinos today. But $\Omega_\nu = \rho_{0\nu}/\rho_{crit}$. Plugging in the numbers we have that $\rho_{crit} = \frac{3H_0^2}{8\pi G}$ with $H_0 = 2.137h \times 10^{-42}GeV$ and $G = 6.67 \times 10^{-11}m^3Kg^{-1}s^{-2} = 6.7 \times 10^{-39}GeV^{-2}$, leading to $\rho_{crit} = 8.13h^2 \times 10^{-11}eV^4$. Meanwhile $T_{0\gamma} \approx 2.72K = 2.34 \times 10^{-13}GeV$ so that we find

$$\Omega_{0\nu} \approx \frac{m_\nu}{94h^2eV} \tag{39}$$

per neutrino flavour.

9. We start from the equation in the problem set. For scales outside the Jeans length, we can ignore the term $k^2c_s^2\delta$ and the equation becomes

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0 \tag{40}$$

Now consider the term $4\pi G\bar{\rho}\delta$. The density $\bar{\rho}$ is the energy density of matter. We re-express it in terms of the total energy density $\bar{\rho}_T$ to get

$$4\pi G\bar{\rho} = 4\pi G\Omega_m\bar{\rho}_T = \frac{3}{2}\Omega_m H^2 \tag{41}$$

Now for both curvature and cosmological constant domination Ω_m is tiny and so we may neglect the term $\frac{3}{2}\Omega_m H^2$ and the equation to be solved becomes

$$\ddot{\delta} + 2H\dot{\delta} = 0 \tag{42}$$

Now consider the two cases

9.a Curvature domination. For curvature domination we have the Friedmann equation

$$3H^2 = -\frac{3K}{a^2} \tag{43}$$

which is only valid for negative curvature $K < 0$. The equation simplifies to

$$\dot{a}^2 = -K \tag{44}$$

which has solution

$$a = \sqrt{|K|}t \tag{45}$$

hence the Hubble parameter evolves as

$$H = \frac{1}{t} \tag{46}$$

Thus the equation for δ becomes

$$\ddot{\delta} + \frac{2}{t}\dot{\delta} = 0 \tag{47}$$

We set $A = \dot{\delta}$ to get

$$\dot{A} = -\frac{2}{t}A \tag{48}$$

which has solution

$$A = \left(\frac{t_0}{t}\right)^2 \tag{49}$$

where t_0 is an integration constant. Thus replacing A with $\dot{\delta}$ we get

$$\dot{\delta} = \left(\frac{t_0}{t}\right)^2 \tag{50}$$

which integrates to

$$\delta = \delta_0 - \frac{t_0^2}{t} \tag{51}$$

9.b Cosmological constant domination. In this case the Hubble parameter is constant $H = H_0$ and the equation for δ becomes

$$\ddot{\delta} + 2H_0\dot{\delta} = 0 \tag{52}$$

We set $A = \dot{\delta}$ to get

$$\dot{A} = -2H_0 A \tag{53}$$

which has solution

$$A = A_0 e^{-2H_0 t} \tag{54}$$

where A_0 is an integration constant. Thus replacing A with $\dot{\delta}$ we get

$$\dot{\delta} = A_0 e^{-2H_0 t} \tag{55}$$

which integrates to

$$\delta = \delta_0 - \frac{A_0}{2H_0}e^{-2H_0 t} \tag{56}$$

where δ_0 is a 2nd integration constant. This is the general solution.

9.c We see that for both curvature and cosmological constant domination, the solution for the density contrast is equal to a constant plus a decaying mode. Thus in both cases, the density contrast stops growing, thus structure formation stops if the Universe enters a curvature or a cosmological constant period.

10.a The inverse metric tensor $g^{\mu\nu}$ will be a perturbation on the background inverse metric tensor $\bar{g}^{\mu\nu}$ which is

$$\bar{g}^{\mu\nu} = \frac{1}{a^2}\begin{pmatrix} -1 & 0 \\ 0 & \gamma^{ij} \end{pmatrix} \tag{57}$$

where $\gamma^{ik}\gamma_{kj} = \delta^i{}_j$. Thus for the total $g^{\mu\nu}$ we should have

$$g^{\mu\nu} = \frac{1}{a^2}\begin{pmatrix} -1 + h^{00} & 0 \\ 0 & \gamma^{ij} + h^{ij} \end{pmatrix} \tag{58}$$

mutiplying $g^{\mu\nu}$ with $g_{\mu\nu}$ we get

$$\begin{aligned}
\delta^\mu{}_\nu = g^{\mu\rho}g_{\rho\nu} &= \frac{1}{a^2}\begin{pmatrix} -1 + h^{00} & 0 \\ 0 & \gamma^{ik} + h^{ik} \end{pmatrix} \times a^2 \begin{pmatrix} -(1+2\Psi) & 0 \\ 0 & (1-2\Phi)\gamma_{kj} + h^{(T)}_{kj} \end{pmatrix} \\
&= \begin{pmatrix} -1 + h^{00} & 0 \\ 0 & \gamma^{ik} + h^{ik} \end{pmatrix} \begin{pmatrix} -(1+2\Psi) & 0 \\ 0 & (1-2\Phi)\gamma_{kj} + h^{(T)}_{kj} \end{pmatrix} \\
&= \begin{pmatrix} 1 + 2\Psi - h^{00} & 0 \\ 0 & \left[\gamma^{ik} + h^{ik}\right]\left[(1-2\Phi)\gamma_{kj} + h^{(T)}_{kj}\right] \end{pmatrix} \\
&= \begin{pmatrix} 1 + 2\Psi - h^{00} & 0 \\ 0 & (1-2\Phi)\gamma^{ik}\gamma_{kj} + \gamma^{ik}h^{(T)}_{kj} + h^{ik}\gamma_{kj} \end{pmatrix}
\end{aligned} \tag{59}$$

Now we can raise the indices on $h^{(T)}_{ij}$ using the background spatial metric tensor γ_{ij}. So we write $h^{(T)}_{kj} = \gamma_{kp}\gamma_{jq}h^{(T)pq}$. We further multiply with γ^{ik} as it appears in the matrix above to get $\gamma^{ik}h^{(T)}_{kj} = \gamma^{ik}\gamma_{kp}\gamma_{jq}h^{(T)pq} = \delta^i_p\gamma_{jq}h^{(T)pq} = \gamma_{kj}h^{(T)ik}$. So substituting this into the matrix we get

$$\begin{aligned}
\delta^\mu{}_\nu = g^{\mu\rho}g_{\rho\nu} &= \begin{pmatrix} 1 + 2\Psi - h^{00} & 0 \\ 0 & \delta^i{}_j - 2\Phi\gamma^{ik}\gamma_{kj} + \gamma_{kj}h^{(T)ik} + h^{ik}\gamma_{kj} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \delta^i{}_j \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned} \tag{60}$$

This means that we must have

$$h^{00} = 2\Psi \tag{61}$$

and

$$h^{ij} = 2\Phi\gamma^{ij} - h^{(T)ij} \tag{62}$$

so that the inverst metric tensor is

$$g^{\mu\nu} = \frac{1}{a^2}\begin{pmatrix} -1 + 2\Psi & 0 \\ 0 & (1+2\Phi)\gamma^{ij} - h^{(T)ij} \end{pmatrix} \tag{63}$$

10.b We need to calculate $u_\mu u_\nu g^{\mu\nu}$. First let's start from the background. Since the background is homogeneous and isotropic, the background 3-velocity must be zero. So $\bar{u} = (\bar{u}_0, \vec{0})$. Now we need to find $\bar{u}_0$. We have

$$-1 = \bar{u}_\mu\bar{u}_\nu\bar{g}^{\mu\nu} = (\bar{u}_0)^2\bar{g}^{00} = -\frac{1}{a^2}(\bar{u}_0)^2 \tag{64}$$

so we solve for $\bar{u}_0$ to get

$$\bar{u}_0 = \pm a \tag{65}$$

We choose the + solution i.e. $\bar{u}_0 = a$. This is purely conventional. Now let us consider the perturbations. We write $u_\mu = \bar{u}_\mu + \delta u_\mu$. We need to find δu_0. We have

$$\begin{aligned}
-1 &= u_\mu u_\nu g^{\mu\nu} \\
&= (\bar{u}_\mu + \delta u_\mu)(\bar{u}_\nu + \delta u_\nu)(\bar{g}^{\mu\nu} + \delta g^{\mu\nu}) \\
&= (\bar{u}_\mu + \delta u_\mu)(\bar{u}_\nu + \delta u_\nu)\bar{g}^{\mu\nu} + (\bar{u}_\mu + \delta u_\mu)(\bar{u}_\nu + \delta u_\nu)\delta g^{\mu\nu} \\
&= (\bar{u}_\mu + \delta u_\mu)\bar{u}_\nu \bar{g}^{\mu\nu} + (\bar{u}_\mu + \delta u_\mu)\delta u_\nu \bar{g}^{\mu\nu} + \bar{u}_\mu \bar{u}_\nu \delta g^{\mu\nu} \\
&= \bar{u}_\mu \bar{u}_\nu \bar{g}^{\mu\nu} + \delta u_\mu \bar{u}_\nu \bar{g}^{\mu\nu} + \bar{u}_\mu \delta u_\nu \bar{g}^{\mu\nu} + \bar{u}_\mu \bar{u}_\nu \delta g^{\mu\nu}
\end{aligned} \tag{66}$$

Since $\bar{u}_\mu \bar{u}_\nu \bar{g}^{\mu\nu} = -1$ we find that

$$0 = \delta u_\mu \bar{u}_\nu \bar{g}^{\mu\nu} + \bar{u}_\mu \delta u_\nu \bar{g}^{\mu\nu} + \bar{u}_\mu \bar{u}_\nu \delta g^{\mu\nu} \tag{67}$$

Then since $\bar{u}_\mu = (a, \vec{0})$ the above expression gives

$$0 = \delta u_\mu \bar{u}_0 \bar{g}^{\mu 0} + \bar{u}_0 \delta u_\nu \bar{g}^{0\nu} + \bar{u}_0 \bar{u}_0 \delta g^{00} \tag{68}$$

$$= a\delta u_\mu \bar{g}^{\mu 0} + a\delta u_\nu \bar{g}^{0\nu} + \Psi \tag{69}$$

But $\bar{g}^{\mu 0}$ is non-zero only for $\mu = 0$ in which case $\bar{g}^{00} = -\frac{1}{a^2}$. Therefore the above expression gives

$$0 = -\frac{2}{a}\delta u_0 + 2\Psi \tag{70}$$

and solving for δu_0 we get

$$\delta u_0 = a\Psi \tag{71}$$

10.c On super-horizon scales we set $k^2 = 0$ so that the equation becomes

$$h^{(T)''} + 2\mathcal{H}h^{(T)'} = 0 \tag{72}$$

The trivial solution is $h^{(T)} = const$. We need the non-trivial solution. Let $A = h^{(T)'}$. Then

$$A' + 2\frac{a'}{a}A = 0 \tag{73}$$

where we have used $\mathcal{H} = \frac{a'}{a}$. Therefore the solution is

$$A = \frac{A_0}{a^2} \tag{74}$$

so that the general solution is

$$h^{(T)} = h_0^{(T)} + A_0 \int \frac{d\eta}{a^2} \tag{75}$$

where $h_0^{(T)}$ is a constant (the trivial solution). We see that the non-trivial solution proportional to A_0 is decaying so we can ignore it. Therefore, the tensor mode on super-horizon scales stays constant in time.

On sub-horizon scales the k^2 term becomes important. The equation to be solved is that of a damped harmonic oscillator. Thus, we expect the solution to be oscillatory with a decaying amplitude.