Astrophysics and Dark Matter - Problems

12th IDPASC School, Granada

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Problem Session 1

Caveat: In some of these problems, you may have to make assumptions that are not given explicitly. Be brave enough to throw away terms that you don't think are important, and be wise enough to go back later and ask whether it was justified.

Problem 1.1 - Hot gas in Galaxy Clusters

Observations of hot X-ray-emitting gas in Galaxy Clusters allows us to trace the underlying mass density (and therefore estimate the mass of DM in clusters). The equation for hydrostatic equilibrium of the gas is:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM_{\mathrm{tot}}(< r)}{r^2} = -\frac{1}{\rho_{\mathrm{gas}}}\frac{\mathrm{d}P_{\mathrm{gas}}}{\mathrm{d}r}\,. \tag{1}$$

Assuming that the gas in the cluster is an ideal gas (with mean mass per gas molecule m_g), show that the enclosed mass can be written as:

$$M_{\mathrm{tot}}(< r) = -\frac{k_B T(r) r}{G m_g}\left[\frac{\mathrm{d}\ln\rho_{\mathrm{gas}}(r)}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T(r)}{\mathrm{d}\ln r}\right]. \tag{2}$$

[Hint: recall that the equation of state for an ideal gas is $pV = Nk_BT$, where N is the total number of gas particles.]

Problem 1.2 - Density Profiles and Rotation Curves

The Hernquist density profile has the following form:

$$\rho(r) = \frac{\rho_0}{(r/a)(1 + r/a)^3}\,. \tag{3}$$

(a) Calculate the enclosed mass as a function of radius $M_{\mathrm{enc}}(r)$ for the Hernquist profile and find an expression for the density parameter ρ_0 in terms of the total mass of the halo and the scale radius a.

(b) Write down the rotational velocity as a function of radius, for stars moving in a Hernquist density profile. How does the rotational velocity at small radii change if we make the DM halo more compact (at fixed mass)?

(c) At large radii, the Hernquist density profile is not as good a fit to simulated DM density profiles as the NFW profile. Derive the scaling of the NFW rotation curve with r at large radii. Show that this rotation curve decays more slowly than the Hernquist case.

[Hint: Here, you can take the limit $r \gg r_s$ and just focus on the scaling with r. If you would like to do this more carefully, you can model the NFW density profile as a broken power-law, where the slope changes abruptly at $r = r_s$.]

Problem 1.3 - Fermionic Dark Matter and the Alhambra

(a) How many people can comfortably fit inside the Alhambra Palace in Granada? *[Hint: Consider what a 'comfortable' distance between two people might be, and estimate the size of the Alhambra.]*

(b) What is the minimum mass of fermionic Dark Matter which can still explain the dense Dark Matter halos in Dwarf Galaxies? *[Hint: Consider what a 'comfortable' distance between two fermions might be, and how densely packed they must be to achieve a given density. Here, it might help to know that the Draco Dwarf Galaxy has a diameter of about 0.7 kpc and a DM halo mass around* $2 \times 10^7\, M_\odot$*.]*

A more refined version of this argument has indeed been used to set a limit on the mass of fermionic dark matter, known as the Tremaine-Gunn Bound.

[Hint: Some unit conversions that you might find useful are that $h/c \approx 4 \times 10^{-89}\, M_\odot\,\mathrm{pc}$ *and* $1\, M_\odot \approx 10^{66}\,\mathrm{eV}$*. Note that here we're using a system of units where* $c = 1$*, such that we can use* eV *both as a unit of energy and as a unit of mass.]*

Problem 1.4 - Freeze-out of Dark Matter

For DM particles initially in thermal equilibrium in the early Universe, the evolution of their number density n_χ can be described by the Boltzmann equation:

$$\dot{n}_\chi + 3Hn_\chi = -\langle \sigma_{\chi\bar{\chi}} v \rangle \left[n_\chi^2 - {n_{\chi,\mathrm{eq}}}^2 \right] . \tag{4}$$

(a) Transform the Boltzmann equation into the Riccati Equation for the Yield $Y = n_\chi / s$:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{\lambda}{x^2} \left[Y(x)^2 - Y_{\mathrm{eq}}(x)^2 \right] , \tag{5}$$

where $x = m_\chi / T$ and $s = (2\pi^2/45) g_{\star s} T^3$ is the entropy density of the Universe. Give an expression for λ in terms of m_χ, $\langle \sigma_{\chi\bar{\chi}} \rangle$ and $H(T = m_\chi)$.

[Hint: recall that $T \propto 1/a$ *and that in radiation domination, we have* $H(T) \propto T^2$*.]*

(b) Assuming that after freeze-out x_f the DM number density is much larger than the equilibrium number density, integrate the yield from x_f to today and find an expression for the present day yield $Y(x_0)$ in terms of x_f and λ.

(c) Show that (assuming x_f doesn't vary too much) the relic DM density is independent of the DM mass and scales as $\langle \sigma v \rangle$.

Problem 1.5 - Free-streaming

Let's estimate the free-streaming scale for warm dark matter (WDM). We'll estimate this as the Dark Matter Jeans length, evaluated at Matter-Radiation Equality (MRE). The physical Jeans length for a collisionless fluid with velocity dispersion σ is given by:

$$\lambda_J(t) = \sqrt{\frac{\pi \sigma(t)^2}{G\bar{\rho}(t)}}\,. \tag{6}$$

The velocity dispersion of the Dark Matter drops with the expansion of the Universe, according to $\sigma \propto a^{-1}$, once the DM freezes out.

(a) Write down an expression for the velocity dispersion at MRE in terms of the Dark Matter mass m_χ, the cosmic temperature today T_0 and the scale factor at MRE $a_{\rm eq}$. *[Hint: We're assuming this is Warm Dark Matter, so let's take $x_f = 1$, corresponding to freeze-out while just relativistic. In light of this, what is the DM velocity dispersion at the moment of freeze-out?]*

(b) Evaluate the physical Jeans length at MRE as a function of the DM mass, assuming $T_0 = 2.7\,\mathrm{K}$, $z_{\rm eq} \approx 3400$ and $\rho_{\rm eq} = 3.3 \times 10^{-19}\,\mathrm{g/cm^3}$. *[Hint: In the lectures, we've been fixing Boltzmann's constant $k_B = 1$. You may have to re-insert the facts of k_B to convert between eV and K.]*

(c) Convert this to the comoving Jeans length $\lambda_J^{\rm com} = (a_{\rm eq})^{-1}\,\lambda_J$ and compare with the expression given during Lecture 1 for the damping scale of Warm Dark Matter.

Problem Session 2

In the first three problems, we will attempt to derive constraints on the Dark Matter self-annihilation cross-section $\langle \sigma v \rangle$ based on observations of the (completely fictional) Dwarf Galaxy 'Tapa-5'.

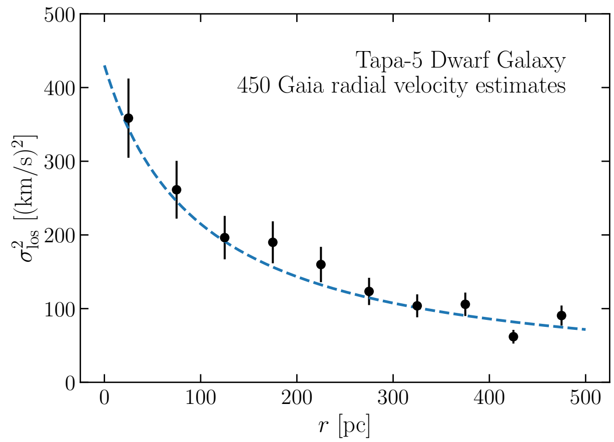

Figure 1: Estimates of the line of sight velocity dispersion $\sigma_{\rm los}$ for the (completely fictional) Tapa-5 Dwarf Galaxy. The dashed blue line is the result of a fit of the form $\sigma^2_{\rm los} = \sigma_0^2\,[a/(r+a)]$. The scale parameter is fixed to $a = 100\,{\rm pc}$ and the fit gives a value of the normalisation factor of $\sigma_0^2 = 430\,({\rm km/s})^2$.

Problem 2.1 - Dynamical Jeans Equations

(a) Show that the spherical Jeans equation:

$$\frac{\partial\left(n(r)\sigma_r^2\right)}{\partial r} + \left[\frac{2\sigma_r^2\beta(r)}{r} + \frac{\partial\Phi}{\partial r}\right] n(r) = 0\,, \tag{7}$$

is satisfied if the DM halo in Tapa-5 has a Hernquist density profile:

$$\rho_H(r) = \frac{M_0}{2\pi a^3}\frac{1}{(r/a)(1+r/a)^3}\,, \tag{8}$$

and find an expression for σ_0^2 in terms of M_0 and a.

[Hint: Here, you can assume that the potential is dominated by the DM halo, and you may find useful the expression for the enclosed mass of a Hernquist profile: $M(r) = M_0 r^2/(a+r)^2$ (see solution to Problem 1.2). For simplicity, you can assume that $\beta(r) = 0$, $\sigma_r = \sigma_{\rm los}$ and that the stellar density $n(r)$ is constant.]

(b) Calculate the numerical value of the total mass of the DM halo M_0.

Problem 2.2 - J-factor for Tapa-5

The J-factor is defined as:

$$J(\Delta\Omega) \equiv \int_{\Delta\Omega} \mathrm{d}\Omega \int_{\mathrm{los}} \rho_\chi^2(\mathbf{r}(\ell,\theta))\,\mathrm{d}\ell = 2\pi \int_0^{\theta_{\max}} \sin\theta\,\mathrm{d}\theta \int_{\mathrm{los}} \rho_\chi^2(\mathbf{r}(\ell,\theta))\,\mathrm{d}\ell \tag{9}$$

Rather than doing the full calculation, we'll do a simple estimate for Tapa-5.

(a) Calculate the value of $\langle\rho_\chi^2\rangle$, where the average is taken over the volume $r \leq a$:

$$\langle\rho_\chi^2\rangle = \frac{1}{V}\int_0^a \rho_\chi^2(r)\,\mathrm{d}^3\mathbf{r}\,. \tag{10}$$

[Hint: In this region, you can assume $\rho \propto 1/r$].

(b) Replacing $\rho_\chi^2 \to \langle\rho_\chi^2\rangle$ in Eq. (9), estimate the J-factor for Tapa-5 out to an opening angle of $\theta_{\max} = 0.5°$.

[Hint: Here, you can assume that Tapa-5 is at a distance $D = 30\,\mathrm{kpc} \gg a$. With this you can assume that $\theta \ll 1$, such that the line-of-sight integrals is dominated by trajectories passing through the centre of the Dwarf. In any case, don't worry too much about the exact geometry of the integral.]

Problem 2.3 - Gamma-ray constraints

Fermi observes in a region of angular size 0.5° around Tapa-5. The resulting upper limit on the gamma-ray flux above 1 GeV is $\Phi_\gamma \gtrsim 10^{-8}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}$.

Assuming DM particle of 100 GeV annihilating directly into a pair of gamma rays, convert the flux upper limit into an upper limit on the DM annihilation cross section $\langle\sigma v\rangle$ (in units of $\mathrm{cm}^3\,\mathrm{s}^{-1}$).

[Hint: Some unit conversions that you may find useful: $1\,M_\odot \approx 10^{57}\,\mathrm{GeV}$ and $1\,\mathrm{pc} \approx 3 \times 10^{18}\,\mathrm{cm}$.]

Bonus Problem - Bird Strikes and Primordial Black Holes

(a) Occasionally, birds will hit aircraft during take-off or landing. Estimate the probability of such a 'bird strike' during a single commercial flight. What is the rate of bird strikes per day across the world?

(b) Primordial Black Holes (PBHs) which cross stars can be captured and eventually can consume and destroy the star. If all of the Dark Matter in the Universe is in the form of PBHs with mass $M_{\mathrm{PBH}} = 10^{-15}\,M_\odot$, what is the probability that the Sun will be destroyed in its lifetime?

[Hint: the local DM density close to the Sun is about $10^{-2}\,M_\odot/\mathrm{pc}^3$.]