Astrophysics and Dark Matter - Problems and Solutions

12th IDPASC School, Granada

September 26, 2023

Problem Session 1

Caveat: In some of these problems, you may have to make assumptions that are not given explicitly. Be brave enough to throw away terms that you don't think are important, and be wise enough to go back later and ask whether it was justified.

Problem 1.1 - Hot gas in Galaxy Clusters

Observations of hot X-ray-emitting gas in Galaxy Clusters allows us to trace the underlying mass density (and therefore estimate the mass of DM in clusters). The equation for hydrostatic equilibrium of the gas is:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{GM_{\mathrm{tot}}(<r)}{r^2} = -\frac{1}{\rho_{\mathrm{gas}}}\frac{\mathrm{d}P_{\mathrm{gas}}}{\mathrm{d}r}\,. \tag{1}$$

Assuming that the gas in the cluster is an ideal gas (with mean mass per gas molecule m_g), show that the enclosed mass can be written as:

$$M_{\mathrm{tot}}(<r) = -\frac{k_B T(r) r}{G m_g}\left[\frac{\mathrm{d}\ln\rho_{\mathrm{gas}}(r)}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T(r)}{\mathrm{d}\ln r}\right]. \tag{2}$$

[Hint: recall that the equation of state for an ideal gas is $pV = Nk_BT$, where N is the total number of gas particles.]

Solution 1.1

We start with the ideal gas equation:

$$PV = Nk_BT \tag{3}$$

$$\Rightarrow P = (N/V)k_BT = (\rho_{\mathrm{gas}}/m_g)k_BT\,. \tag{4}$$

Calculating the derivative:

$$\frac{\mathrm{d}P_{\mathrm{gas}}}{\mathrm{d}r} = \frac{k_B}{m_g}\left(\frac{\mathrm{d}\rho_{\mathrm{gas}}}{\mathrm{d}r}T + \rho_{\mathrm{gas}}\frac{\mathrm{d}T}{\mathrm{d}r}\right) \tag{5}$$

$$= \frac{k_BT}{m_g}\left(\frac{\mathrm{d}\rho_{\mathrm{gas}}}{\mathrm{d}r} + \frac{\rho_{\mathrm{gas}}}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right). \tag{6}$$

Using $\mathrm{d}\ln x/\mathrm{d}r = (1/x)\mathrm{d}x/\mathrm{d}r$, we have:

$$\frac{1}{\rho_{\mathrm{gas}}}\frac{\mathrm{d}P_{\mathrm{gas}}}{\mathrm{d}r} = \frac{k_BT}{m_g}\left(\frac{\mathrm{d}\ln\rho_{\mathrm{gas}}}{\mathrm{d}r} + \frac{\mathrm{d}\ln T}{\mathrm{d}r}\right). \tag{7}$$

With this, we finally have:

$$M_{\mathrm{tot}}(<r) = -\frac{k_B T(r) r}{G m_g}\left[\frac{\mathrm{d}\ln\rho_{\mathrm{gas}}(r)}{\mathrm{d}\ln r} + \frac{\mathrm{d}\ln T(r)}{\mathrm{d}\ln r}\right], \tag{8}$$

where we have also used $r\mathrm{d}x/\mathrm{d}r = \mathrm{d}x/\mathrm{d}\ln r$.

Problem 1.2 - Density Profiles and Rotation Curves

The Hernquist density profile has the following form:

$$\rho(r) = \frac{\rho_0}{(r/a)(1+r/a)^3} \,. \tag{9}$$

(a) Calculate the enclosed mass as a function of radius $M_{\rm enc}(r)$ for the Hernquist profile and find an expression for the density parameter ρ_0 in terms of the total mass of the halo and the scale radius a.

(b) Write down the rotational velocity as a function of radius, for stars moving in a Hernquist density profile. How does the rotational velocity at small radii change if we make the DM halo more compact (at fixed mass)?

(c) At large radii, the Hernquist density profile is not as good a fit to simulated DM density profiles as the NFW profile. Derive the scaling of the NFW rotation curve with r at large radii. Show that this rotation curve decays more slowly than the Hernquist case.

[Hint: Here, you can take the limit $r \gg r_s$ and just focus on the scaling with r. If you would like to do this more carefully, you can model the NFW density profile as a broken power-law, where the slope changes abruptly at $r = r_s$.]

Solution 1.2

(a) The enclosed mass for the Hernquist profile is:

$$M_H(r) = \int_0^r \frac{\rho_0}{(r/a)(1+r/a)^3} 4\pi r^2 \,\mathrm{d}r \,. \tag{10}$$

Here, it's useful to change variables to $x = r/a$ in order to render the integral dimensionless:

$$M_H(r) = 4\pi\rho_0 a^3 \int_0^x \frac{x}{(1+x)^3} \,\mathrm{d}x \,. \tag{11}$$

By writing the integrand as $x/(1+x)^3 = 1/(1+x)^2 - 1/(1+x)^3$, we can perform the integral straightforwardly to obtain:

$$M_H(r) = 4\pi\rho_0 a^3 \left[\frac{-(1+2x)}{2(1+x)^2}\right]_0^{(r/a)} \tag{12}$$

$$= 2\pi\rho_0 a^3 \frac{x^2}{(1+x)^2} \qquad \text{with } x = r/a. \tag{13}$$

The total mass of the halo is obtained taking $x \to \infty$, to obtain $M_0 = 2\pi\rho_0 a^3$.

(b) The rotational velocity is given by:

$$v_{\rm rot}(r) = \sqrt{\frac{GM_{\rm enc}(r)}{r}} \tag{14}$$

$$= \sqrt{GM_0 \frac{r}{(r+a)^2}} \tag{15}$$

At small radii, the rotational velocity goes as $v_{\rm rot}(r) = \sqrt{GM_0 r/a^2}$. If we make the DM halo more compact at fixed M_0, we reduce a, increasing the rotational velocity close to the centre. This makes sense: making the halo more compact increases the central mass and therefore the rotational velocity.

(c) The NFW profile is given by:

$$\rho_{\rm NFW}(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2}\,. \tag{16}$$

At large radii, the density profile goes as:

$$\rho_{\rm NFW}(r) \to \frac{\rho_s}{(r/r_s)^3}\,. \tag{17}$$

The enclosed mass will receive a contribution from the central region of the DM halo. Let's call this central contribution M_c. At large radii $r \gg r_s$, the enclosed mass should vary as:

$$M_{\rm enc}(r) \approx M_c + \int_{r_s}^{r} 4\pi\rho_s \frac{r_s^3}{r}\,\mathrm{d}r \tag{18}$$

$$= M_c + 4\pi\rho_s r_s^3 \ln(r/r_s)\,. \tag{19}$$

At large radii then, the rotational velocity scales as $v_{\rm rot}(r) \sim \sqrt{\ln(r)/r}$, dropping more slowly than the Hernquist rotation curve ($v_{\rm rot} \sim \sqrt{1/r}$).

Here, it's worth noting that the NFW profile does admit an analytic expression for the mass profile (though the calculation isn't particularly instructive):

$$M_{\rm NFW}(r) = 4\pi\rho_s r_s^3 \left[\ln(1+x) - \frac{x}{1+x}\right]\,, \tag{20}$$

with $x = r/r_s$. In fact, the mass of NFW halos is formally infinite at large radii.

Problem 1.3 - Fermionic Dark Matter and the Alhambra

(a) How many people can comfortably fit inside the Alhambra Palace in Granada? *[Hint: Consider what a 'comfortable' distance between two people might be, and estimate the size of the Alhambra.]*

(b) What is the minimum mass of fermionic Dark Matter which can still explain the dense Dark Matter halos in Dwarf Galaxies? *[Hint: Consider what a 'comfortable' distance between two fermions might be, and how densely packed they must be to achieve a given density. Here, it might help to know that the Draco Dwarf Galaxy has a diameter of about 0.7 kpc and a DM halo mass around* $2 \times 10^7\,M_\odot$.*]*

A more refined version of this argument has indeed been used to set a limit on the mass of fermionic dark matter, known as the Tremaine-Gunn Bound.

[Hint: Some unit conversions that you might find useful are that $h/c \approx 4 \times 10^{-89}\,M_\odot\,\mathrm{pc}$ *and*

$1\,M_\odot \approx 10^{66}\,\mathrm{eV}$. *Note that here we're using a system of units where $c = 1$, such that we can use* eV *both as a unit of energy and as a unit of mass.]*

Solution 1.3

Of course, there are different ways to try and answer these questions, so your answers may vary, but the important part of orders of magnitude.

(a) How much space does a person comfortably need? Let's be nice and assume we won't be stacking people on top of each other. So let's say that a person comfortably needs one arm-span of space. We'll suggestively call this a person's 'Compton Armlength' $\lambda \sim 2\,\mathrm{m}$. What is the area of the Alhambra? A sensible guess might be $A \sim L^2$, with $L = 500\,\mathrm{m}$. The maximum comfortable number density of people is $n \sim 1/\lambda^2$, giving a total of $N = n \times A = (L/\lambda)^2 \approx 62,500$. A little googling suggests that the Alhambra gets around 8,000 visitors a day, so they must have plenty of space.

(b) Fermionic particles obey an exclusion principle, meaning that two identical particles cannot occupy the same state. A sensible estimate for when two particles start to 'overlap' would be if their Compton wavelength $\lambda_c = h/(m_f c)$ is comparable to their separation. In fact, we should be a bit more careful and look at the full density of states available in phase space, because the particles could have the same position but different momenta. But this will be an okay first estimate.

So let's say that each Fermion of mass m_f should be separated by at least $\lambda_c = h/(m_f c)$, giving a number density of $n \sim 1/\lambda_c^3$ and a maximum mass density $\rho_f \sim m_f/\lambda_c^3$. For the Draco Dwarf galaxy, the average density of the DM profile can be estimated as:

$$\rho_{\mathrm{Draco}} \sim M/R^3 \sim 10^{-2}\,M_\odot/\mathrm{pc}^3\,, \tag{21}$$

where we've used $M \approx 2 \times 10^7\,M_\odot$ and $R = 0.7\,\mathrm{kpc}$. Setting $\rho_f \sim \rho_{\mathrm{Draco}}$, we can infer the minimum fermion mass as:

$$m_f \gtrsim \left(\frac{h^3 \rho_{\mathrm{Draco}}}{c^3}\right)\,. \tag{22}$$

Plugging in the unit conversions given in the question, we obtain:

$$m_f \gtrsim 0.1\,\text{eV}\,. \tag{23}$$

In fact, the true bound from phase space arguments is a few orders of magnitude stronger (at the level of keV), but this isn't too bad for a rather simple back-of-the-envelope calculation.

Problem 1.4 - Freeze-out of Dark Matter

For DM particles initially in thermal equilibrium in the early Universe, the evolution of their number density n_χ can be described by the Boltzmann equation:

$$\dot{n}_\chi + 3Hn_\chi = -\langle \sigma_{\chi\bar{\chi}} v\rangle \left[n_\chi^2 - {n_{\chi,\text{eq}}}^2\right]\,. \tag{24}$$

(a) Transform the Boltzmann equation into the Riccati Equation for the Yield $Y = n_\chi/s$:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{\lambda}{x^2}\left[Y(x)^2 - Y_\text{eq}(x)^2\right]\,, \tag{25}$$

where $x = m_\chi/T$ and $s = (2\pi^2/45)g_{\star s}T^3$ is the entropy density of the Universe. Give an expression for λ in terms of m_χ, $\langle\sigma_{\chi\bar{\chi}}\rangle$ and $H(T = m_\chi)$.

[Hint: recall that $T \propto 1/a$ and that in radiation domination, we have $H(T) \propto T^2$.]

(b) Assuming that after freeze-out x_f the DM number density is much larger than the equilibrium number density, integrate the yield from x_f to today and find an expression for the present day yield $Y(x_0)$ in terms of x_f and λ.

(c) Show that (assuming x_f doesn't vary too much) the relic DM density is independent of the DM mass and scales as $\langle\sigma v\rangle$.

Solution 1.4

(a) We begin with the Boltzmann equation:

$$\dot{n}_\chi + 3Hn_\chi = -\langle \sigma_{\chi\bar{\chi}} v\rangle \left[n_\chi^2 - {n_{\chi,\text{eq}}}^2\right]\,. \tag{26}$$

First, let's change our derivative with respect to time into a derivative with respect to $x = m_\chi/T$:

$$\frac{\mathrm{d}n_\chi}{\mathrm{d}x} = \frac{\mathrm{d}n_\chi}{\mathrm{d}t}\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]^{-1}\,. \tag{27}$$

We need to evaluate:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{m_\chi}{T^2}\frac{\mathrm{d}T}{\mathrm{d}t} = -x\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t}\,. \tag{28}$$

Let's write $T = T_0/a$. We also write $H = H(T = m_\chi)(T/m_\chi)^2$, which is valid during radiation domination. Then we have:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -(T_0/a^2)\dot{a} = -TH(T)\,. \tag{29}$$

With this, we find $\mathrm{d}x/\mathrm{d}t = xH(T)$, and therefore:

$$\frac{\mathrm{d}n_\chi}{\mathrm{d}x} = \frac{1}{Hx}\frac{\mathrm{d}n_\chi}{\mathrm{d}t} \,. \tag{30}$$

We'll now transform from n_χ to $Y = n_\chi/s$. We have:

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}x} &= \frac{1}{s}\frac{\mathrm{d}n_\chi}{\mathrm{d}x} - \frac{n_\chi}{s^2}\frac{\mathrm{d}s}{\mathrm{d}x} \\ &= \frac{1}{s}\frac{\mathrm{d}n_\chi}{\mathrm{d}x} + 3\frac{n_\chi}{sx} \\ &= \frac{1}{Hsx}\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3\frac{n_\chi}{sx} \\ &= \frac{1}{Hsx}\left(\frac{\mathrm{d}n_\chi}{\mathrm{d}t} + 3Hn_\chi\right) \,, \end{aligned} \tag{31}$$

where in the second line we've used the straightforward relation that $\mathrm{d}s/\mathrm{d}x = (\mathrm{d}s/\mathrm{d}T) \times (\mathrm{d}T/\mathrm{d}x) = -3s/x$ and in the third line we've substituted from Eq. (30). The sharp reader will now notice that the term in brackets in the last line of Eq. (31) is simply the left hand side of the Boltzmann equation, Eq. (26). Substituting for the right hand side, we obtain:

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}x} &= \frac{1}{Hsx}\left(-\langle\sigma_{\chi\bar{\chi}}v\rangle\left[n_\chi^2 - {n_{\chi,\mathrm{eq}}}^2\right]\right) \\ &= -\frac{\langle\sigma_{\chi\bar{\chi}}v\rangle s}{Hx}\left[Y^2 - {Y_{\mathrm{eq}}}^2\right] \\ &= -\frac{\langle\sigma_{\chi\bar{\chi}}v\rangle m_\chi^2}{H(T=m_\chi)T^2x}\left(\frac{2\pi^2}{45}g_{\star,s}T^3\right)\left[Y^2 - {Y_{\mathrm{eq}}}^2\right] \\ &= -\frac{\langle\sigma_{\chi\bar{\chi}}v\rangle m_\chi^3}{H(T=m_\chi)}\left(\frac{2\pi^2}{45}g_{\star,s}\right)\frac{1}{x^2}\left[Y^2 - {Y_{\mathrm{eq}}}^2\right] \,. \end{aligned} \tag{32}$$

This is the Riccati Equation, with:

$$\lambda = \left(\frac{2\pi^2}{45}g_{\star,s}\right)\frac{\langle\sigma_{\chi\bar{\chi}}v\rangle m_\chi^3}{H(T=m_\chi)} \,. \tag{33}$$

(b) After freeze-out, the yield is always much larger than the equilibrium yield, meaning that the Riccati Equation reduces to:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{\lambda}{x^2}Y(x)^2 \,. \tag{34}$$

We can rearrange and integrate explicitly:

$$\int_{Y(x_f)}^{Y(x_0)} \frac{1}{Y^2}\,\mathrm{d}Y = -\int_{x_f}^{x_0}\frac{\lambda}{x^2}\,\mathrm{d}x \tag{35}$$

$$\Rightarrow \left[\frac{1}{Y(x_f)} - \frac{1}{Y(x_0)}\right] = \lambda\left[\frac{1}{x_0} - \frac{1}{x_f}\right] \,. \tag{36}$$

Freeze-out occurs deep in the radiation era, so $x_f \ll x_0$, and we can neglect the first term on the right hand side. Perhaps harder to justify, we will also neglect $Y(x_f)$. We know that $Y(x_0)$ is smaller than $Y(x_f)$, so the second term on the left hand side should dominate. We can also inspect the numerical solution and see that $Y(x_f) \gg Y(x_0)$. With this, we get the approximate solution:

$$Y(x_0) = \frac{x_f}{\lambda} \,. \tag{37}$$

(c) Noting that $H(T) \propto T^2$ deep in the radiation era, we see that $H(T = m_\chi) \propto m_\chi^2$. With this, we find that $\lambda \sim \langle \sigma_{\chi\bar{\chi}} v \rangle m_\chi$.

The present day DM density goes as $\rho_{\rm DM} \sim m_\chi Y(x_0) \sim m_\chi x_f/\lambda \sim x_f/\langle \sigma_{\chi\bar{\chi}} v \rangle$, almost independent of the DM mass.

Problem 1.5 - Free-streaming

Let's estimate the free-streaming scale for warm dark matter (WDM). We'll estimate this as the Dark Matter Jeans length, evaluated at Matter-Radiation Equality (MRE). The physical Jeans length for a collisionless fluid with velocity dispersion σ is given by:

$$\lambda_J(t) = \sqrt{\frac{\pi \sigma(t)^2}{G\bar{\rho}(t)}} \,. \tag{38}$$

The velocity dispersion of the Dark Matter drops with the expansion of the Universe, according to $\sigma \propto a^{-1}$, once the DM freezes out.

(a) Write down an expression for the velocity dispersion at MRE in terms of the Dark Matter mass m_χ, the cosmic temperature today T_0 and the scale factor at MRE $a_{\rm eq}$. *[Hint: We're assuming this is Warm Dark Matter, so let's take $x_f = 1$, corresponding to freeze-out while just relativistic. In light of this, what is the DM velocity dispersion at the moment of freeze-out?]*

(b) Evaluate the physical Jeans length at MRE as a function of the DM mass, assuming $T_0 = 2.7\,{\rm K}$, $z_{\rm eq} \approx 3400$ and $\rho_{\rm eq} = 3.3 \times 10^{-19}\,{\rm g/cm^3}$. *[Hint: In the lectures, we've been fixing Boltzmann's constant $k_B = 1$. You may have to re-insert the facts of k_B to convert between eV and K.]*

(c) Convert this to the comoving Jeans length $\lambda_J^{\rm com} = (a_{\rm eq})^{-1}\,\lambda_J$ and compare with the expression given during Lecture 1 for the damping scale of Warm Dark Matter.

Solution 1.5

(a) For Warm DM, we'll take $x_f = 1$, such that the DM freezes out at $T_f = m_\chi$. At this point, we'll assume that the DM is still relativistic, and from this moment on, its velocity dispersion decays towards being non-relativistic. So we'll assume $\sigma \sim c$ at T_f. At MRE, the velocity dispersion is $\sigma(t_{\rm eq}) = c \times (a_f/a_{\rm eq})$ where a_f the scale factor at freeze-out. The temperature scales as $T(a) = (a_0/a) T_0$, so we can write $\sigma(t_{\rm eq}) = c \times (T_0/T_f) \times a_{\rm eq}^{-1}$, where we've used that $a_0 = 1$. Finally, setting $T_f = m_\chi$, we have $\sigma(t_{\rm eq}) = c \times (T_0/m_\chi) \times a_{\rm eq}^{-1}$.

(b) The physical Jeans length is at MRE is then:

$$\lambda_J(t_{\rm eq}) = (1 + z_{\rm eq}) \frac{k_B T_0}{m_\chi} \sqrt{\frac{\pi c^2}{G \bar{\rho}_{\rm eq}}} \,. \tag{39}$$

Here, we've replaced $a_{\rm eq} = (1 + z_{\rm eq})^{-1}$ and we've re-inserted a factor of $k_B = 8.6 \times 10^{-5}\,\text{eV/K}$ such that m_χ/k_B is a temperature.

Substituting in the values given in the question, we find:

$$\lambda_J(t_{\rm eq}) \approx \left(\frac{\text{keV}}{m_\chi}\right) 51\,\text{pc}\,. \tag{40}$$

Indeed, this corresponds to the physical length scales on which we might want to suppress power (e.g. Dwarf Galaxy scales, and the centres of galaxies).

(c) The comoving Jeans Length is:

$$\lambda_J^{\rm com} = (1 + z_{\rm eq})\lambda_J(t_{\rm eq}) = \left(\frac{\text{keV}}{m_\chi}\right) 0.17\,\text{Mpc}\,. \tag{41}$$

This is not so far from the more detailed estimate given in the lecture:

$$R_S \approx 0.47 \left(\frac{\text{keV}}{m_\chi}\right)^{1.15} \text{Mpc}\,. \tag{42}$$

One possible source of the slightly different scaling with m_χ is that we have made a very simple assumption about the freeze-out time $T_f = m_\chi$, while in reality, x_f varies (slowly) with the DM mass.

Problem Session 2

In the first three problems, we will attempt to derive constraints on the Dark Matter self-annihilation cross-section $\langle \sigma v \rangle$ based on observations of the (completely fictional) Dwarf Galaxy 'Tapa-5'.

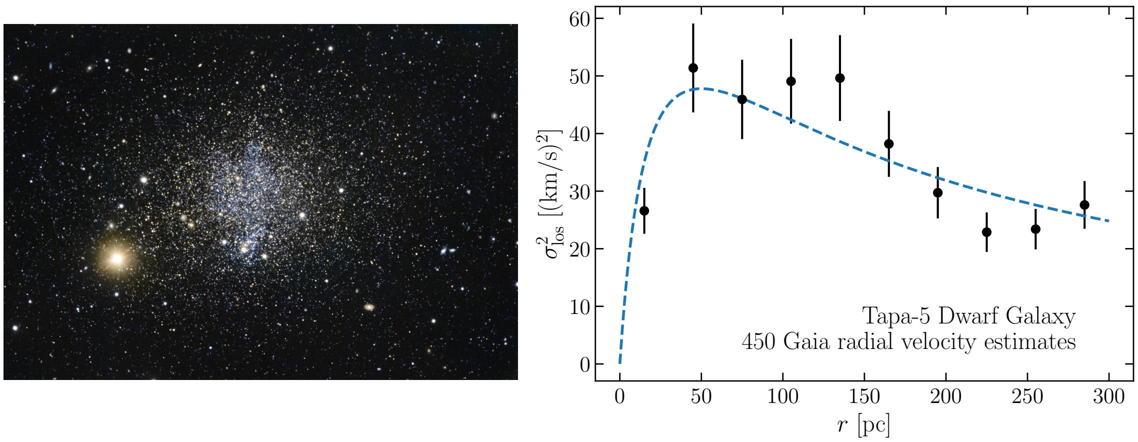

Figure 1: Estimates of the line of sight velocity dispersion $\sigma_{\rm los}$ for the (completely fictional) Tapa-5 Dwarf Galaxy. The dashed blue line is the result of a fit of the form $\sigma^2_{\rm los} = \sigma^2_0\,[a/(r+a)]$. The scale parameter is fixed to $a = 100\,\text{pc}$ and the fit gives a value of the normalisation factor of $\sigma^2_0 = 430\,(\text{km/s})^2$.

Problem 2.1 - Dynamical Jeans Equations

(a) Show that the spherical Jeans equation:

$$\frac{\partial\left(n(r)\sigma_r^2\right)}{\partial r} + \left[\frac{2\sigma_r^2\beta(r)}{r} + \frac{\partial\Phi}{\partial r}\right] n(r) = 0\,, \tag{43}$$

is satisfied if the DM halo in Tapa-5 has a Hernquist density profile:

$$\rho_H(r) = \frac{M_0}{2\pi a^3}\frac{1}{(r/a)(1+r/a)^3}\,, \tag{44}$$

and find an expression for σ_0^2 in terms of M_0 and a.

[Hint: Here, you can assume that the potential is dominated by the DM halo, and you may find useful the expression for the enclosed mass of a Hernquist profile: $M(r) = M_0 r^2/(a+r)^2$ (see solution to Problem 1.2). For simplicity, you can assume that $\beta(r) = 0$, $\sigma_r = \sigma_{\rm los}$ and that the stellar density $n(r)$ is constant.]

(b) Calculate the numerical value of the total mass of the DM halo M_0.

Solution 2.1

(a) Let's begin by simplifying things. We'll assume that the system has an isotropic velocity distribution, such that $\beta(r) = 0$. We'll also set $\sigma_r \sim \sigma_{\rm los}$ (which shouldn't be too bad an approximation for isotropic systems). Finally, the number density of stars (which act as tracers of the underlying DM gravitational potential) is taken to be constant, so we can take it outside the derivative and ultimately cancel it from both terms in the Spherical Jeans Equation.

This leaves us with a rather simple form:

$$\frac{\partial \sigma_{\rm los}^2}{\partial r} = -\frac{\partial \Phi}{\partial r}\,. \tag{45}$$

The velocity dispersion has the form:

$$\sigma_{\rm los}^2(r) = \sigma_0^2 \frac{a}{a+r} \tag{46}$$

$$\Rightarrow \frac{\partial \sigma_{\rm los}^2}{\partial r} = -\sigma_0^2 \frac{a}{(a+r)^2}\,. \tag{47}$$

On the right hand side of the Jeans Equation, we just have the derivative of the gravitational potential, which we identify as the gravitational acceleration due to the mass enclosed at a given radius.[a] We thus have:

$$-\frac{\partial \Phi}{\partial r} = -\frac{GM(r)}{r^2} = -\frac{GM_0}{(a+r)^2}\,. \tag{48}$$

From this, we can make the identification:

$$\sigma_0^2 = \frac{GM_0}{a}\,. \tag{49}$$

(b) We can now calculate the total mass as:

$$M_0 = a\sigma_0^2/G \approx 10^7\, M_\odot\,, \tag{50}$$

where we've used the numerical value $G \approx 4.3 \times 10^{-3}\,(\mathrm{km/s})^2 \mathrm{pc}\, M_\odot^{-1}$.

[a] Another way to see this is that the gravitational potential is calculated by integrating the gravitational force from infinity to some point r. The derivative of the potential then is simply the gravitational force. Note that in a spherically symmetric system, this force only depends on the mass *interior* to the radius r.

Problem 2.2 - J-factor for Tapa-5

The J-factor is defined as:

$$J(\Delta\Omega) \equiv \int_{\Delta\Omega} \mathrm{d}\Omega \int_{\rm los} \rho_\chi^2(\mathbf{r}(\ell,\theta))\,\mathrm{d}\ell = 2\pi \int_0^{\theta_{\rm max}} \sin\theta\,\mathrm{d}\theta \int_{\rm los} \rho_\chi^2(\mathbf{r}(\ell,\theta))\,\mathrm{d}\ell \tag{51}$$

Rather than doing the full calculation, we'll do a simple estimate for Tapa-5.

(a) Calculate the value of $\langle \rho_\chi^2 \rangle$, where the average is taken over the volume $r \leq a$:

$$\langle \rho_\chi^2 \rangle = \frac{1}{V} \int_0^a \rho_\chi^2(r)\, \mathrm{d}^3\mathbf{r}\,. \tag{52}$$

[Hint: In this region, you can assume $\rho \propto 1/r$].

(b) Replacing $\rho_\chi^2 \to \langle \rho_\chi^2 \rangle$ in Eq. (51), estimate the J-factor for Tapa-5 out to an opening angle of $\theta_{\mathrm{max}} = 0.5°$.

[Hint: Here, you can assume that Tapa-5 is at a distance $D = 30\,\mathrm{kpc} \gg a$. With this you can assume that $\theta \ll 1$, such that the line-of-sight integrals is dominated by trajectories passing through the centre of the Dwarf. In any case, don't worry too much about the exact geometry of the integral.]

Solution 2.2

(a) Here, the idea to to replace the DM halo with a constant density region with radius r and density squared $\langle \rho_\chi^2 \rangle$. The mean density squared of the Hernquist halo (assuming in the inner region that $\rho \sim 1/r$) is:

$$\langle \rho_\chi^2 \rangle = \frac{1}{V} \int_0^a \rho_0^2 \frac{a^2}{r^2}\, \mathrm{d}^3\mathbf{r} \tag{53}$$

$$= \frac{3}{4\pi a^3} \int_0^a \rho_0^2 \frac{a^2}{r^2}\, 4\pi r^2 \mathrm{d}r \tag{54}$$

$$= \frac{3\rho_0^2}{a^3} \int_0^a \mathrm{d}r \tag{55}$$

$$= 3\rho_0^2\,. \tag{56}$$

Note that here we've been careful to include the volume element $4\pi r^2\,\mathrm{d}r$.

(b) We now replace $\rho_\chi^2 \to \langle \rho_\chi^2 \rangle$ to obtain an expression for the J-factor:

$$J(\Delta\Omega) = 6\pi\rho_0^2 \int_0^{\theta_{\mathrm{max}}} \sin\theta\, \mathrm{d}\theta \int \mathrm{d}\ell\,. \tag{57}$$

The final $\mathrm{d}\ell$ integral is over line-of-sight (los) distances ℓ which lie within the constant density core of the DM halo (which in principle depends also on θ). Because we've assumed that the halo density is constant, this just reduces to an integral over the volume of the halo enclosed with $\theta < \theta_{\mathrm{max}}$. The geometry is illustrated in Fig. 2.

We don't want to worry too much about the specifics of this geometric integral (the exact calculation is a pain). There are a few ways to do this, but let's notice that $a \ll D$ and $\theta_{\mathrm{max}} \ll 1$, meaning that the lines of sight which pass through the DM halo are roughly parallel. Expanding $\sin\theta \approx \theta$, we'll then perform the two integrals separately:

$$J(\Delta\Omega) = 6\pi\rho_0^2 \left[\int_0^{\theta_{\mathrm{max}}} \theta\, \mathrm{d}\theta\right] \times \left[\int_{D-a}^{D+a} \mathrm{d}\ell\right]. \tag{58}$$

Geometrically, this is equivalent to approximating the volume as a cylinder of length $2a$ (the diameter of the uniform density halo) and cross-sectional radius $R \approx \theta_{\rm max} D$.

Performing the integrals straightforwardly, we obtain:

$$J(\Delta\Omega) \approx 6\pi \rho_0^2 a \theta_{\rm max}^2 = \frac{3}{2\pi} \theta_{\rm max}^2 M_0^2 / a^5 \,. \tag{59}$$

Substituting the opening angle in radians $\theta_{\rm max} = 0.5^\circ / 180^\circ \approx 3 \times 10^{-3}$, we obtain:

$$J \approx 0.04\, M_\odot^2\, {\rm pc}^{-5} \approx 1.6 \times 10^{20}\, {\rm GeV}^2\, {\rm cm}^{-2} \,. \tag{60}$$

Note that this is a couple of orders of magnitude larger than typical Dwarf J-factors.

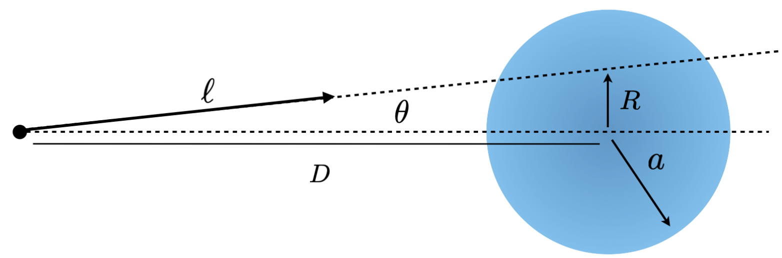

Figure 2: Geometry for calculating the J-factor.

Problem 2.3 - Gamma-ray constraints

Fermi observes in a region of angular size 0.5° around Tapa-5. The resulting upper limit on the gamma-ray flux above 1 GeV is $\Phi_\gamma \gtrsim 10^{-8}\,{\rm cm}^{-2}\,{\rm s}^{-1}$.

Assuming DM particle of 100 GeV annihilating directly into a pair of gamma rays, convert the flux upper limit into an upper limit on the DM annihilation cross section $\langle \sigma v \rangle$ (in units of ${\rm cm}^3\,{\rm s}^{-1}$).

[Hint: Some unit conversions that you may find useful: $1\, M_\odot \approx 10^{57}\,{\rm GeV}$ *and* $1\,{\rm pc} \approx 3 \times 10^{18}\,{\rm cm}$.*]*

Solution 2.3

(a) The prompt gamma-ray annihilation flux is given by:

$$\frac{{\rm d}\Phi_\gamma}{{\rm d}E_\gamma} = \frac{1}{4\pi} \frac{\langle \sigma_{\rm ann} v \rangle}{2 m_\chi^2} \frac{{\rm d}N_\gamma}{{\rm d}E_\gamma} \times \int_{\Delta\Omega} {\rm d}\Omega \int_{\rm los} \rho_\chi^2(\mathbf{r}(\ell,\theta)){\rm d}\ell \equiv \frac{1}{4\pi} \frac{\langle \sigma_{\rm ann} v \rangle}{2 m_\chi^2} \frac{{\rm d}N_\gamma}{{\rm d}E_\gamma} J \,. \tag{61}$$

We have a constraint on the DM flux above 1 GeV, so let's integrate over gamma-ray energies:

$$\Phi_{>1\,{\rm GeV}} = \frac{2}{4\pi} \frac{\langle \sigma_{\rm ann} v \rangle}{2 m_\chi^2} J \,, \tag{62}$$

where the factor of 2 comes from $\int \mathrm{d}N_\gamma/\mathrm{d}E_\gamma\,\mathrm{d}E_\gamma = 2$ (i.e. 2 photons per annihilation). Note that for annihilation into two photons, each gamma ray carries away an energy $E_\gamma = m_\chi$, so the integral over energies $E_\gamma > 1\,\mathrm{GeV}$ includes these photons for the assumed DM mass of 100 GeV.

Rearranging, we find:

$$\langle \sigma_{\mathrm{ann}} v \rangle \lesssim \frac{4\pi m_\chi^2 \Phi_{>1\,\mathrm{GeV}}}{J} \,. \tag{63}$$

Substituting the value of J given in the previous problem and the WIMP mass of $m_\chi = 100\,\mathrm{GeV}$, we obtain:

$$\langle \sigma_{\mathrm{ann}} v \rangle \lesssim 7.7 \times 10^{-24}\,\mathrm{cm}^3\,\mathrm{s}^{-1} \,. \tag{64}$$

Note that this is a bit weaker that real constraints (partly because the value we assumed for the limit on the gamma ray flux was a few orders of magnitude weaker than more realistic observed values).

Bonus Problem - Bird Strikes and Primordial Black Holes

(a) Occasionally, birds will hit aircraft during take-off or landing. Estimate the probability of such a 'bird strike' during a single commercial flight. What is the rate of bird strikes per day across the world?

(b) Primordial Black Holes (PBHs) which cross stars can be captured and eventually can consume and destroy the star. If all of the Dark Matter in the Universe is in the form of PBHs with mass $M_{\mathrm{PBH}} = 10^{-15}\,M_\odot$, what is the probability that the Sun will be destroyed in its lifetime?

[Hint: the local DM density close to the Sun is about $10^{-2}\,M_\odot/\mathrm{pc}^3$*.]*

Bonus Solution

(a) Let's start with a very simple estimate and see where we go wrong. The number of birds on Earth is probably around $N_\mathrm{b} \approx 10^10$ (one bird per person?). The radius of the Earth is $R_\mathrm{E} \approx 6000\,\mathrm{km}$, giving a surface area of $A_\mathrm{E} \approx \pi R_\mathrm{E}^2 \approx 10^8\,\mathrm{km}^2$. If we distribute the birds uniformly across the surface of the Earth, we get a bird surface density of $N_\mathrm{b}/A_\mathrm{E}$. The surface area of a runway (roughly the area covered by an aircraft taking off) must be about $A_\mathrm{a} \approx 50\,\mathrm{m} \times 2\,\mathrm{km} \approx 10^{-1}\,\mathrm{km}$. This area contains $A_\mathrm{a} N_\mathrm{b}/A_\mathrm{E}$, coming out at about 10 bird strikes per take-off!

This is clearly too many, and the thing that's missing is the vertical direction! Let's also distribute our birds vertically in the lowest $L = 1\,\mathrm{km}$ of the atmosphere. The volumetric bird density is then $N_\mathrm{b}/(A_\mathrm{E} L)$. The cross sectional area of an aircraft is perhaps $100\,\mathrm{m}^2 = 10^{-4}\,\mathrm{km}^2$, meaning that as it takes off along a 1 km runway, it sweeps out a volume $V_\mathrm{a} \approx 10^{-4}\,\mathrm{km}^2$. This gives a bird strike probability of $V_\mathrm{a} \times N_\mathrm{b}/(A_\mathrm{E} L) \approx 1\%$. This seems a bit more reasonable (though perhaps still a bit high; I guess birds must not be uniformly distributed...).

Some quick googling suggests that there are about 100,000 flights per day, suggesting around 1000 bird strikes per day. More googling suggest that there are about 50 bird strikes per day in the USA alone, so maybe our estimate is not so far off...

(b) As with birds, so with PBHs. Let's take our PBHs and distribute them uniformly. If the local DM density is $\rho_0 \approx 10^{-2}\, M_\odot/\mathrm{pc}^3$, this gives us a number density $n_{\mathrm{PBH}} = \rho_0/M_{\mathrm{PBH}} \approx 10^{13}\,\mathrm{pc}^{-3}$.

Now what is the volume swept out by the Sun during its lifetime of $t_\odot \approx 5 \times 10^9\,\mathrm{yr} \approx 10^{17}\,\mathrm{s}$? The cross sectional area of the Sun is around $A_\odot \approx (10^6\,\mathrm{km})^2 = 10^{12}\,\mathrm{km}^2$. The velocity of the Sun is around $v_\odot = 200\,\mathrm{km/s}$, such that it sweeps out a volume $V = A_\odot \times v_\odot \times t_\odot \approx 10^{31}\,\mathrm{km}^3$ during its lifetime.

This means that a total of $n_{\mathrm{PBH}} \times V \approx 10^{44}\,(\mathrm{km/pc})^3 \approx 3000$ PBHs should pass through the Sun during its lifetime! In fact, it turns out that even if a PBH passes through a star, it generally does not lose enough energy to be captured (i.e. the capture probability is much smaller than 1). So while this estimate of the number of PBH encounters seems troubling, in reality we don't need to worry too much about the Sun being swallowed by a PBH.