Symmetries in physics

Renato Fonseca

renatofonseca@ugr.es

High-Energy Physics Group, University of Granada

12th International IDPASC school, Granada, 18-28 September 2023

Symmetry and group theory

Something symmetric is something that does not change under some kind of transformation

$$T(\text{"something"}) = \text{"something"}$$

Some shape, a matrix, a function, ...

Let us consider the set of all such T's : S.

'Do nothing' must be one of the T's
($\mathrm{Id} \in S$)

$$\mathrm{Id}(\text{"something"}) = \text{"something"}$$

For each transformation $T \in S$, it's inverse ($\equiv T^{-1}$) must also be a symmetry

$$T^{-1}(\text{"something"}) = \text{"something"}$$

If T_1 and T_2 are symmetries then application of both transformations must also produce no change.

$$(T_1 \circ T_2)(\text{"something"}) = \text{"something"}$$

Symmetry and group theory

If we include the requirement that the composition of transformations is associative, $\boxed{T_1\circ(T_2\circ T_3) = (T_1\circ T_2)\circ T_3,}$ we have the definition of a group

A group is a set G and a binary operation $\bullet$ $(G\times G \xrightarrow{\bullet} G)$ obeying

- $e\in G : e\cdot g = g\cdot e = g \quad \forall g\in G$
- For all $g\in G$ there is an inverse, $\exists g^{-1}$, such that $g\cdot g^{-1} = g^{-1}\cdot g = e$
- $\forall g_1, g_2, g_3\in G \quad g_1\cdot(g_2\cdot g_3) = (g_1\cdot g_2)\cdot g_3$

Group theory and physics

As you know, the importance group theory in fundamental physics is enormous

- Lorentz group
- Poincaré group
- Baryon and Lepton numbers
- Supersymmetry
- Gauge symmetry
- Flavor symmetries
- Conservation laws (Noether's theorem)
- BRST symmetry
- Anomalies
- Gell-Mann's Eightfold Way
- C, P, T
- ...
- Exact/Approximate/Spontaneously broken symmetries

Books

Group Theory in Physics, Wu-Ki Tung

Lie Algebras in Particle Physics: From isospin to unified theories, Howard Georgi

Group Theory – A physicist's survey, Pierre Ramond

Lie groups, Lie algebras and some of their applications, Robert Gilmore

... (many more)

I will go through some of the basic concepts in group theory but at the same time I will consider that you had some exposure to the topic before

Example of groups

Finite groups

Z_n : $\{1, a, a^2, \cdots, a^{n-1}\}$ with $a^n = 1$, n elements , abelian = Commutative

S_n : group formed by the $n!$ permutations of n objects

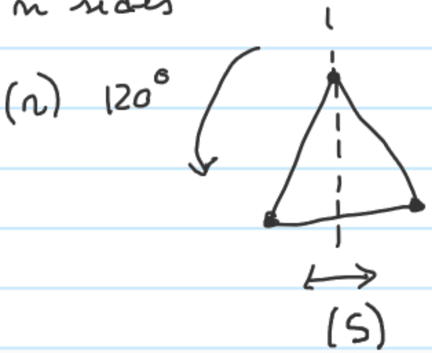
$\Pi_{123}(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\}$; $\Pi_{321}(\{x_1, x_2, x_3\}) = \{x_3, x_2, x_1\}$ + 4 more

↳ identity

All finite groups are a subgroup of some S_n with n = size of group

A_n : Alternating group; composed of all even permutations of n objects (Subgroup of S_n)

D_n : Dihedral group; composed of the symmetries of the regular polygon with n sides



$r^3 = s^2 = 1$

$(rs)^2 = 1$

Defines D_3 with 6 elements

Example of groups

Infinite discrete groups

One example would be the integers ($\mathbb{Z}$) and addition ($+$) as the group operation.

All these were discrete groups. There are also continuous groups (Nature seems to prefer these)

Example of groups

Continuous groups

We will be very interested in these, particularly the Lie groups (and their algebra)

$(\mathbb{R},+)$: The real numbers with addition

$(\mathbb{R}-\{0\},\times)$: The non-null real numbers with multiplication

$U(1)$: Set of complex numbers with norm 1, with usual multiplication

Example of groups

There are also the matrix group, i.e. certain sets of invertible matrices and the usual dot product.

GL(n): The n-dimensional general linear group, composed of all $n \times n$ invertible matrices.

SL(n): The n-dimensional special linear group; GL(n) matrices with $\det = 1$

U(n): Unitary group: $n \times n$ matrices U such that $U^\dagger U = \mathbb{1}$

SU(n): Special unitary group: same as U(n) with $\det(U) = 1$

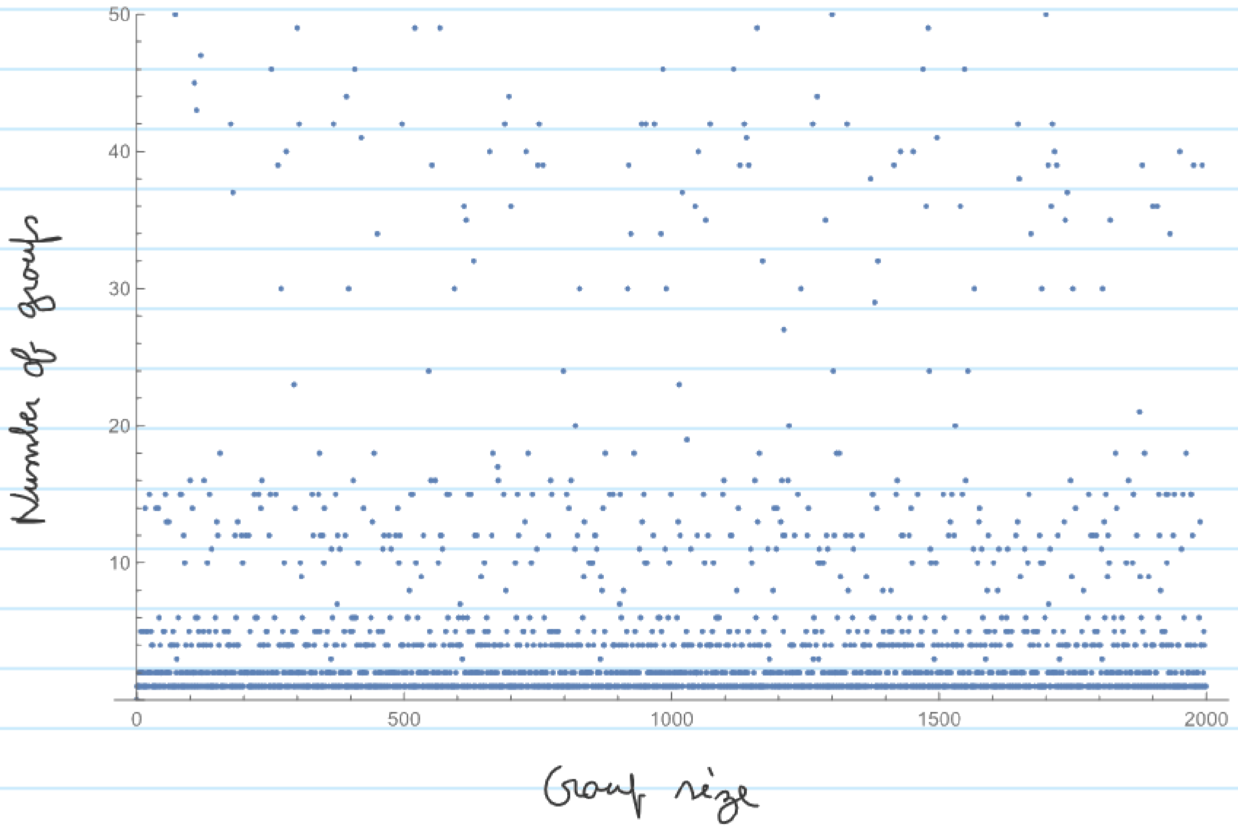
O(n): Orthogonal group: group of n-dimensional orthogonal matrices $(O^T O = \mathbb{1})$

SO(n): Special orthogonal group: as in O(n) and also $\det(O) = 1$

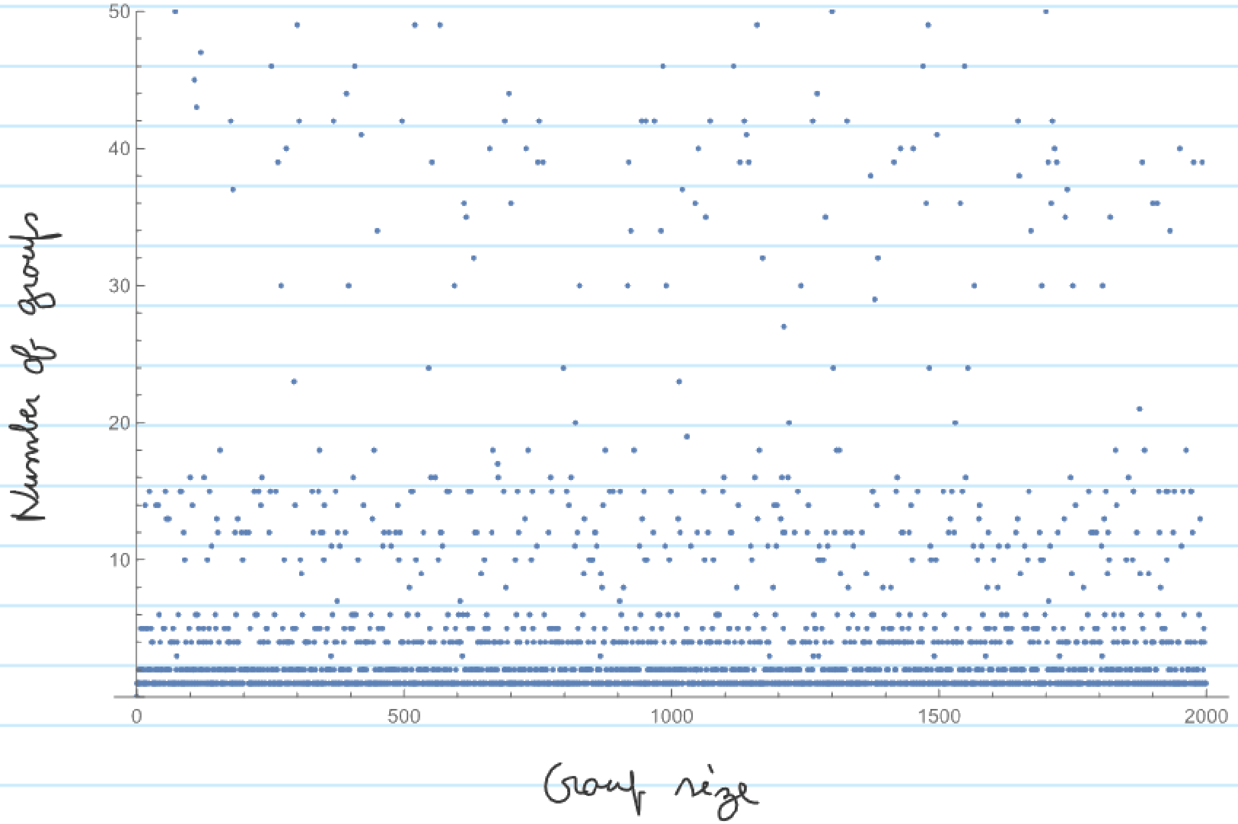
SP(2n): Symplectic group: matrices M such that $M^T \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$

→ The classical compact groups

How many finite groups?



How many finite groups?



It does not seem that much. However I truncated the vertical axis

There are 49.910.531.351 groups with 2000 elements or less.

(More than 99% have size $1024 = 2^{10}$)

Specifying a group

How do we specify a group? There are several ways.

Multiplication table (Brute force method) $g_i \cdot g_j = g_{f(i,j)}$

↖ provide this function for all i, j

* * *

Some property e.g. all square matrices M such that $\dim(M) = n$ and $M^\dagger M = \mathbb{1}$ (the U(n) group)

* * *

Presentation e.g. $\underbrace{\langle r \mid r^n = 1 \rangle}_{Z_n \text{ group}}$; $\underbrace{\langle \overset{\text{generators}}{r, s} \mid \overbrace{r^n = s^2 = (rs)^2 = 1}^{\text{relations}} \rangle}_{D_n \text{ group}}$

getting all elements of D_3 (6 of them)

0 gen.	1 gen.	2 gen.	3 generators → nothing new → stop	
1	r	r^2	$\cancel{r^3} = 1$	$\cancel{r.s.r} = s$
	s	$r.s$	$\cancel{r^2 s} = s r$	$\cancel{s r^2} = r.s$
		$s.r$	$\cancel{r s^2} = r$	$\cancel{s.r.s} = r^2$
		$\cancel{s^2} = 1$		

Direct and semi-direct product

The direct product group $\overset{\text{group}}{G\times H}$ has an underlying set which is just the Cartesian product of the sets of G and H. Group multiplication is inherited.

$$(\underset{\in G}{g},\underset{\in H}{h}) \in G\times H \qquad \boxed{(g_1,h_1)\cdot(g_2,h_2) = (g_1\times g_2,\, h_1 * h_2)}$$

Examples: $S_3\times SO(10)$ $\quad Z_2\times Z_3 = Z_6$ $\quad Z_2\times Z_2 \neq Z_4$

This is a special case of an indirect product $G\rtimes_\varphi H$. The idea is that $h_2\in H$ can be used to change g_1: $g_1 \longrightarrow \varphi_{h_2}(g_1)$

↑ Some appropriate function* of h_2

$$(g,h)\in G\rtimes_\varphi H \qquad \boxed{(g_1,h_1)\cdot(g_2,h_2) = (\varphi_{h_2}(g_1)\times g_2,\, h_1 * h_2)}$$

Direct and semi-direct product

$(n,h)\in N\rtimes_{\varphi}H$ $\quad$ $\boxed{(n_1,h_1)\cdot(n_2,h_2)=\left(n_1\cdot\varphi_{h_1}(n_2),\,h_1*h_2\right)}$

An example will help. Consider D_n: $\langle r,s\mid r^n=s^2=1,\ \underbrace{srs^{-1}=r^{-1}}_{\text{Equivalent to what we had before}}\rangle$

An element of D_n $g=\underbrace{rsrrs\cdots sr}_{\text{Word with } r\text{'s},\, s\text{'s}}$ can alway be written as $g=r^is^j$ $\left.\begin{matrix} i=0,1,\dots,n-1\\ j=0,1\end{matrix}\right]$ 2n elements

Let us multiply two elements of D_n:

$$\underbrace{r^is^j}_{g}\cdot\underbrace{r^{i'}s^{j'}}_{g'}=\begin{cases} r^i\overbrace{(s\,r^{i'}s^{-1})}^{r^{-i'}}s\,s^{j'} & \text{if } j=1\\ r^{i+i'}s^{j'} & \text{if } j=0\end{cases}=\underbrace{\begin{cases} r^{i-i'}s^{j'+1} & j=1\\ r^{i+i'}s^{j'} & j=0\end{cases}}$$

We have $g\cdot g'$ in a canonical form

Compare with taking the group elements as tuples (r^i,s^j):

$$(r^i,s^j)\cdot(r^{i'},s^{j'})=(r^i\underbrace{\varphi_{s^j}}_{s^jr^{i'}s^{-j}}(r^{i'}),\,s^j\cdot s^{j'})$$

$$\boxed{D_n=\mathbb{Z}_n\rtimes\mathbb{Z}_2}$$

$\mathbb{Z}_n$ here is a normal subgroup of D_n

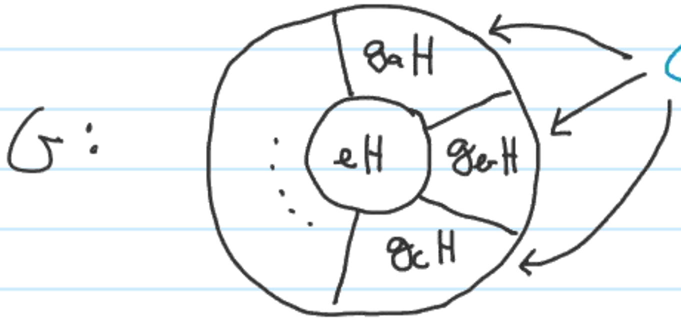
Cosets, normal subgroup, factor group

Consider a subgroup H of G. It can be used to split G in $\frac{|G|}{|H|}$ **cosets**:

$$G = eH + g_a H + g_b H + \cdots$$

Non overlapping (right) cosets

There is some arbitrariness in picking $e, g_a, g_b, \cdots$ which are representatives of each coset



Cosets (only eH is a subgroup)

Cosets, normal subgroup, factor group

Consider a subgroup H of G and that of all $g \in G$, $h \in H$ we have $ghg^{-1} \in H$.

I.e. $\boxed{gHg^{-1} = H}$ These are called normal subgroups.

$$\boxed{H \triangleleft G}$$

The cosets $g_a H$ are particularly interesting: they have a group structure.

$$(g_a H)\cdot(g_b H) = \underbrace{g_a g_b}_{g_c}\underbrace{(g_b^{-1} H g_b)}_{H} H = g_c H \quad \left(eH = \text{id.}\ ;\ g^{-1}H \text{ inverse of } gH \right)$$

This is called the quotient/factor group: $\boxed{G/H}$

Note Even when H is not a normal subgroup of G, one can use G/H to denote the cosets of H

Automorphisms (the symmetry of symmetries)

For $x, g \in G$ $\varphi_g(x) \equiv g^{-1} x g$ is a function which conjugates x by g.

Why is this interesting?

$$\varphi_g(x) \cdot \varphi_g(x') = \varphi_g(x \cdot x')$$

i.e. for any $g \in G$ we may define a funtion $G \xrightarrow{\varphi_g} G$ which preserves the group structure

Such functions are called automorphisms of G. They are symmetries of G.

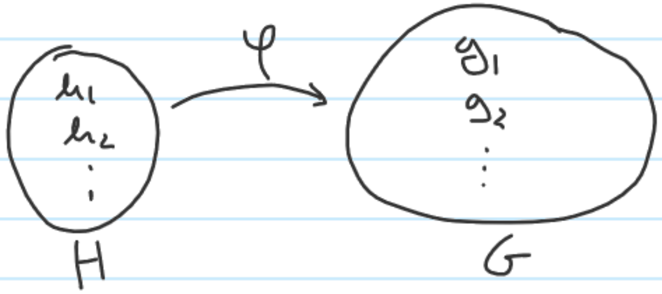
The set of all automorphisms of a group G forms a group: $\mathrm{Aut}(G)$

Some of these symmetries are trivial $\rightsquigarrow$ φ_g for some g. They form the group of inner automorphisms $\mathrm{Inn}(G)$

The remaining automorphisms are called outer automorphisms: $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$

Subgroup embeddings

If H is a group and it's elements can be mapped to some bigger group G and this map preserves the group structure of H, then H is a subgroup of G.

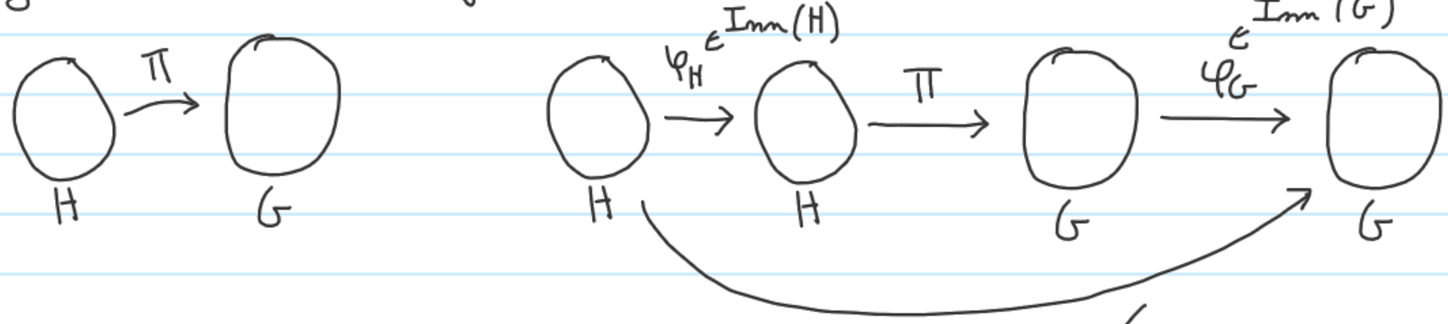


$\varphi(h_i)\in G$; $\varphi(h_i\cdot h_j)=\varphi(h_i)*\varphi(h_j)$

- This map is an embedding of H in G.
- We all know intuitively what is a subgroup. I bring this up because sometimes H can be embedded in G in more than one way [e.g. SU(2) in SU(3)]

Subgroup embeddings

Imagine two embeddings π and π' of H in G. When do we consider π and π' equivalent?



$\pi' = \varphi_G \circ \pi \circ \cancel{\varphi_H}$

not needed

π and π' are equivalent iff $\exists \varphi \in \mathrm{Inn}(G)$ such that $\pi' = \varphi \circ \pi$.

But this is hard to check. E. Dynkin came up with the concept of linear equivalence of 2 embeddings:

If $\forall \rho$ rep of G $\rho \circ \pi \underset{\substack{\uparrow \\ \text{similar}}}{\sim} \rho \circ \pi'$ then π' and π are linearly equivalent

Much to unpack here. Just keep in mind that an $H \subset G$ can have many embeddings.

Simple groups

Simple groups are those which do not have (non-trivial) normal subgroups.

In a way, they can be compared to prime numbers:

Every natural number has a unique prime decomposition ⟷ Every group can be "decomposed" into simple groups (not unique though)

Eg: $167706 = 2\cdot 3^2 \cdot 7 \cdot 11^3$

The way this works

- Start with G and pick one of its maximal* normal subgroups H_1;
- Do the same for H_1, by find a maximal normal subgr. H_2

...

$$G \triangleright H_1 \triangleright H_2 \triangleright \cdots \triangleright H_k \triangleright e$$

Trivial group

Simple group

G/H_1 $\quad$ H_1/H_2

Simple groups

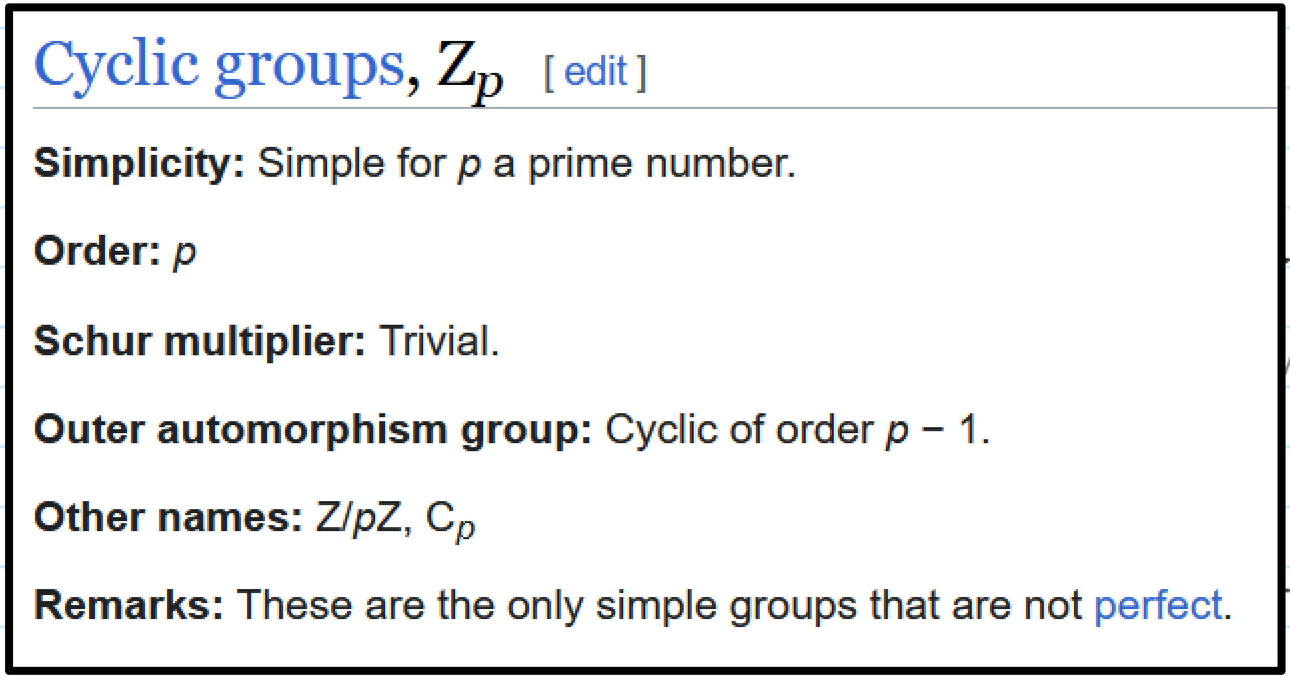
E.g.:

$$\begin{tikzcd}
S_4 & \triangleright & A_4 & \triangleright & \mathbb{Z}_2\times\mathbb{Z}_2 & \triangleright & \mathbb{Z}_2 & \triangleright & e \\
 & S_4/A_4=\mathbb{Z}_2 \arrow[u, dashed] & & A_4/\mathbb{Z}_2\times\mathbb{Z}_2=\mathbb{Z}_3 \arrow[u, dashed] & & \mathbb{Z}_2\times\mathbb{Z}_2/\mathbb{Z}_2=\mathbb{Z}_2 \arrow[u, dashed] & & \mathbb{Z}_2/e=\mathbb{Z}_2 \arrow[u, dashed] &
\end{tikzcd}$$

In green are the <u>factor groups</u>; they are <u>simple</u> (because the H_i are maximal normal subgroups)

So now that we are convinced that simple groups are the "building blocks" for all finite groups, what is the list of simple groups?

Simple groups



Cyclic groups, Z_p [edit]

Simplicity: Simple for *p* a prime number.

Order: *p*

Schur multiplier: Trivial.

Outer automorphism group: Cyclic of order *p* − 1.

Other names: Z/*p*Z, C_p

Remarks: These are the only simple groups that are not perfect.

$Z_2 \triangleright e$

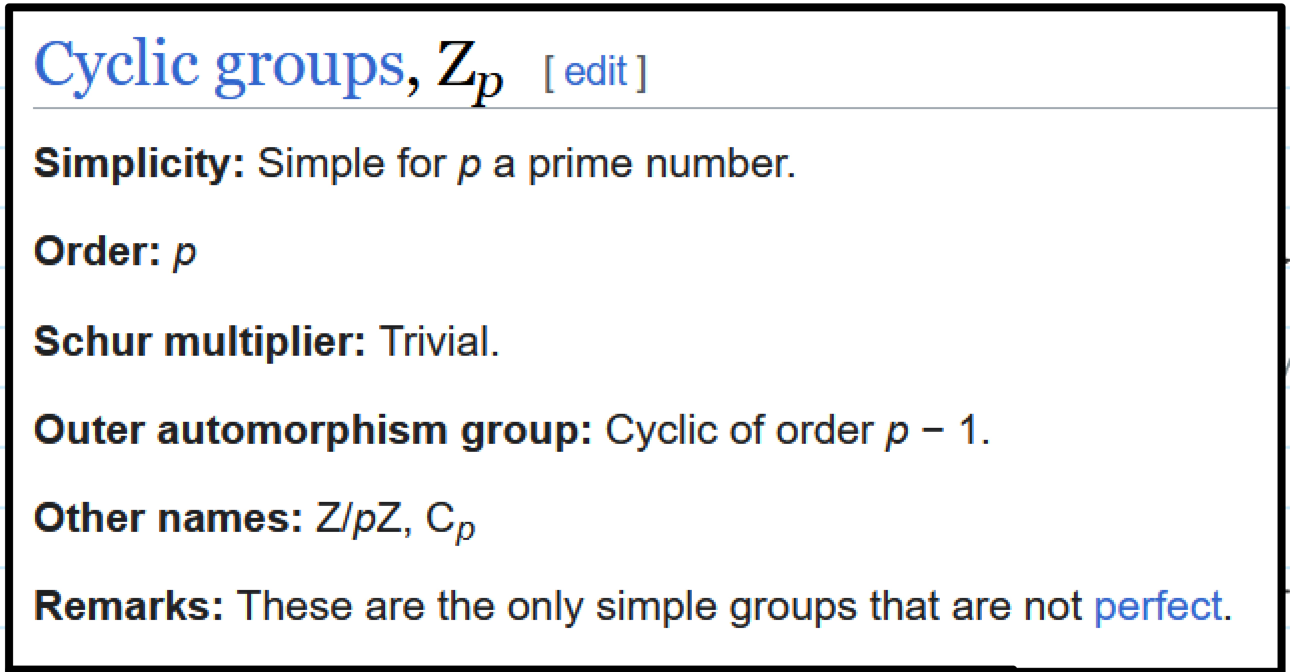
$/Z_2 = Z_2$

$Z_2/e = Z_2$

simple (because the H_i are maximal normal subgroups)

So now that we are convinced that simple groups are the "building blocks" for all finite groups, what is the list of simple groups?

Simple groups



Cyclic groups, Z_p [edit]

Simplicity: Simple for *p* a prime number.

Order: *p*

Schur multiplier: Trivial.

Outer automorphism group: Cyclic of order $p - 1$.

Other names: Z/*p*Z, C_p

Remarks: These are the only simple groups that are not perfect.

Alternating groups, A_n, $n > 4$ [edit]

$Z_2 \triangleright e$

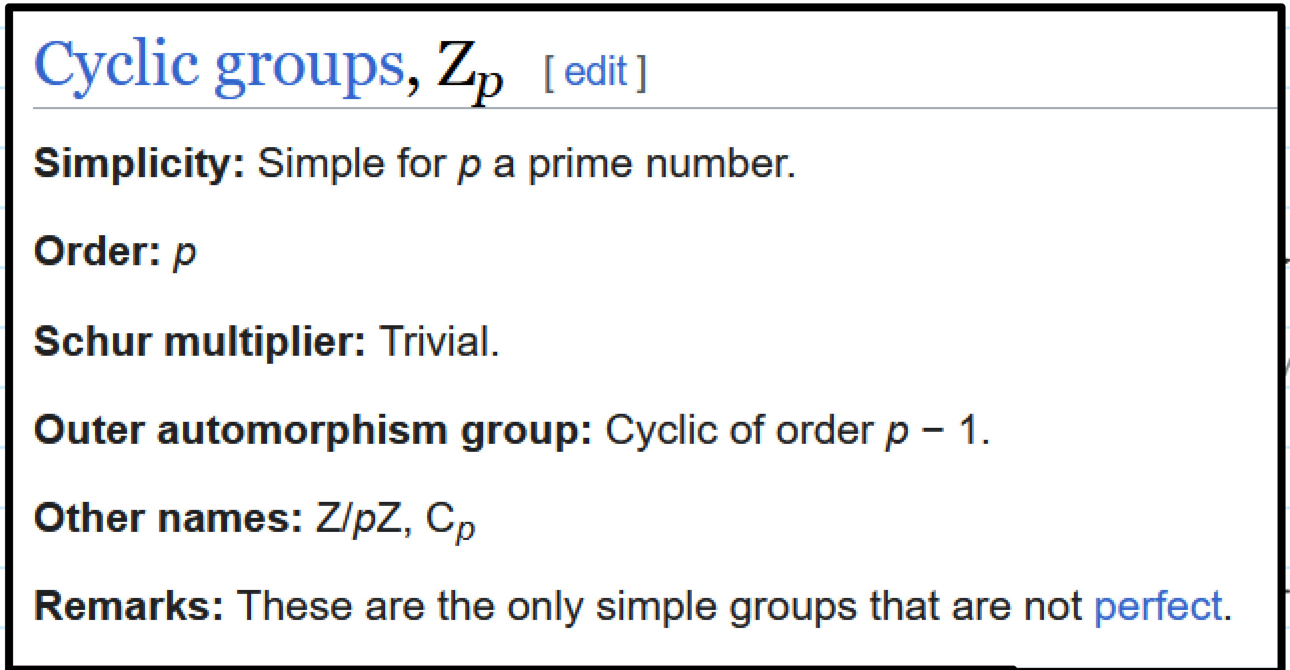
$/Z_2 = Z_2$

$Z_2 / e = Z_2$

simple (because the H_i are maximal normal subgroups)

So now that we are convinced that simple groups are the "building blocks" for all finite groups, what is the list of simple groups?

Simple groups



Cyclic groups, Z_p [edit]

Simplicity: Simple for *p* a prime number.

Order: *p*

Schur multiplier: Trivial.

Outer automorphism group: Cyclic of order *p* − 1.

Other names: Z/*p*Z, C_p

Remarks: These are the only simple groups that are not perfect.

$\mathbb{Z}_2 \triangleright e$

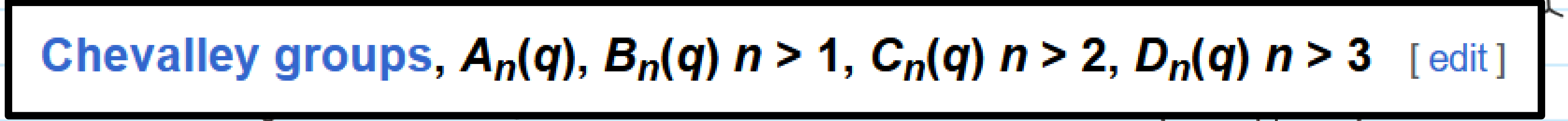
$/\mathbb{Z}_2 = \mathbb{Z}_2$

$\mathbb{Z}_2 / e = \mathbb{Z}_2$

simple (because the H_i are maximal normal subgroups)

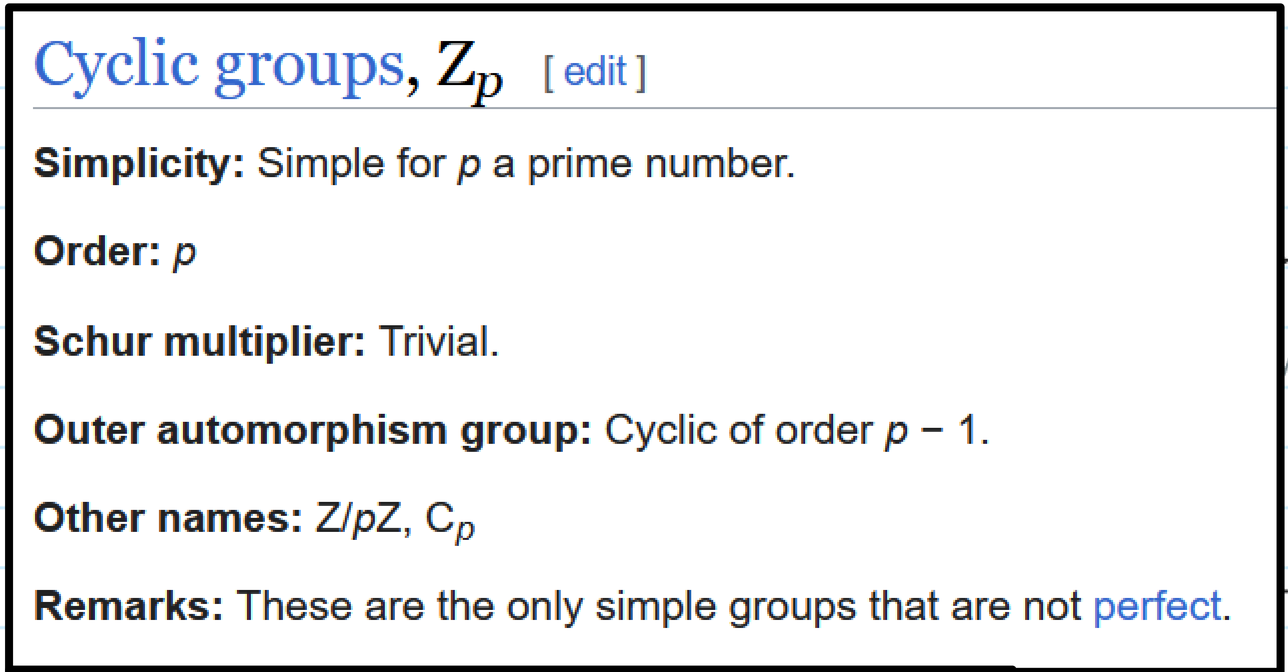
the "building blocks" for

Alternating groups, A_n, $n > 4$ [edit]



Chevalley groups, $A_n(q)$, $B_n(q)$ $n > 1$, $C_n(q)$ $n > 2$, $D_n(q)$ $n > 3$ [edit]

Simple groups



Cyclic groups, Z_p [edit]

Simplicity: Simple for *p* a prime number.

Order: *p*

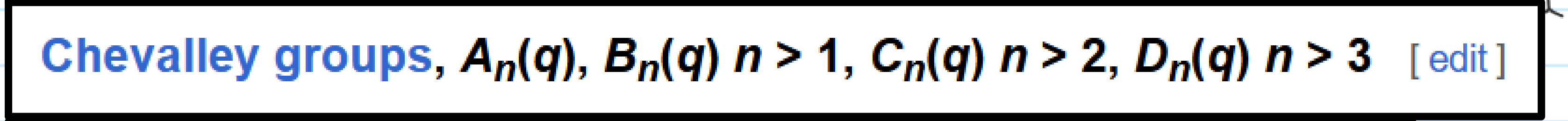
Schur multiplier: Trivial.

Outer automorphism group: Cyclic of order *p* − 1.

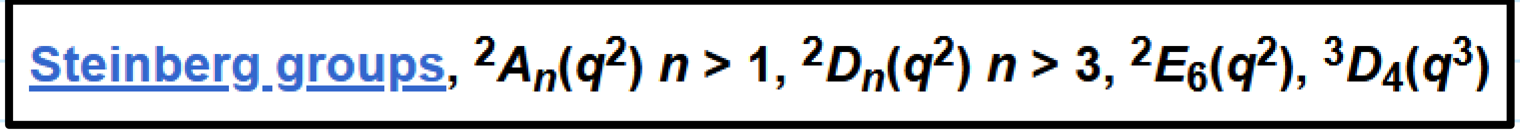
Other names: Z/*p*Z, C_p

Remarks: These are the only simple groups that are not perfect.

Alternating groups, A_n, $n > 4$ [edit]



Chevalley groups, $A_n(q)$, $B_n(q)$ $n > 1$, $C_n(q)$ $n > 2$, $D_n(q)$ $n > 3$ [edit]



Steinberg groups, ${}^2A_n(q^2)$ $n > 1$, ${}^2D_n(q^2)$ $n > 3$, ${}^2E_6(q^2)$, ${}^3D_4(q^3)$

$Z_2 \triangleright e$

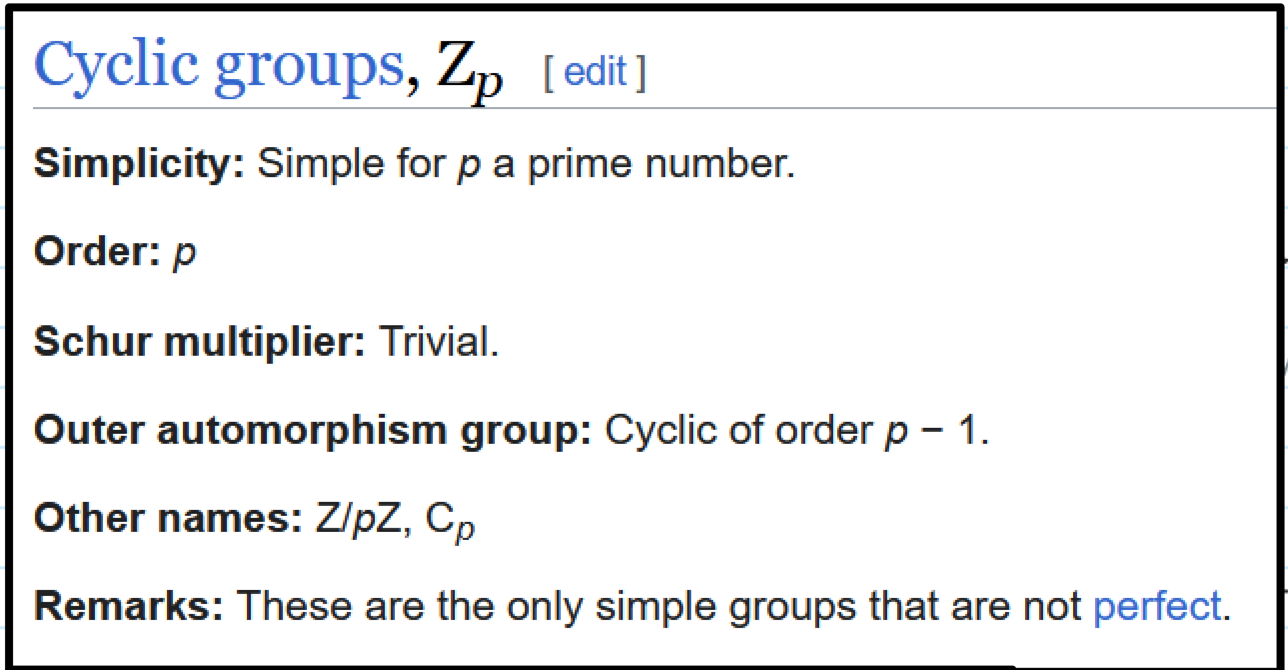
$/Z_2 = Z_2$

$Z_2/e = Z_2$

simple (because the H_i are maximal normal subgroups)

"building blocks" for

Simple groups



Cyclic groups, Z_p [edit]

Simplicity: Simple for p a prime number.

Order: p

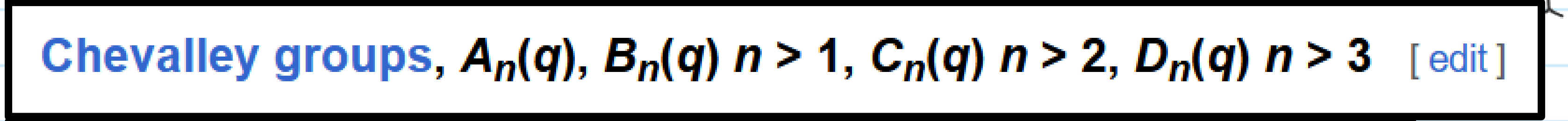
Schur multiplier: Trivial.

Outer automorphism group: Cyclic of order $p - 1$.

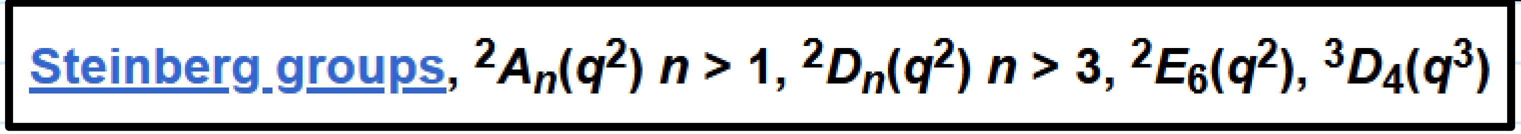
Other names: Z/pZ, C_p

Remarks: These are the only simple groups that are not perfect.

Alternating groups, A_n, $n > 4$ [edit]



Chevalley groups, $A_n(q)$, $B_n(q)$ $n > 1$, $C_n(q)$ $n > 2$, $D_n(q)$ $n > 3$ [edit]



Steinberg groups, ${}^2A_n(q^2)$ $n > 1$, ${}^2D_n(q^2)$ $n > 3$, ${}^2E_6(q^2)$, ${}^3D_4(q^3)$

Suzuki groups, ${}^2B_2(2^{2n+1})$

$\mathbb{Z}_2 \triangleright$

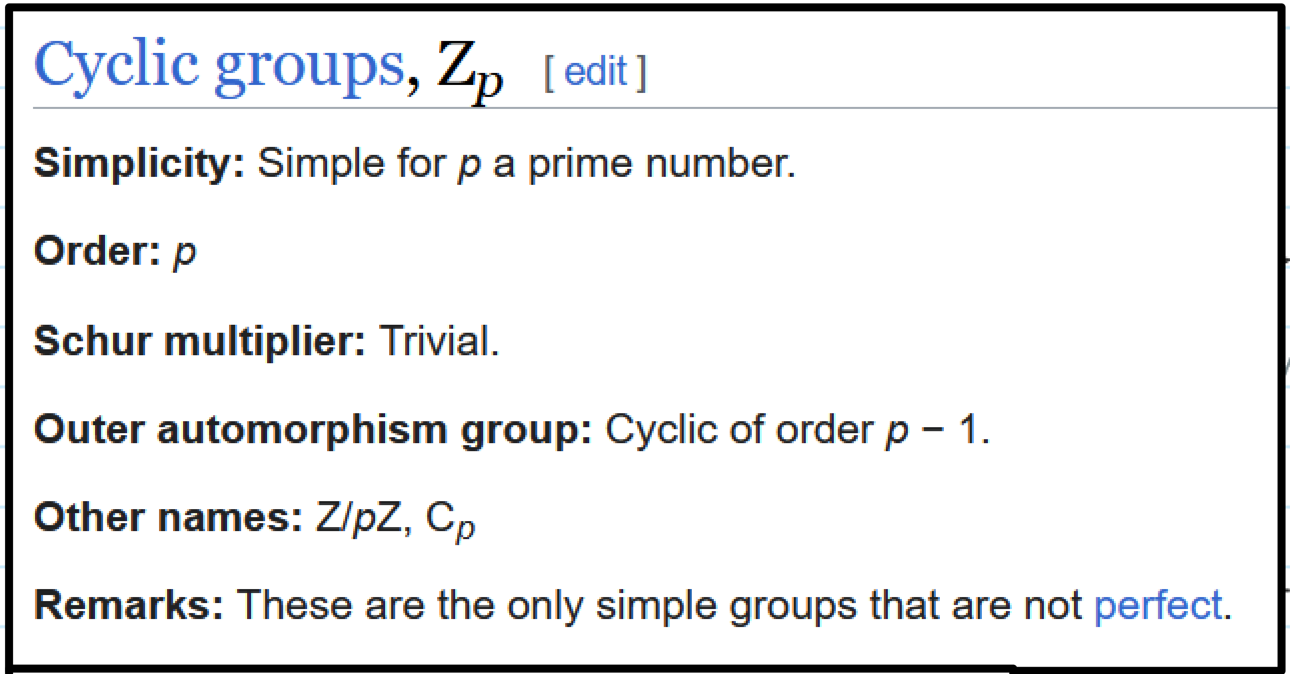
$/\mathbb{Z}_2 = \mathbb{Z}_2$

$\mathbb{Z}_2/e = \mathbb{Z}_2$

simple (because the H_i are maximal normal subgroups)

the "building blocks" for

Simple groups



Cyclic groups, Z_p [edit]

Simplicity: Simple for p a prime number.

Order: p

Schur multiplier: Trivial.

Outer automorphism group: Cyclic of order $p - 1$.

Other names: Z/pZ, C_p

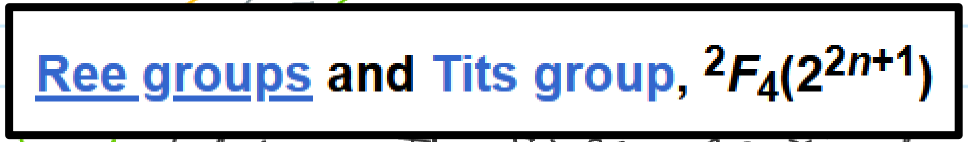
Remarks: These are the only simple groups that are not perfect.

$Z_2 \triangleleft$

$/Z_2 = Z_2$

Z / $= Z$

Suzuki groups, $^2B_2(2^{2n+1})$

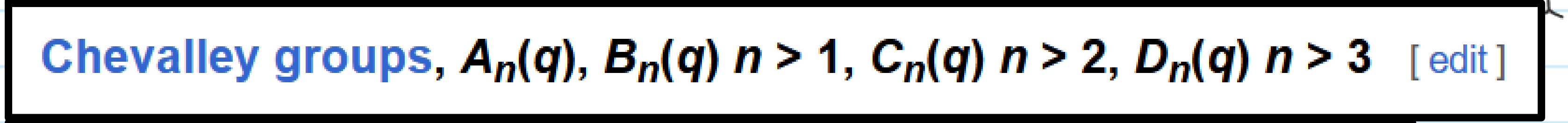


Ree groups and Tits group, $^2F_4(2^{2n+1})$

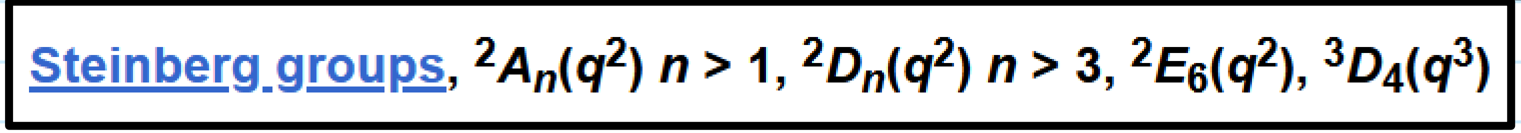
simple (because the H_i are maximal normal subgroup)

Alternating groups, A_n, $n > 4$ [edit]

the "building blocks" for

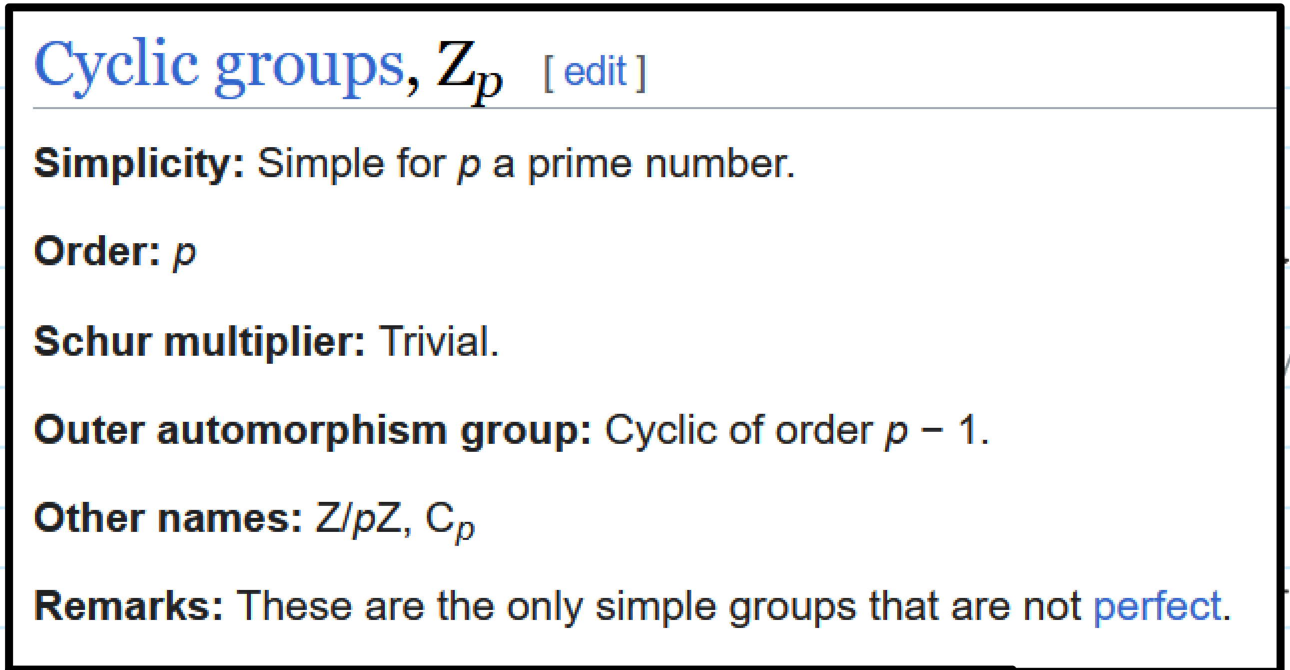


Chevalley groups, $A_n(q)$, $B_n(q)$ $n > 1$, $C_n(q)$ $n > 2$, $D_n(q)$ $n > 3$ [edit]



Steinberg groups, $^2A_n(q^2)$ $n > 1$, $^2D_n(q^2)$ $n > 3$, $^2E_6(q^2)$, $^3D_4(q^3)$

Simple groups



Cyclic groups, Z_p [edit]

Simplicity: Simple for p a prime number.

Order: p

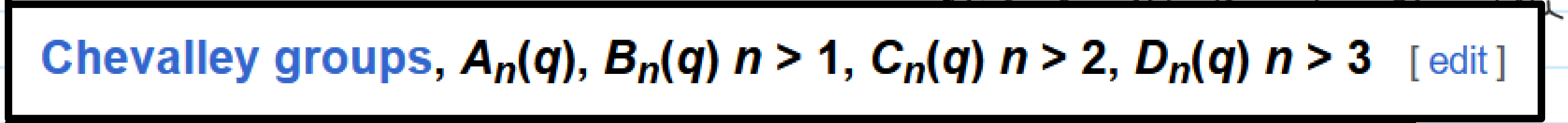
Schur multiplier: Trivial.

Outer automorphism group: Cyclic of order $p - 1$.

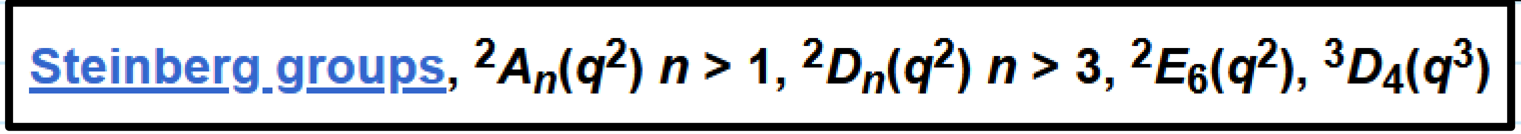
Other names: Z/pZ, C_p

Remarks: These are the only simple groups that are not perfect.

Alternating groups, A_n, $n > 4$ [edit]



Chevalley groups, $A_n(q)$, $B_n(q)$ $n > 1$, $C_n(q)$ $n > 2$, $D_n(q)$ $n > 3$ [edit]

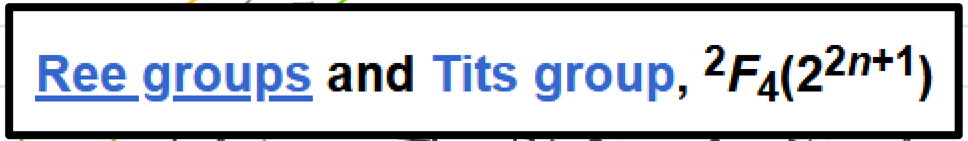


Steinberg groups, $^2A_n(q^2)$ $n > 1$, $^2D_n(q^2)$ $n > 3$, $^2E_6(q^2)$, $^3D_4(q^3)$

$Z_2 \triangleright$

$/Z_2 = Z_2$

Suzuki groups, $^2B_2(2^{2n+1})$



Ree groups and Tits group, $^2F_4(2^{2n+1})$

simple (because the H_i are maximal

ups)

Ree groups, $^2G_2(3^{2n+1})$

the "building blocks" for

Simple groups

Cyclic groups, Z_p

Simplicity: Simple for p

Order: p

Schur multiplier: Trivial

Outer automorphism g

Other names: Z/pZ, C_p

Remarks: These are the

Alternating grou

Chevalley groups,

Steinberg groups, 2

Mathieu groups	M_{11}	7920
	M_{12}	95 040
	M_{22}	443 520
	M_{23}	10 200 960
	M_{24}	244 823 040
Janko groups	J_1	175 560
	J_2	604 800
	J_3	50 232 960
	J_4	86 775 571 046 077 562 880
Conway groups	Co_3	495 766 656 000
	Co_2	42 305 421 312 000
	Co_1	4 157 776 806 543 360 000
Fischer groups	Fi_{22}	64 561 751 654 400
	Fi_{23}	4 089 470 473 293 004 800
	Fi_{24}'	1 255 205 709 190 661 721 292 800
Higman–Sims group	HS	44 352 000
McLaughlin group	McL	898 128 000
Held group	He	4 030 387 200
Rudvalis group	Ru	145 926 144 000
Suzuki sporadic group	Suz	448 345 497 600
O'Nan group	O'N	460 815 505 920
Harada–Norton group	HN	273 030 912 000 000
Lyons group	Ly	51 765 179 004 000 000
Thompson group	Th	90 745 943 887 872 000
Baby Monster group	B	4 154 781 481 226 426 191 177 580 544 000 000
Monster group	M	808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

groups, ${}^2B_2(2^{2n+1})$

and Tits group, ${}^2F_4(2^{2n+1})$

se the H_i are maximal

ups)

groups, ${}^2G_2(3^{2n+1})$

"building block" for

!

Orbit and stability group

Take some set S and a group G acting on it. That means that there is some function $\varphi: G\times S \to S$, i.e. $\varphi(g,s)=s'$. It feels much better to just write $\boxed{g(s)=s'}$.

$\in G$ $\in S$

This G-group action on S must obey $e(s)=s'$ and $g(g'(s))=(gg')(s)$

I will only mention two concepts on this matter: orbit and stability group

Very intuitive: pick some $s\in S$;

$$\boxed{Gs=\{g_i s\} \longleftarrow \text{the orbit of } s\in S}$$

↑ use all elements of G

the set of all $g_i \in G$ which leave s invariant ($\boxed{g_i(s)=s}$) is the <u>stability group</u> of s, $K(s)$ (also called <u>little group</u>).

Orbit and stability group

E.g. $\underbrace{\text{Consider rotations}}_{G=SO(3)}$ in $\underset{S}{\mathbb{R}^3}$. Take $s=(0,0,1)\in S$ (north pole).

- Then z-axis rotations leave s invariant, so $K(s)=SO(2)$
- Gs = set of all points at a distance 1 from the origin. I.e. the orbit of s is the sphere S_2.

stability group $= SO(2)$; Orbit $= S_2$; $\underbrace{SO(3)/SO(2)}_{\text{Set of Cosets}} = S_2$
(Little group)

E.g #2: Take the Poincaré group (more on this later) and consider a massive particle. Go to rest frame where $P^\mu=(M,0,0,0)$.

Boosts change P^μ, so the stability group is $\underbrace{SO(3)}_{\text{3D rotations}}$ ⇝ tagged with a spin

For massless particles, let us choose $P^\mu=(E,0,0,E)$. Not so intuitive, but the stability group here is E_2 = group of symmetries of a 2D plane. Its representations can be tagged with a "helicity".

Representations

It is very useful to study the action of a group G on a vector field.

For that we associate to every $g \in G$ a linear operator on V. I.e. a matrix.

$$g \in G \xrightarrow{\rho} \rho(g) \in GL(V)$$

Recall: group with all invertible matrices action on V

This mapping must be an homomorphism:

$$\rho(g_1)\rho(g_2) = \rho(g_1 \cdot g_2) \rightsquigarrow \text{preserve } G \text{ structure}$$

We call "representation" to the mapping ρ itself, but also to the matrices $\rho(g_i)$ and often even to the vector space V.

"Field ψ (i.e. the V) is an irrep of the gauge group"

The vector space V for us is usually $\mathbb{R}^n$ or $\mathbb{C}^n$.

Representations

$Z_3 = \{e, a, a^2\}$

- $\rho(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\rho(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\rho(a^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $V = \mathbb{R}^3$
- $\sigma(e) = \sigma(a) = \sigma(a^2) = (1)$ $V = \mathbb{R}^1$ is a trivial representation
- $\varphi(e) = (1)$ $\varphi(a) = (\omega)$ $\varphi(a^2) = (\omega^2)$ $V = \mathbb{R}^1$, $\omega \equiv \exp\left(i \frac{2\pi}{3}\right)$ $\omega^3 = 1$

$\mathbb{R}$ with "+" as the group operation

- $\rho(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ $\rightsquigarrow$ $\rho(x_1)\rho(x_2) = \begin{pmatrix} 1 & x_1 + x_2 \\ 0 & 1 \end{pmatrix} = \rho(x_1 + x_2)$ $V = \mathbb{R}^2$

Representations

Similar / Equivalent representations

$g \longrightarrow \rho(g)$ vs $g \longrightarrow \overbrace{S^{-1}\rho(g)S}^{\equiv\, \rho'(g)}$ for all $g \in G$

Same matrix

ρ and ρ' are equivalent

Unitary representations

$\rho(g)$ is a unitary matrix : $\rho^{\dagger}(g)\rho(g) = \mathbb{1}_m$

Faithful representation

If there are no $g_1 \neq g_2 \in G$ such that $\rho(g_1) = \rho(g_2)$ then the representation is faithful (one can actually use the $\rho(g_i)$ to define the group)

Representations

⇝ Some representations are special and have their own name :

→ Natural rep. of S_n = n-dimensional (reducible)

→ Fundamental irrep of $SU(n)$ = n

→ Spinor representation of $SO(n)$

→ Adjoint of a Lie group ($\doteq 3$ in $SU(2)$, $= 8$ in $SU(3)$, ...)

→ Regular representation

Real/pseudo-real/complex representation

Real / Complex / Pseudo-real representations

- If there is a basis where the representation matrices are all real,

$$\exists_B : B\rho(g_i)B^{-1} = \left(B\rho(g_i)B^{-1}\right)^* \quad \text{for all } g_i$$

then the representation is real.

- If ρ and ρ^* are not equivalent,

$$\nexists_B : \rho(g_i)^* = B\rho(g_i)B^{-1} \quad \text{for all } g_i$$

then the representation is complex.

- There is a third possibility. ρ and ρ^* might be isomorphic but it is impossible to make the matrices real.

The representation is pseudo-real.

E.g. SU(2) : 1, 3, 5, 7, ··· are real ; 2, 4, 6, ··· are pseudoreal

$SU(3): \underbrace{3, \overline{3}, 6, \overline{6}, \cdots}_{\text{complex}} ; \underbrace{8, 27, \cdots}_{\text{real}}$

S_n : all irrep's are real

Irreducible representations

Take $Z_2=\{e,a\}$ and $\rho(e)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\rho(a)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$.

With $S=\begin{pmatrix}1/\sqrt{2}&-1/\sqrt{2}\\1/\sqrt{2}&1/\sqrt{2}\end{pmatrix}$ we get an equivalent representation

$$S^{-1}\rho(e)S=\begin{pmatrix}1&0\\0&1\end{pmatrix} \qquad S^{-1}\rho(a)S=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

which is (block) diagonal.

Any representation whose matrices cannot be simultaneously block diagonalized

i.e. $\rho(g_i)=\begin{pmatrix}\blacksquare&0\\0&\blacksquare\end{pmatrix}$ for all $g_i\in G$ is not possible

is said to be irreducible (an irrep)

They are very important: they are the building blocks of all representations

Irreducible representations

E.g.

$$\begin{pmatrix} Q\\ u^c\\ d^c\\ L\\ e^c\\ H\end{pmatrix} \xrightarrow[\text{Lorentz}]{\text{Gauge group}} \begin{pmatrix} \boxed{6\times6} & & & & 0 & \\ & \boxed{3\times3} & & & & \\ & & \boxed{3\times3} & & & \\ & & & \boxed{2\times2} & & \\ & 0 & & & \boxed{1\times1} & \\ & & & & & \boxed{2\times2} \end{pmatrix} \times \begin{pmatrix} Q\\ u^c\\ d^c\\ L\\ e^c\\ H\end{pmatrix}$$

The representation matrices are block diagonal

One could change basis, for example handling $Q_{1,1}+H_2-\sqrt{2}\,e^c$ but that looks like a very bad idea; rep matrices become more complicated

$\rightsquigarrow$ Often we call irreps by their dimention (maybe adding bar's and primes's)

Z_2: 1, 1' A_4: 1, 1', 1'', 3 SU(3): 1, 3, $\bar{3}$, 8, 10, 15, 15', ...

$\rightsquigarrow$ In other cases, some label can be used:

- → Half-integer j for the irreps of SU(2)
- → Partitions / Young diagrams for S_n (e.g. $S_3 \rightsquigarrow 1 = \square\square\square$, $1' = \begin{array}{c}\square\\ \square\\ \square\end{array}$, $2 = \begin{array}{cc}\square & \square\\ \square & \end{array}$)
- → Mass m and spin s for massive irreps of the Poincaré group

...

Application: quark/lepton flavor symmetries

Finite groups have been used to try to explain flavor in the SM.

Take the up quark masses:

$$\overline{u}_L \underbrace{\begin{pmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{pmatrix}}_{y^u = M^u/\langle H^0 \rangle \text{ can be anything}} u_R \langle H^0 \rangle$$

Here is an idea: what if the laws of physics are invariant under permutations of the quarks?

Application: quark/lepton flavor symmetries

① $S_3^L \times S_3^R$ ⤳ We may permute u_L's and u_R's independently.

$$M^u = A \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow$$ two massless quarks (not a bad approximation: $m_t >> m_{u,c}$)

② S_3 ⤳ u_L's and u_R's transform in the same way

$$u_L \longrightarrow T_i \, u_L \qquad T_i = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \ldots$$

$$u_R \longrightarrow T_i \, u_R$$

$$M^u = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} \xrightarrow{\text{change of basis}} \begin{pmatrix} A-B & 0 & 0 \\ 0 & A-B & 0 \\ 0 & 0 & A+2B \end{pmatrix}$$ 2 degenerate masses

$\boxed{\text{This last result was predictable}}$ Let's see why...

Application: quark/lepton flavor symmetries

Ⓐ symmetry requires $T_i^\dagger M^u T_i = M^u \Leftrightarrow [M^u, T^i] = 0$

Ⓑ Our T^i is reducible: natural rep of $S_n = 1 + (n-1)$.

So there is a basis where

$$T_i = \begin{pmatrix} \boxed{\begin{matrix} \times & \times \\ \times & \times \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{\times} \end{pmatrix} \begin{matrix} \leftarrow 2 \\ \\ \leftarrow 1 \end{matrix}$$

Ⓒ The T^i are not a random collection of 6 3-dim matrices; They represent a group.

Schur lemmas: If U and U' are irreps of some group G and there is a matrix A such that

$$A \cdot U'(g) = U(g) \cdot A \qquad \forall g \in G$$

Then $\begin{cases} A \propto \mathbb{1} & \text{if } U = U' \\ A = 0 & \text{if } U \neq U' \end{cases}$

$\uparrow$ "equivalent" / "inequivalent"

Application: quark/lepton flavor symmetries

$$\overbrace{\begin{pmatrix} R_2 & 0 \\ 0 & R_1 \end{pmatrix}}^{T^i}\overbrace{\begin{pmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{pmatrix}}^{M^u} - \overbrace{\begin{pmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{pmatrix}}^{M^u}\overbrace{\begin{pmatrix} R_2 & 0 \\ 0 & R_1 \end{pmatrix}}^{T^i} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{lcl} R_2 M_{22} - M_{22} R_2 = 0 & \longrightarrow & M_{22} = A' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ R_2 M_{21} - M_{21} R_1 = 0 & \longrightarrow & M_{21} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ R_1 M_{12} - M_{12} R_2 = 0 & \longrightarrow & M_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix} \\ R_1 M_{11} - M_{11} R_1 = 0 & \longrightarrow & M_{11} = \begin{pmatrix} B' \end{pmatrix} \end{array}\right\} M^u = \begin{pmatrix} A' & & \\ & A' & \\ & & B' \end{pmatrix}$$

So just by knowing that our rep $= R_1 + R_2 + \cdots \Rightarrow M^u = \begin{pmatrix} \boxed{\alpha_1 \mathbb{1}} & 0 & 0 \\ 0 & \boxed{\alpha_2 \mathbb{1}} & 0 \\ 0 & 0 & \ddots \end{pmatrix}$

Sizes of blocks given by irrep sizes; off-diagonal blocks are possible with repeated irreps

Application: quark/lepton flavor symmetries

We have discussed masses only; mixing can also be predicted by symmetry.

For some time the θ_{13} angle in leptons was compatible with 0.

[Daya Bay and other reactor experiments later showed that $\theta_{13} > 0$; $\theta_{13} \sim 8^\circ - 9^\circ$]

The whole lepton mixing matrix was compatible with a tri-bimaximal form:
(TBM)

$$|U_{PMNS}| = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \\ \sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \end{pmatrix}$$

Harrison, Perkins, Scott 0202074

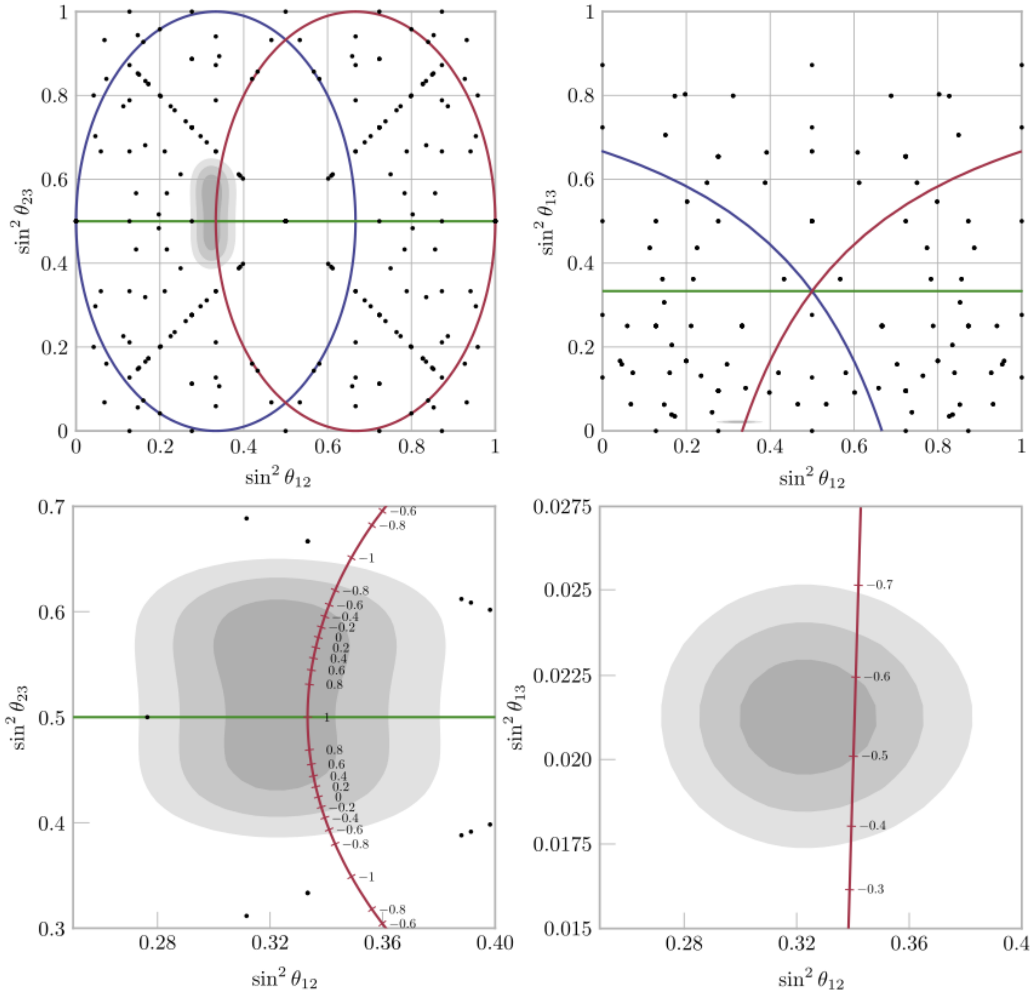
Very symmetric looking; A_4 was used extensively to get it. But a convincing case was made that S_4 was in fact the TBM symmetry group.

Lam 0809.1185

With the same mindset we classified all mixing patterns predicted by all finite group involving genuine 3 generation mixing.

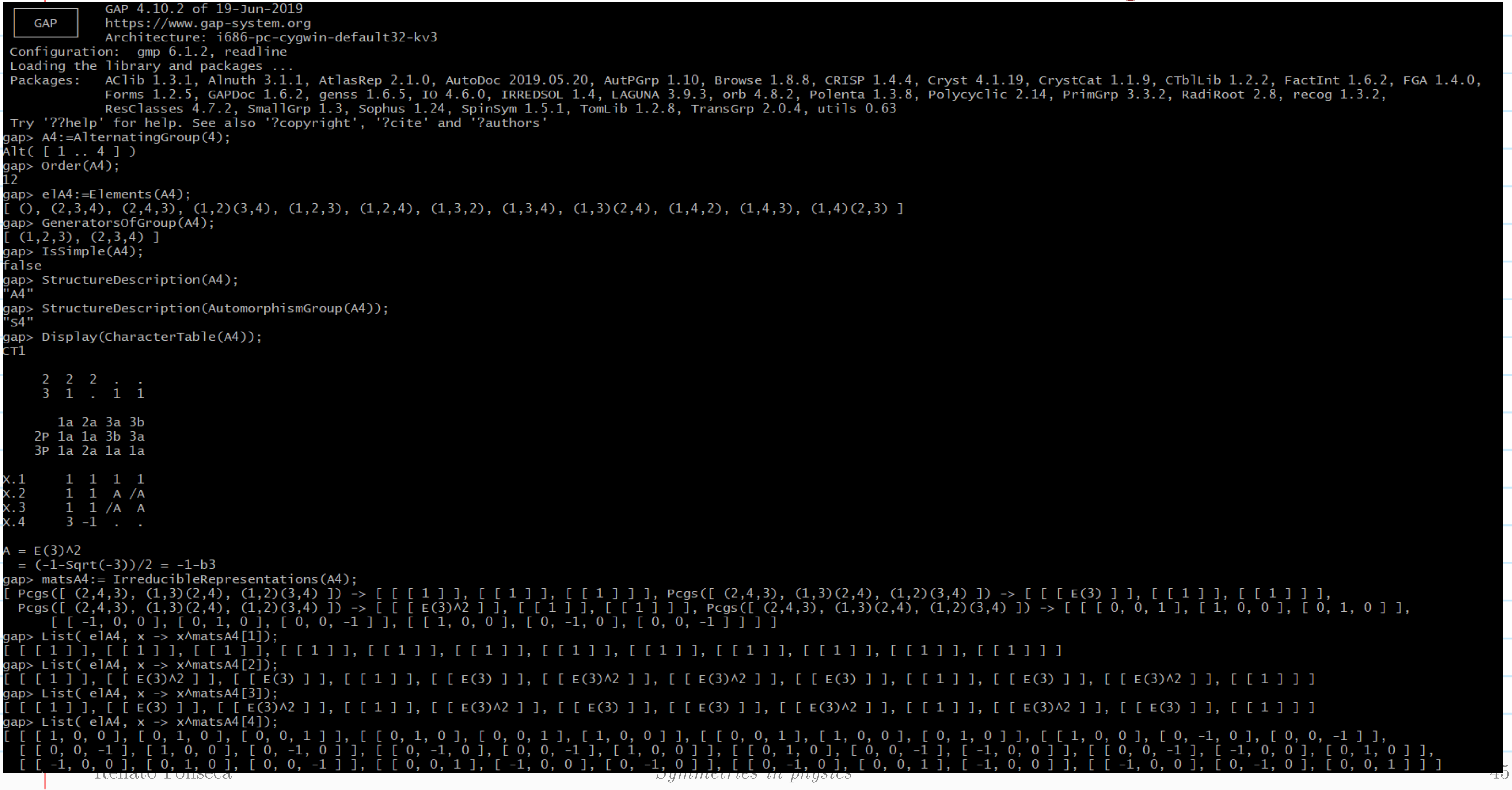
RF, Grimus 1405.3678

Application: quark/lepton flavor symmetries



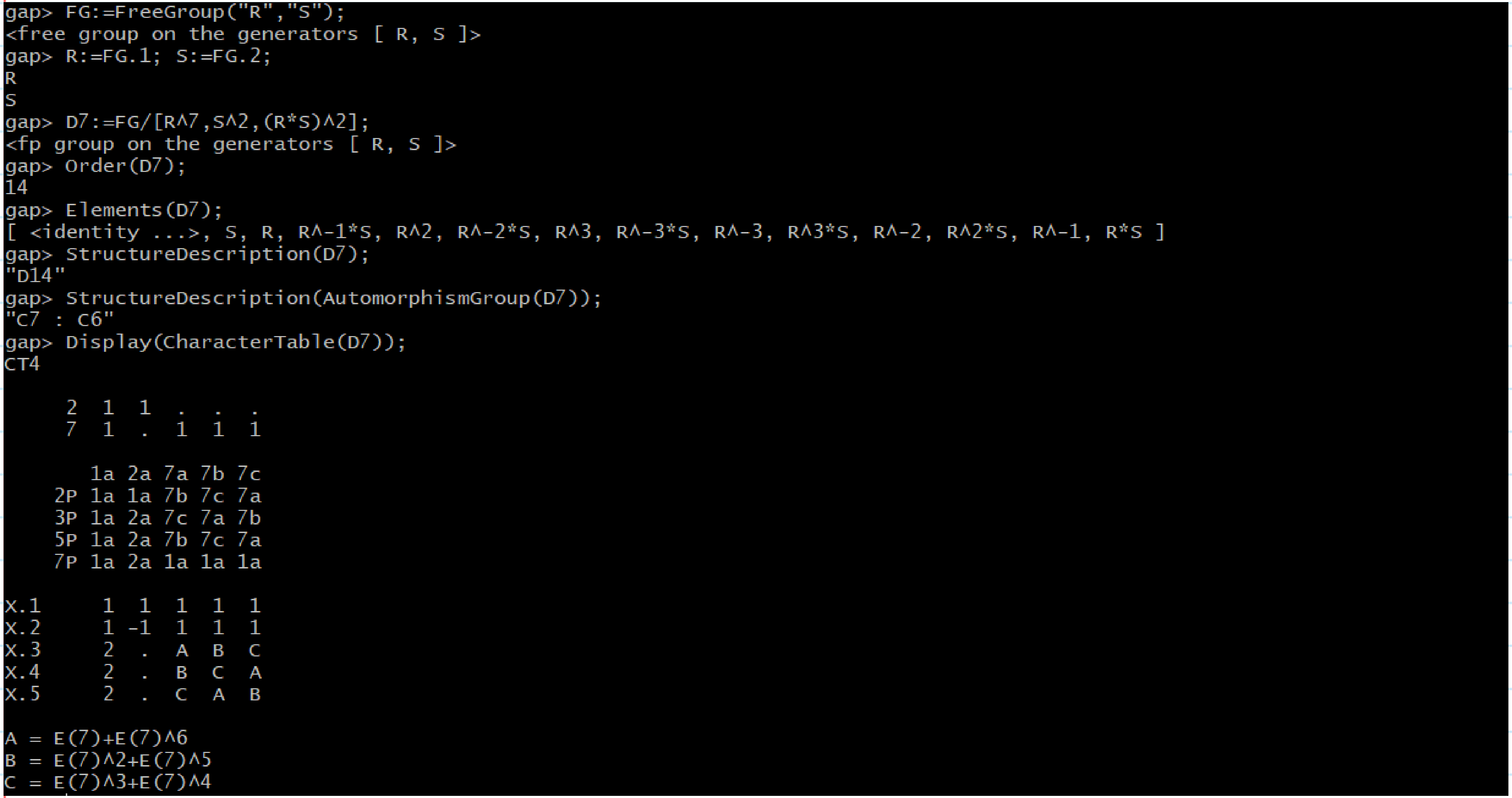
The lines correspond essentially to the $\Delta(6n^2)$ groups which had already been identified in other works as giving a good fit to the data.

GAP examples: Predefined group



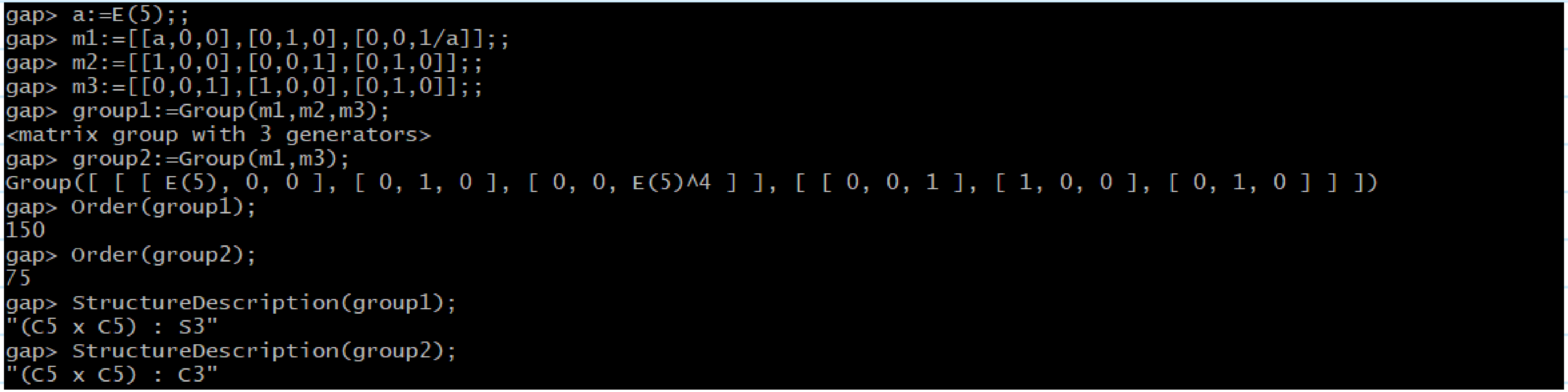
```
 ┌───────┐   GAP 4.10.2 of 19-Jun-2019
 │  GAP  │   https://www.gap-system.org
 └───────┘   Architecture: i686-pc-cygwin-default32-kv3
 Configuration:  gmp 6.1.2, readline
 Loading the library and packages ...
 Packages:   AClib 1.3.1, Alnuth 3.1.1, AtlasRep 2.1.0, AutoDoc 2019.05.20, AutPGrp 1.10, Browse 1.8.8, CRISP 1.4.4, Cryst 4.1.19, CrystCat 1.1.9, CTblLib 1.2.2, FactInt 1.6.2, FGA 1.4.0,
             Forms 1.2.5, GAPDoc 1.6.2, genss 1.6.5, IO 4.6.0, IRREDSOL 1.4, LAGUNA 3.9.3, orb 4.8.2, Polenta 1.3.8, Polycyclic 2.14, PrimGrp 3.3.2, RadiRoot 2.8, recog 1.3.2,
             ResClasses 4.7.2, SmallGrp 1.3, Sophus 1.24, SpinSym 1.5.1, TomLib 1.2.8, TransGrp 2.0.4, utils 0.63
 Try '??help' for help. See also '?copyright', '?cite' and '?authors'
gap> A4:=AlternatingGroup(4);
Alt( [ 1 .. 4 ] )
gap> Order(A4);
12
gap> elA4:=Elements(A4);
[ (), (2,3,4), (2,4,3), (1,2)(3,4), (1,2,3), (1,2,4), (1,3,2), (1,3,4), (1,3)(2,4), (1,4,2), (1,4,3), (1,4)(2,3) ]
gap> GeneratorsOfGroup(A4);
[ (1,2,3), (2,3,4) ]
gap> IsSimple(A4);
false
gap> StructureDescription(A4);
"A4"
gap> StructureDescription(AutomorphismGroup(A4));
"S4"
gap> Display(CharacterTable(A4));
CT1

     2  2  2  .  .
     3  1  .  1  1

       1a 2a 3a 3b
    2P 1a 1a 3b 3a
    3P 1a 2a 1a 1a

X.1      1  1  1  1
X.2      1  1  A /A
X.3      1  1 /A  A
X.4      3 -1  .  .

A = E(3)^2
  = (-1-Sqrt(-3))/2 = -1-b3
gap> matsA4:= IrreducibleRepresentations(A4);
[ Pcgs([ (2,4,3), (1,3)(2,4), (1,2)(3,4) ]) -> [ [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ] ], Pcgs([ (2,4,3), (1,3)(2,4), (1,2)(3,4) ]) -> [ [ [ E(3) ] ], [ [ 1 ] ], [ [ 1 ] ] ],
  Pcgs([ (2,4,3), (1,3)(2,4), (1,2)(3,4) ]) -> [ [ [ E(3)^2 ] ], [ [ 1 ] ], [ [ 1 ] ] ], Pcgs([ (2,4,3), (1,3)(2,4), (1,2)(3,4) ]) -> [ [ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ],
      [ [ -1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, -1 ] ], [ [ 1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, -1 ] ] ] ]
gap> List( elA4, x -> x^matsA4[1]);
[ [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ], [ [ 1 ] ] ]
gap> List( elA4, x -> x^matsA4[2]);
[ [ [ 1 ] ], [ [ E(3)^2 ] ], [ [ E(3) ] ], [ [ 1 ] ], [ [ E(3) ] ], [ [ E(3)^2 ] ], [ [ E(3)^2 ] ], [ [ E(3) ] ], [ [ 1 ] ], [ [ E(3) ] ], [ [ E(3)^2 ] ], [ [ 1 ] ] ]
gap> List( elA4, x -> x^matsA4[3]);
[ [ [ 1 ] ], [ [ E(3) ] ], [ [ E(3)^2 ] ], [ [ 1 ] ], [ [ E(3)^2 ] ], [ [ E(3) ] ], [ [ E(3) ] ], [ [ E(3)^2 ] ], [ [ 1 ] ], [ [ E(3)^2 ] ], [ [ E(3) ] ], [ [ 1 ] ] ]
gap> List( elA4, x -> x^matsA4[4]);
[ [ [ 1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, 1 ] ], [ [ 0, 1, 0 ], [ 0, 0, 1 ], [ 1, 0, 0 ] ], [ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ], [ [ 1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, -1 ] ],
  [ [ 0, 0, -1 ], [ 1, 0, 0 ], [ 0, -1, 0 ] ], [ [ 0, -1, 0 ], [ 0, 0, -1 ], [ 1, 0, 0 ] ], [ [ 0, 1, 0 ], [ 0, 0, -1 ], [ -1, 0, 0 ] ], [ [ 0, 0, -1 ], [ -1, 0, 0 ], [ 0, 1, 0 ] ],
  [ [ -1, 0, 0 ], [ 0, 1, 0 ], [ 0, 0, -1 ] ], [ [ 0, 0, 1 ], [ -1, 0, 0 ], [ 0, -1, 0 ] ], [ [ 0, -1, 0 ], [ 0, 0, 1 ], [ -1, 0, 0 ] ], [ [ -1, 0, 0 ], [ 0, -1, 0 ], [ 0, 0, 1 ] ] ]
```

GAP examples: From a presentation



```
gap> FG:=FreeGroup("R","S");
<free group on the generators [ R, S ]>
gap> R:=FG.1; S:=FG.2;
R
S
gap> D7:=FG/[R^7,S^2,(R*S)^2];
<fp group on the generators [ R, S ]>
gap> Order(D7);
14
gap> Elements(D7);
[ <identity ...>, S, R, R^-1*S, R^2, R^-2*S, R^3, R^-3*S, R^-3, R^3*S, R^-2, R^2*S, R^-1, R*S ]
gap> StructureDescription(D7);
"D14"
gap> StructureDescription(AutomorphismGroup(D7));
"C7 : C6"
gap> Display(CharacterTable(D7));
CT4

     2  1  1  .  .  .
     7  1  .  1  1  1

       1a 2a 7a 7b 7c
    2P 1a 1a 7b 7c 7a
    3P 1a 2a 7c 7a 7b
    5P 1a 2a 7b 7c 7a
    7P 1a 2a 1a 1a 1a

X.1     1  1  1  1  1
X.2     1 -1  1  1  1
X.3     2  .  A  B  C
X.4     2  .  B  C  A
X.5     2  .  C  A  B

A = E(7)+E(7)^6
B = E(7)^2+E(7)^5
C = E(7)^3+E(7)^4
```

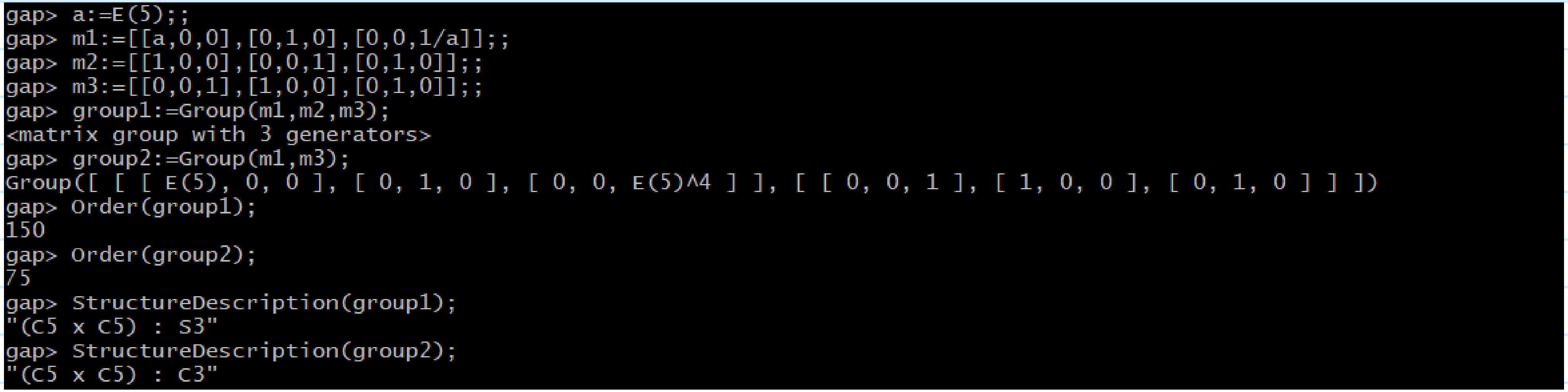
GAP examples: From matrices



```
gap> a:=E(5);;
gap> m1:=[[a,0,0],[0,1,0],[0,0,1/a]];;
gap> m2:=[[1,0,0],[0,0,1],[0,1,0]];;
gap> m3:=[[0,0,1],[1,0,0],[0,1,0]];;
gap> group1:=Group(m1,m2,m3);
<matrix group with 3 generators>
gap> group2:=Group(m1,m3);
Group([ [ [ E(5), 0, 0 ], [ 0, 1, 0 ], [ 0, 0, E(5)^4 ] ], [ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ] ])
gap> Order(group1);
150
gap> Order(group2);
75
gap> StructureDescription(group1);
"(C5 x C5) : S3"
gap> StructureDescription(group2);
"(C5 x C5) : C3"
```

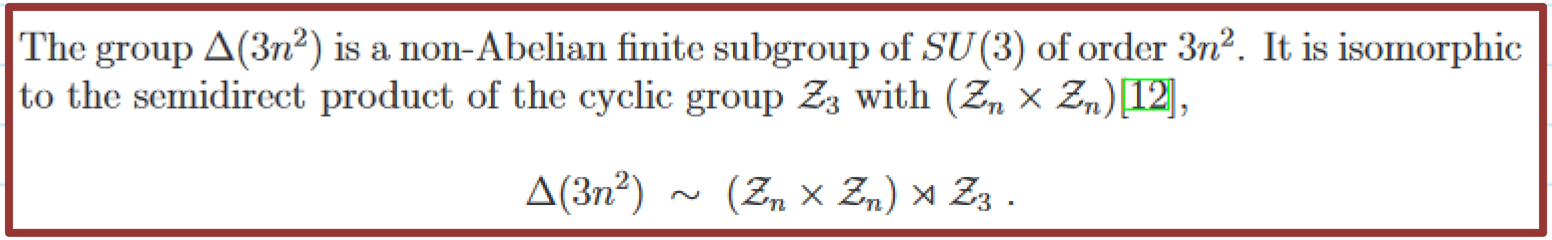
What are these groups?

GAP examples: From matrices



```
gap> a:=E(5);;
gap> m1:=[[a,0,0],[0,1,0],[0,0,1/a]];;
gap> m2:=[[1,0,0],[0,0,1],[0,1,0]];;
gap> m3:=[[0,0,1],[1,0,0],[0,1,0]];;
gap> group1:=Group(m1,m2,m3);
<matrix group with 3 generators>
gap> group2:=Group(m1,m3);
Group([ [ [ E(5), 0, 0 ], [ 0, 1, 0 ], [ 0, 0, E(5)^4 ] ], [ [ 0, 0, 1 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ] ])
gap> Order(group1);
150
gap> Order(group2);
75
gap> StructureDescription(group1);
"(C5 x C5) : S3"
gap> StructureDescription(group2);
"(C5 x C5) : C3"
```

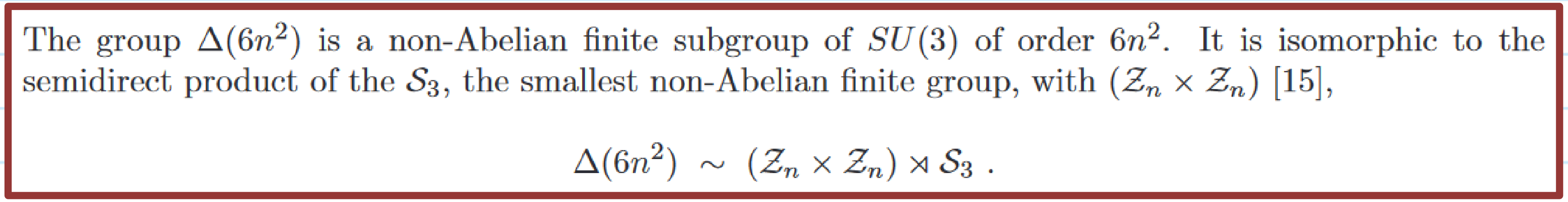
What are these groups?



The group $\Delta(3n^2)$ is a non-Abelian finite subgroup of $SU(3)$ of order $3n^2$. It is isomorphic to the semidirect product of the cyclic group $\mathcal{Z}_3$ with $(\mathcal{Z}_n \times \mathcal{Z}_n)$[12],

$$\Delta(3n^2) \sim (\mathcal{Z}_n \times \mathcal{Z}_n) \rtimes \mathcal{Z}_3 \ .$$

[0701188 Luhn, Nasri, Ramond]

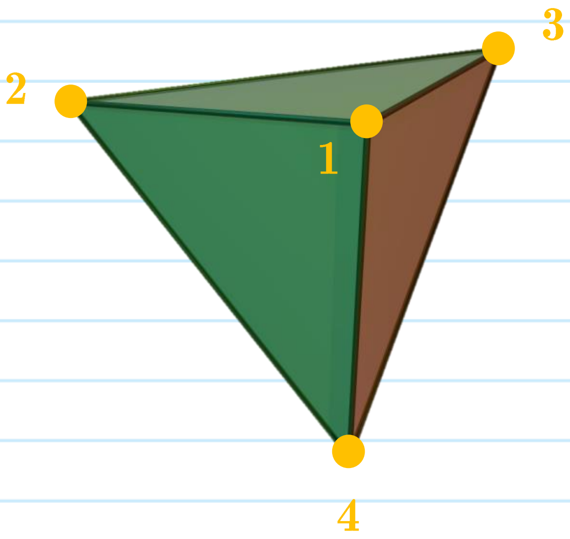


The group $\Delta(6n^2)$ is a non-Abelian finite subgroup of $SU(3)$ of order $6n^2$. It is isomorphic to the semidirect product of the $\mathcal{S}_3$, the smallest non-Abelian finite group, with $(\mathcal{Z}_n \times \mathcal{Z}_n)$ [15],

$$\Delta(6n^2) \sim (\mathcal{Z}_n \times \mathcal{Z}_n) \rtimes \mathcal{S}_3 \ .$$

[0809.0639 Escobar, Luhn]

GAP examples: tetrahedron symmetries



What are the symmetries of a tetrahedron?

we take the set of points $\{P_1, P_2, P_3, P_4\} \equiv T$, $P_i \in \mathbb{R}^3$

What rotations ($\in SO(3)$) leave T invariant?

The orbit-stabilizer theorem can help count the size of the group (very intuitive)

$G \equiv$ group of symmetries
$\uparrow$
$\subset SO(3)$

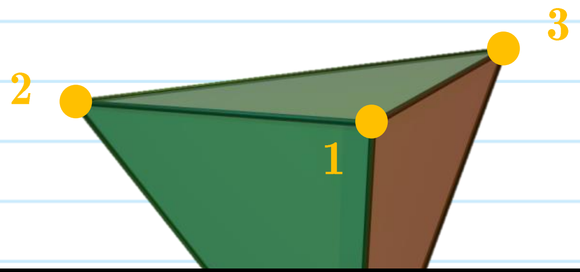
$$|G| = \underbrace{|\text{Orbit of } P_1 \text{ under } G|}_{4}\ \underbrace{|\text{Subgroup of } G \text{ which stabilizes } P_1|}_{3}$$

one can move $P_1 \to P_{1,2,3,4}$ with a rotation

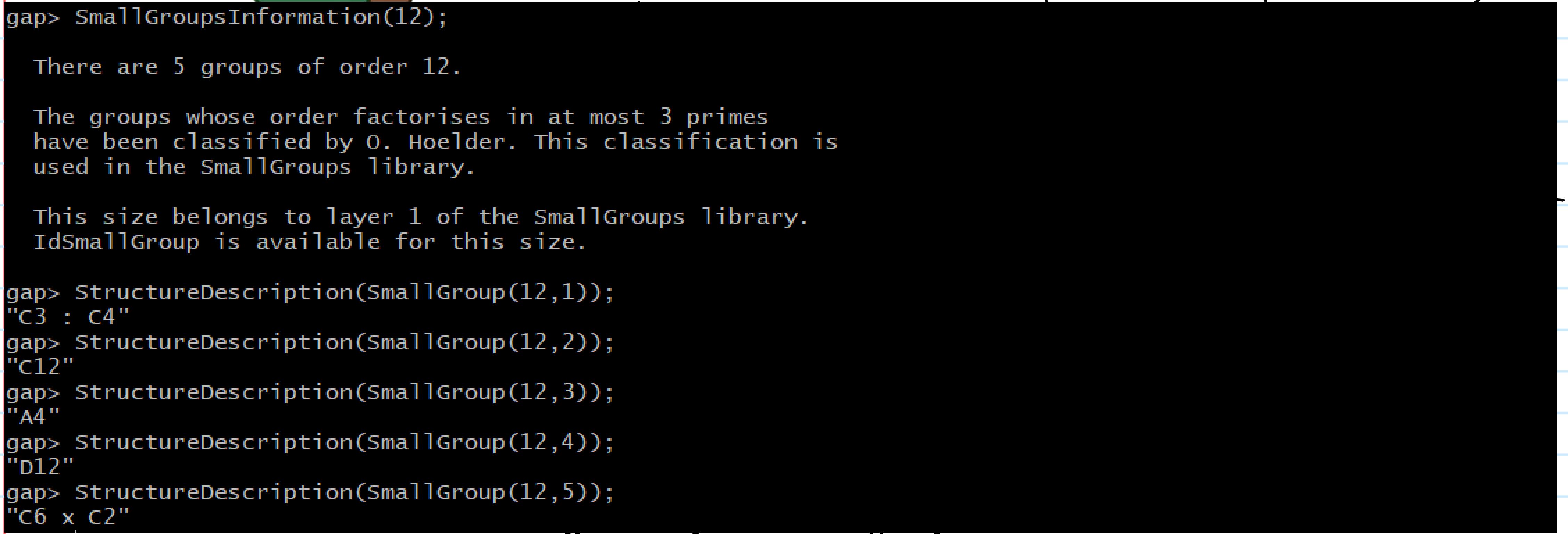
120° rotations change $P_{2,3,4}$ leaving P_1 fixed

$\boxed{|G|=12}$ What group?

GAP examples: tetrahedron symmetries

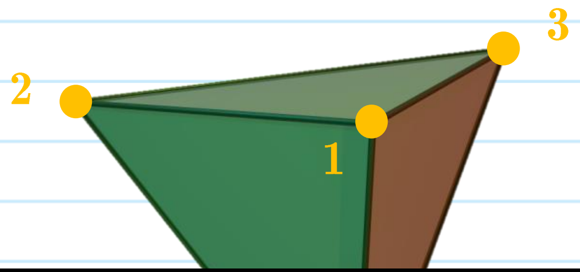


What are the symmetries of a tetrahedron?

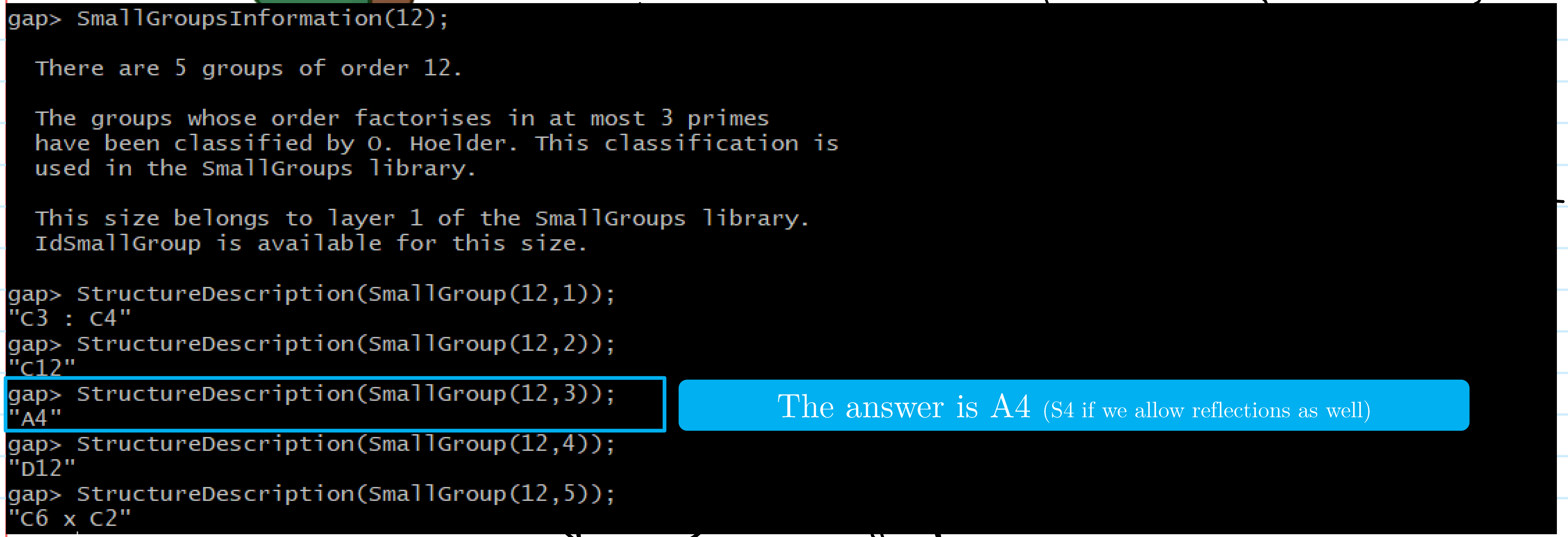


```
gap> SmallGroupsInformation(12);

  There are 5 groups of order 12.

  The groups whose order factorises in at most 3 primes
  have been classified by O. Hoelder. This classification is
  used in the SmallGroups library.

  This size belongs to layer 1 of the SmallGroups library.
  IdSmallGroup is available for this size.

gap> StructureDescription(SmallGroup(12,1));
"C3 : C4"
gap> StructureDescription(SmallGroup(12,2));
"C12"
gap> StructureDescription(SmallGroup(12,3));
"A4"
gap> StructureDescription(SmallGroup(12,4));
"D12"
gap> StructureDescription(SmallGroup(12,5));
"C6 x C2"
```

GAP examples: tetrahedron symmetries



What are the symmetries of a tetrahedron?



```
gap> SmallGroupsInformation(12);

  There are 5 groups of order 12.

  The groups whose order factorises in at most 3 primes
  have been classified by O. Hoelder. This classification is
  used in the SmallGroups library.

  This size belongs to layer 1 of the SmallGroups library.
  IdSmallGroup is available for this size.

gap> StructureDescription(SmallGroup(12,1));
"C3 : C4"
gap> StructureDescription(SmallGroup(12,2));
"C12"
gap> StructureDescription(SmallGroup(12,3));
"A4"
gap> StructureDescription(SmallGroup(12,4));
"D12"
gap> StructureDescription(SmallGroup(12,5));
"C6 x C2"
```

The answer is A4 (S4 if we allow reflections as well)

Continuous groups

The symmetries of space time

You know that any coordinate transformation $x^\mu \to x'^\mu$ which leaves the metric invariant will not affect the laws of physics (otherwise we must account for curvature).

So what are the transformations that leave the Minkowski metric invariant? (Isometries; associated to Killing vectors)

The symmetries of space time

You know that any coordinate transformation $x^\mu \to x'^\mu$ which leaves the metric invariant will not affect the laws of physics (otherwise we must account for curvature).

So what are the transformations that leave the Minkowski metric invariant? (Isometries; associated to Killing vectors)

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\sigma}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\sigma\rho} = g_{\mu\nu} \qquad \text{for } g = \eta \qquad ?$$

The full answer: $x'^\mu = \underline{\Lambda^\mu{}_\nu x^\nu} + b^\mu$

Translations

Homogeneous;
The Lorentz group O(1,3)
=
Rotations + Boosts

Poincaré group

The Lorentz group

Corresponds to all matrices Λ such that $\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$

Λ has 6 free parameters

$$\underset{\substack{\uparrow \\ \text{3 free parameters;} \\ SO(3)}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \times & \\ 0 & & & \end{pmatrix}} \qquad \underbrace{\begin{pmatrix} \cosh\alpha_x & \sinh\alpha_x & 0 & 0 \\ \sinh\alpha_x & \cosh\alpha_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, (Y), (Z)}_{\text{3 boosts}}$$

Under an infinitesimal transformation $\Lambda(\delta\omega) = \mathbb{1} - \frac{i}{2}\delta\omega^{\mu\nu}\underline{J_{\mu\nu}}$

$$[J_{\mu\nu}, J_{\lambda\sigma}] = i\left(J_{\lambda\nu}g_{\mu\sigma} - J_{\sigma\nu}g_{\mu\lambda} + J_{\mu\lambda}g_{\nu\sigma} - J_{\mu\sigma}g_{\nu\lambda}\right)$$

The Lorentz group

Let us separate the 6 anti-symmetric $J_{\mu\nu}$ in 3+3 generators of rotations and boosts:

$$\underbrace{J_k \equiv \frac{1}{2}\varepsilon^{kmn} J_{mn}}_{\text{Rotations}} \; ; \; \underbrace{K_m \equiv J_{m0}}_{\text{Boosts}} \quad (k,m,n=1,2,3)$$

You can go ahead and rewrite $[J_{\mu\nu}, J_{\rho\sigma}]$ in terms of J_k and K_m. But it is more instructive to consider

$$\boxed{\begin{aligned} A_k^R &\equiv \frac{1}{2}\left(J_k + i\, K_k\right) \\ A_k^L &\equiv \frac{1}{2}\left(J_k - i\, K_k\right) \end{aligned}}$$

$$\left(A^R_{1,2,3}\right)_{\mu\nu} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i/2 \\ 0 & 0 & -i/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & i/2 \\ -1/2 & 0 & 0 & 0 \\ 0 & i/2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1/2 \\ 0 & 0 & -i/2 & 0 \\ 0 & i/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(A^L_{1,2,3}\right)_{\mu\nu} = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i/2 \\ 0 & 0 & i/2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & i/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & -i/2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -i/2 & 0 \\ 0 & i/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}$$

The Lorentz group

We can rotate these matrices $\left(A_i^{L/R} \to V^\dagger A_i^{L/R} \underset{\uparrow\ \text{find } V}{V}\right)$ such that

they become $\underbrace{\mathbb{1}_2 \otimes \left(\tfrac{1}{2}\sigma_i\right) \text{ and } \left(\tfrac{1}{2}\sigma_i\right) \otimes \mathbb{1}_2}$.

↑ These are the representation matrices of a bi-doublet of $SU(2)$

$A \sim (2,2)$ of $SU(2)_L \times SU(2)_R$ $\left[(2,2) = (\underset{\substack{\uparrow\\ j_R}}{1/2}, \underset{\substack{\uparrow\\ j_L}}{1/2}) \text{ using spins}\right]$

It is also very easy to check that

$$[A_i^{R/L}, A_j^{L/R}] = 0$$

$$[A_i^{R/L}, A_j^{R/L}] = i\,\varepsilon_{ijk} A^{R/L}$$

Finite dimensional representations: one can infer them from those of $SU(2)^2$:

(j_R, j_L) $\quad$ $j_{R/L} = 0, 1/2, 1, 3/2, \ldots$

The Lorentz group

Scalar $\phi = (0,0)$; Weyl R-fermion $= (\frac{1}{2},0)$; Weyl L-fermion $= (0,\frac{1}{2})$; $A_\mu = (\frac{1}{2},\frac{1}{2})$

Dirac spinor $= \Psi_R + \Psi_L = (\frac{1}{2},0) + (0,\frac{1}{2})$; $F_{\mu\nu} \sim [D_\mu, D_\nu] = \underbrace{(1,0)+(0,1)}_{3+3 \text{ real components}}$

We can also immediately use what we know from SU(2) to figure out what is and what is not Lorentz invariant:

ϕ^n ✓ $\Psi_R \Psi_R$ ✓ $\Psi_L \Psi_L$ ✓ $A_\mu A^\mu$ ✓* $\Psi_R \Psi_L A_\mu$ ✓*

Obviously one must check if these are invariant under the remaining model sym

BUT there are 3 things to keep in mind

The Lorentz group

① Careful with the conjugation of representations

$\psi_R^* \neq (1/2, 0) \quad \psi_L^* \neq (0, 1/2)$ contrary to expectations. In fact

$\psi_R^* = (0, 1/2) \sim \psi_L \quad \psi_L^* = (1/2, 0) \sim \psi_R$ Why?

The restricted Lorentz group $SO(1,3)^+$ is not the same as $SU(2) \times SU(2)$

$$\exp(i\underset{\in\mathbb{R}}{\underset{\uparrow}{\alpha_i}} J_i + i\underset{\in\mathbb{R}}{\underset{\uparrow}{\beta_i}} K_i) \qquad \exp(i\underset{\in\mathbb{R}}{\underset{\uparrow}{\gamma_i}} A_i^R + i\underset{\in\mathbb{R}}{\underset{\uparrow}{\delta_i}} A_i^L)$$

$$A_K^R \equiv \frac{1}{2}(J_K + \underline{i} K_K)$$

$$A_K^L \equiv \frac{1}{2}(J_K - \underline{i} K_K)$$

Consequences: $\boxed{1}$ $(j_R, j_L)^* = (j_L, j_R)$

$\boxed{2}$ The Lorentz group is not compact. It has no finite and unitary representations

The Lorentz group

② We did not consider the full Lorentz group

The moment we considered infinitesimal transformations (J_i, K_i) we left behind any disconnected part of the group.

The full Lorentz group $O(1,3)$ has 4 disconnected parts:

$\boxed{\Lambda^0{}_0 = 1 \,,\ \det(\Lambda) = 1}$ ⇝ Forms a group, the proper Lorentz group $SO(1,3)^+$ which we just considered.

$+$

$\boxed{\Lambda^0{}_0 = -1 \,,\ \det(\Lambda) = 1}$ $+$ $\boxed{\Lambda^0{}_0 = 1 \,,\ \det(\Lambda) = -1}$ $+$ $\boxed{\Lambda^0{}_0 = -1 \,,\ \det(\Lambda) = -1}$

We missed the space and time reversal symmetries.

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$O(1,3) = \underbrace{SO(1,3)^+}_{\text{Proper Lorentz group}} \rtimes \underbrace{\{\mathbb{1}, P, T, PT\}}_{Z_2 \times Z_2}$$

The Lorentz group

What are the representations of the full $O(1,3)$?

Both P and T revert the sign of boosts K_i, preserving rotations:

$$P J_i P^{-1} = J_i \qquad P K_i P^{-1} = -K_i \quad \Rightarrow \text{ e.g. } (j_R, j_L) \xrightarrow{P} (j_L, j_R)$$

$$T J_i T^{-1} = J_i \qquad T K_i T^{-1} = -K_i$$

So irreps of the full $O(1,3)$ are

(j,j) and $(j_R, j_L) + (j_L, j_R)$ $\quad j_R \neq j_L$

$\boxed{\text{Note}}$ In quantum mechanics the implementation of time involves some thinking ⇝ anti-unitary operator

$$|\psi(t,\vec{x})\rangle \xrightarrow{T} \eta \cancel{|\psi(-t,\vec{x})\rangle} \, |\psi^*(-t,x)\rangle$$

The Lorentz group

③ Ignore P and T. We didn't really study $SO(1,3)^+$ but rather $Spin(3) = SL(2)$

$SU(2) \neq SO(3)$ [$SU(2)$ is the universal covering group]

And we took $SU(2)$ for the rotation group (half spin representations).

So we used the double cover of the proper Lorentz group ($SO(1,3)^+$) which is called $Spin(1,3)$ and it is isomorphic to $SL(2,C)$

$\uparrow$

Complex 2×2 invertible matrices with $\det = 1$

$$SO(1,3)^+ = SL(2,C)/Z_2$$

The mapping between the two groups is usually presented as follows:

The Lorentz group

$$x^\mu \longleftrightarrow \underbrace{\sigma_\mu x^\mu}_{\equiv X} = \begin{pmatrix} x^0+x^3 & -x^1-ix^2 \\ -x^1+ix^2 & x^0-x^3 \end{pmatrix}$$

$$\det(X) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \qquad X^\dagger = X$$

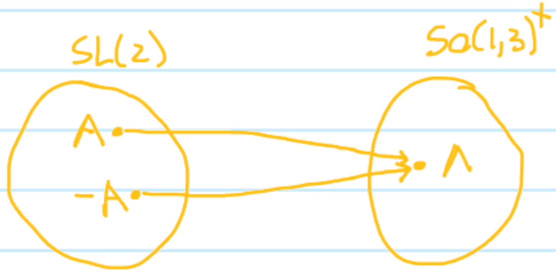
An $SO(1,3)^+$ transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ induces a transformation in X:

$$X \to X' \equiv A X \underset{\hookleftarrow \text{ depends on } \Lambda}{A^\dagger}$$

$\det(X') = \det(X)$ so $|\det(A)|^2 = 1$; let's settle on just $\det(A) = 1$

So $A \in SL(2) \leftarrow$ 6 generators, just like $SO(1,3)^+$.

But crucially



$SL(2)$ is simply connected; so it is the universal covering group;

It also means that $\exp(\langle \text{Algebra of } SO(1,3)^+ \rangle) = SL(2)$.

The Poincaré group

To the Lorentz group we add translations:

$$x'^\mu = \underline{\Lambda^\mu{}_\nu x^\nu} + \boxed{b^\mu}$$

Homogeneous;
The Lorentz group O(1,3)
=
Rotations + Boosts

An infinitesimal one: $T(\delta b) = \mathbb{1} - i\,\delta b^\mu P_\mu$ ⟵ The 4 generators

We must add them to the 6 $J_{\alpha\beta}$ of O(1,3).

NEW

$$\overbrace{[P_\mu, P_\nu] = 0}^{\text{Translations commute}} \; ; \; \overbrace{[P_\mu, J_{\lambda\sigma}] = i\left(P_\lambda g_{\mu\sigma} - P_\sigma g_{\mu\lambda}\right)}^{\text{Translations do not commute with rotations / boosts}}$$

$$[J_{\mu\nu}, J_{\lambda\sigma}] = i\left(J_{\lambda\nu} g_{\mu\sigma} - J_{\sigma\nu} g_{\mu\lambda} + J_{\mu\lambda} g_{\nu\sigma} - J_{\mu\sigma} g_{\nu\lambda}\right)$$

The Poincaré group

Eugene Wigner (1902 - 1995) classified the representations of the group.

In the standard approach to this topic one uses two Casimir operators to establish two invariants, which can then be used to tag the representations.

$C_1 = P^\mu P_\mu$ and $C_2 = W_\mu W^\mu$ with $W^\lambda = \frac{1}{2} \varepsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} P_\sigma$

↑ Pauli-Lubanski vector

$[J_\mu J^\mu, P^\nu] \neq 0$ so one must use C_2 instead

$\underline{p^\mu p_\mu = M^2 > 0}$

Consider one particular p^μ, e.g. $\tilde{p}^\mu = (M, \vec{0})$. Acting on it with the Lorentz group allows us to get any other p^μ such that $p^\mu p_\mu = M^2$.

The Poincaré group

Note that the stability group (little group) of $\tilde{p}^\mu$ are the 3D rotations, i.e. $SO(3)$.

So we need more labels for each vector/state.

$$\Big|\underset{\substack{\uparrow \\ \text{eigenvalue of } P^\mu}}{p^\mu}, \overset{\substack{\text{spin; eigenvalue of } J^2 \\ \downarrow}}{s}, \overset{\substack{\text{eigenvalue of } J_3 \\ \swarrow}}{\lambda}\Big\rangle \xrightarrow[\text{or}]{} \Big|\overbrace{M, s}^{\text{fixed}}, \underbrace{\vec{p}, \lambda}_{\text{affected by the group action}}\Big\rangle$$

$$P^\mu |M, s, \vec{p}, \lambda\rangle = p^\mu |M, s, \vec{p}, \lambda\rangle$$

$$J^2 |M, s, \vec{p}, \lambda\rangle = s(s+1) |M, s, \vec{p}, \lambda\rangle$$

$$J_3 |M, s, \vec{p}, \lambda\rangle = \lambda |M, s, \vec{p}, \lambda\rangle$$

$|M, s, \vec{p}, \lambda\rangle$ form an infinite dimensional representation $\left\{ \begin{array}{l} \lambda = s, s-1, \cdots, -s+1, s \\ \vec{p} \in \mathbb{R}^3 \end{array} \right.$

The Poincaré group

$P_\mu P^\mu = 0$ with $P_\mu \neq 0$

A particular choice of P_μ would be $\tilde{P}_\mu = (E, 0, 0, E)$. With a boost and a rotation one can get to any other 4-vector such that $P_\mu P^\mu = 0$.

But before we do that let's try to figure out what is the little group of $\tilde{P}_\mu$. This will also help the full vector space associated with each irrep.

For $\tilde{P}_\mu$

$$W^0 = W^3 = E\, J_3 \leftarrow \text{Generator of } z \text{ rotations}$$

$$\left.\begin{array}{l} W_1 = E\,(J_1 + K_2) \\ W_2 = E\,(J_2 - K_1) \end{array}\right) \text{Mixture of rotations and boosts involving } x, y$$

Out of the 4 components only 3 are independent (e.g. J_3, W_1, W_2). What group do they generate?

The Poincaré group

$$\boxed{\begin{array}{l} C_2\,|w,\lambda\rangle = w^2\,|w,\lambda\rangle \\ J_3\,|w,\lambda\rangle = \lambda\,|w,\lambda\rangle \end{array}}$$

$w \geq 0$ a continuous parameter

$\lambda = 0, \pm 1, \pm 2, \ldots$

When $w>0$ you can check that $W_{1,2}$ acting on $|w,\lambda\rangle$ change the value of λ and $|\lambda|$ is unbounded (imagine the $SU(2)$ $J_{+/-}$ but with no limit on new vectors created).

The "continuous spin representations" because of $w \in \mathbb{R}^+$

$|w>0,\lambda\rangle$ is an infinite irrep. No field known in the representation.

We are left with $|w=0,\lambda\rangle$. For this case

$$\exp(i\alpha_1 W_1 + i\alpha_2 W_2)\,|0,\lambda\rangle = |0,\lambda\rangle \Longleftarrow$$

W_1, W_2 produce no effect; $|0,\lambda\rangle$ is an unfaithful irrep of E_2

$$\exp(i\alpha_3 J_3)\,|0,\lambda\rangle = e^{i\lambda\alpha_3}\,|0,\lambda\rangle$$

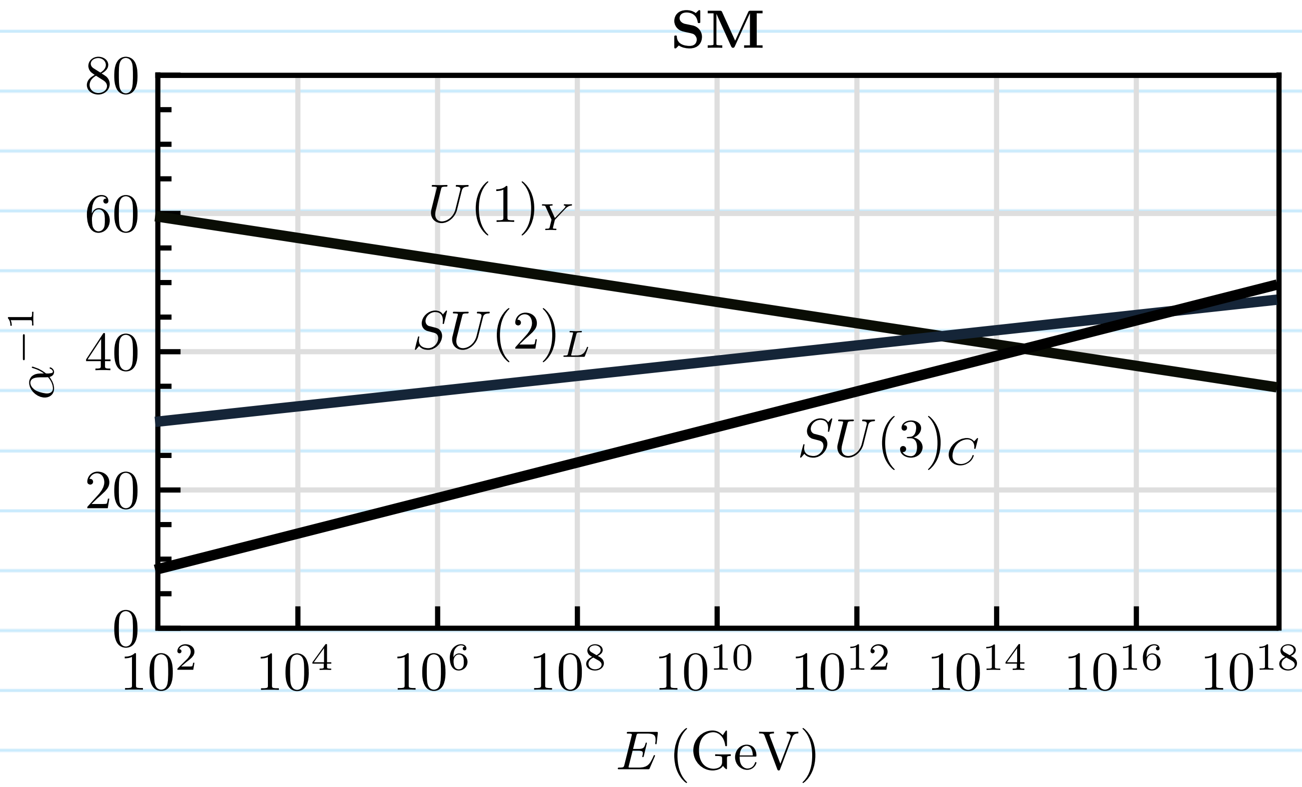
λ = the helicity; how fast the state changes under z rotations $\longrightarrow U(1)$

The gauge group

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$SO(3,1)^+$
Q	$\mathbf{3}$	$\mathbf{2}$	$\frac{1}{6}$	$\left(\frac{1}{2},0\right)$
u^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\frac{1}{3}$	$\left(\frac{1}{2},0\right)$
d^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-\frac{2}{3}$	$\left(\frac{1}{2},0\right)$
L	$\mathbf{1}$	$\mathbf{2}$	$-\frac{1}{2}$	$\left(\frac{1}{2},0\right)$
e^c	$\mathbf{1}$	$\mathbf{1}$	1	$\left(\frac{1}{2},0\right)$
H	$\mathbf{1}$	$\mathbf{2}$	$\frac{1}{2}$	$(0,0)$
F_G	$\mathbf{8}$	$\mathbf{1}$	0	$(1,0)$
F_W	$\mathbf{1}$	$\mathbf{3}$	0	$(1,0)$
F_B	$\mathbf{1}$	$\mathbf{1}$	0	$(1,0)$

The Standard Model has a SU(3) x SU(2) x U(1) gauge group.
Can it be the remnant of a larger group?

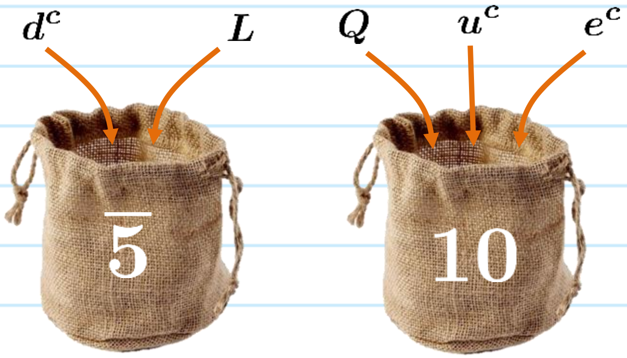
Grand unification?



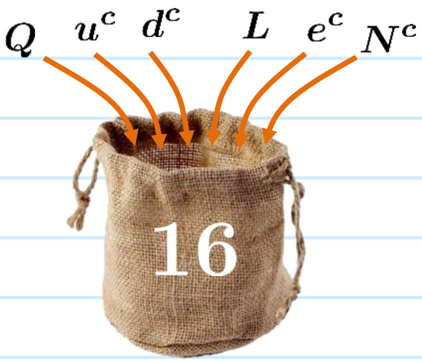
Running of the gauge couplings suggests that at high scales they have a similar value

Grand unification?

$SU(5)$

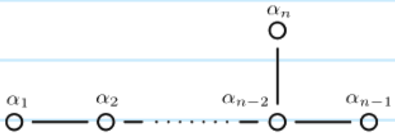
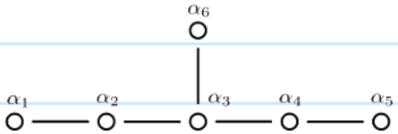
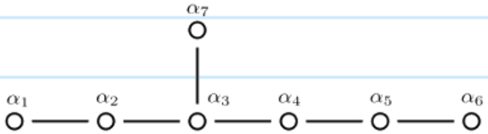
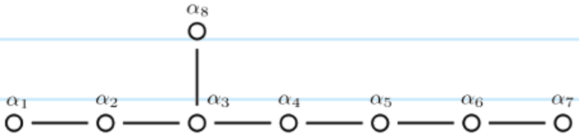


$SO(10)$



SU(5) and SO(10) are strong candidates for the bigger group. The Standard Model fermions would need to be part of larger multiplets.

Landscape of possible groups

Simple algebra	Dynkin diagram	Cartan matrix
SU(n+1) A_n		$\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$
SO(2n+1) B_n		$\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -2 \\ 0 & & & -1 & 2 \end{pmatrix}$
SP(2n) C_n		$\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ 0 & & & -2 & 2 \end{pmatrix}$
SO(2n) D_n		$\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & -1 \\ & & -1 & 2 & 0 \\ 0 & & -1 & 0 & 2 \end{pmatrix}$
E_6		$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$
E_7		$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$
E_8		$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$
F_4		$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$
G_2		$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

These are the complex simple Lie algebras

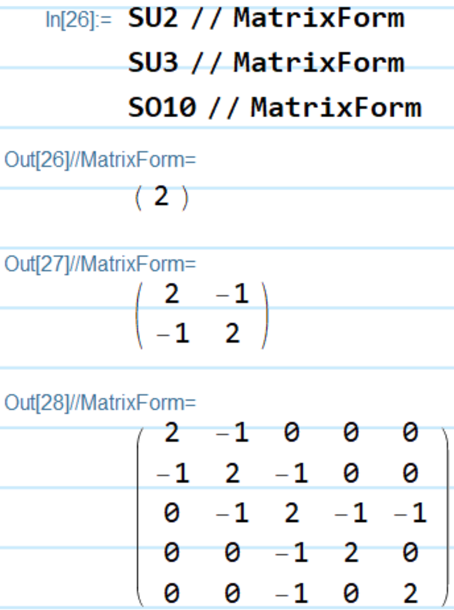
Examples: some small irreps

GAP also has code for continuous groups.

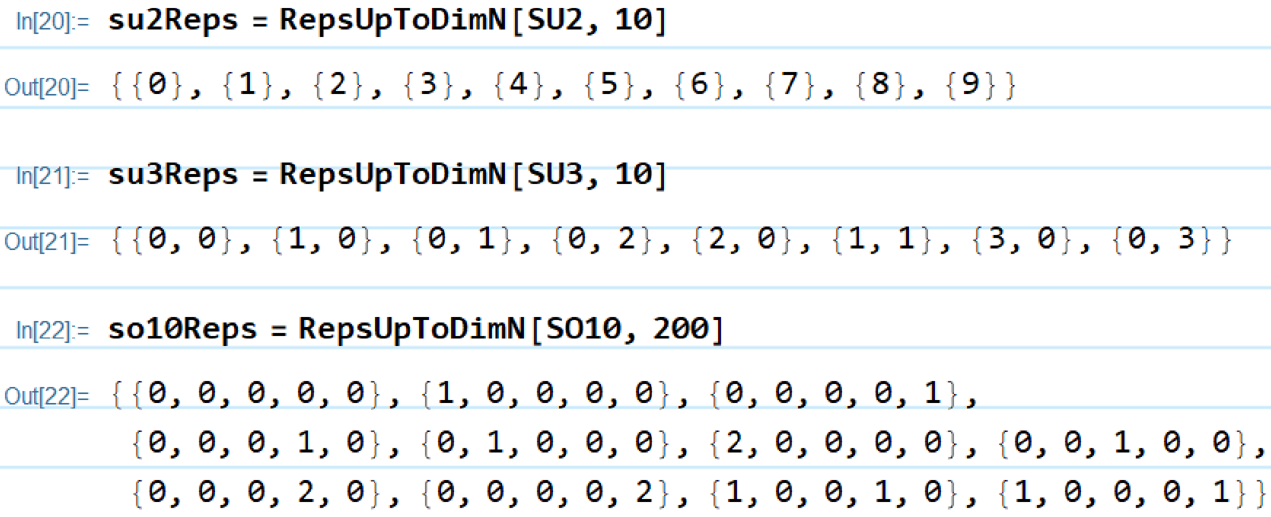
For particle physics we also have some packages: LieART 2 Feger, Kephart, Saskowski 1912.10929

GroupMath R.F. 2011.01764

Examples with GroupMath



```
In[26]:= SU2 // MatrixForm
         SU3 // MatrixForm
         SO10 // MatrixForm

Out[26]//MatrixForm=
         ( 2 )

Out[27]//MatrixForm=
         (  2  -1 )
         ( -1   2 )

Out[28]//MatrixForm=
         (  2  -1   0   0   0 )
         ( -1   2  -1   0   0 )
         (  0  -1   2  -1  -1 )
         (  0   0  -1   2   0 )
         (  0   0  -1   0   2 )
```

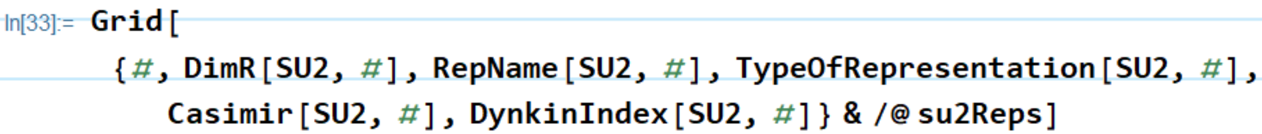


```
In[20]:= su2Reps = RepsUpToDimN[SU2, 10]

Out[20]= {{0}, {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}}

In[21]:= su3Reps = RepsUpToDimN[SU3, 10]

Out[21]= {{0, 0}, {1, 0}, {0, 1}, {0, 2}, {2, 0}, {1, 1}, {3, 0}, {0, 3}}

In[22]:= so10Reps = RepsUpToDimN[SO10, 200]

Out[22]= {{0, 0, 0, 0, 0}, {1, 0, 0, 0, 0}, {0, 0, 0, 0, 1},
          {0, 0, 0, 1, 0}, {0, 1, 0, 0, 0}, {2, 0, 0, 0, 0}, {0, 0, 1, 0, 0},
          {0, 0, 0, 2, 0}, {0, 0, 0, 0, 2}, {1, 0, 0, 1, 0}, {1, 0, 0, 0, 1}}
```

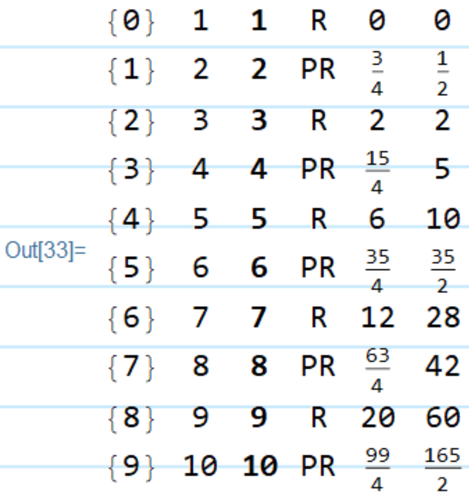
These are the Cartan matrices we saw earlier

Some of their representations

Examples: some small irreps

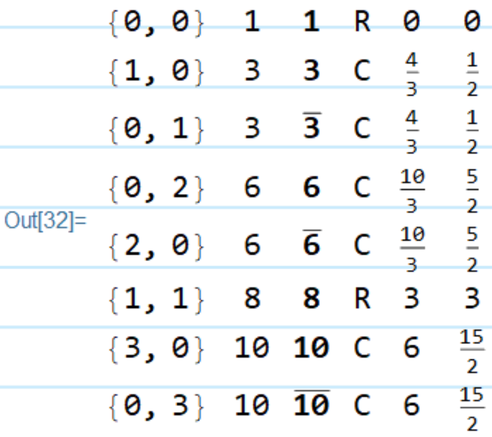


```
In[33]:= Grid[
          {#, DimR[SU2, #], RepName[SU2, #], TypeOfRepresentation[SU2, #],
             Casimir[SU2, #], DynkinIndex[SU2, #]} & /@ su2Reps]
```



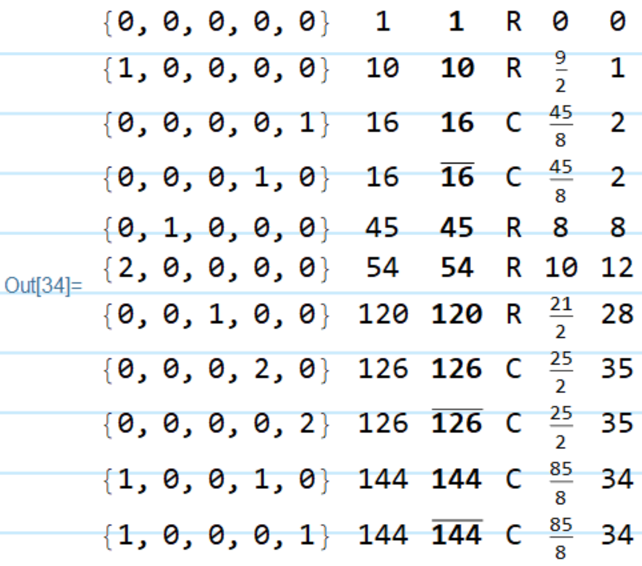
```
         {0}   1   1  R    0     0
         {1}   2   2  PR   3/4   1/2
         {2}   3   3  R    2     2
         {3}   4   4  PR   15/4  5
         {4}   5   5  R    6     10
Out[33]= {5}   6   6  PR   35/4  35/2
         {6}   7   7  R    12    28
         {7}   8   8  PR   63/4  42
         {8}   9   9  R    20    60
         {9}  10  10  PR   99/4  165/2
```

SU(2)



```
         {0, 0}   1   1   R  0     0
         {1, 0}   3   3   C  4/3   1/2
         {0, 1}   3   3̄   C  4/3   1/2
         {0, 2}   6   6   C  10/3  5/2
Out[32]= {2, 0}   6   6̄   C  10/3  5/2
         {1, 1}   8   8   R  3     3
         {3, 0}  10  10   C  6     15/2
         {0, 3}  10  1̄0̄   C  6     15/2
```

SU(3)

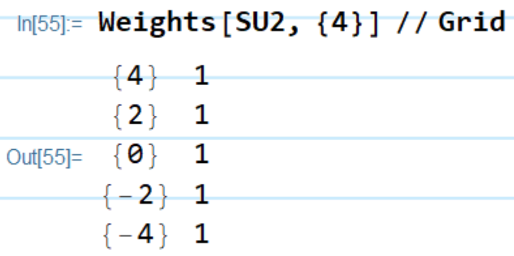


```
         {0, 0, 0, 0, 0}   1    1    R  0     0
         {1, 0, 0, 0, 0}  10   10    R  9/2   1
         {0, 0, 0, 0, 1}  16   16    C  45/8  2
         {0, 0, 0, 1, 0}  16   1̄6̄    C  45/8  2
         {0, 1, 0, 0, 0}  45   45    R  8     8
Out[34]= {2, 0, 0, 0, 0}  54   54    R  10    12
         {0, 0, 1, 0, 0}  120  120   R  21/2  28
         {0, 0, 0, 2, 0}  126  126   C  25/2  35
         {0, 0, 0, 0, 2}  126  1̄2̄6̄   C  25/2  35
         {1, 0, 0, 1, 0}  144  144   C  85/8  34
         {1, 0, 0, 0, 1}  144  1̄4̄4̄   C  85/8  34
```

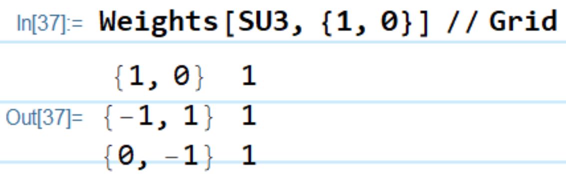
SO(10)

Some properties of the irreps

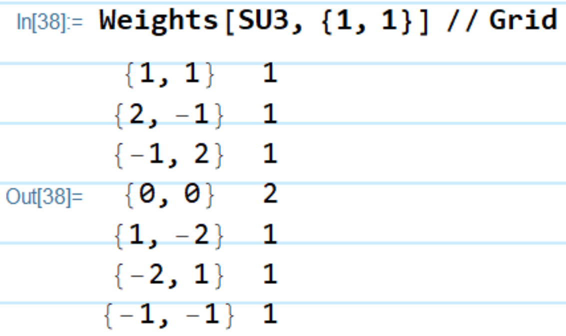
Examples: weights



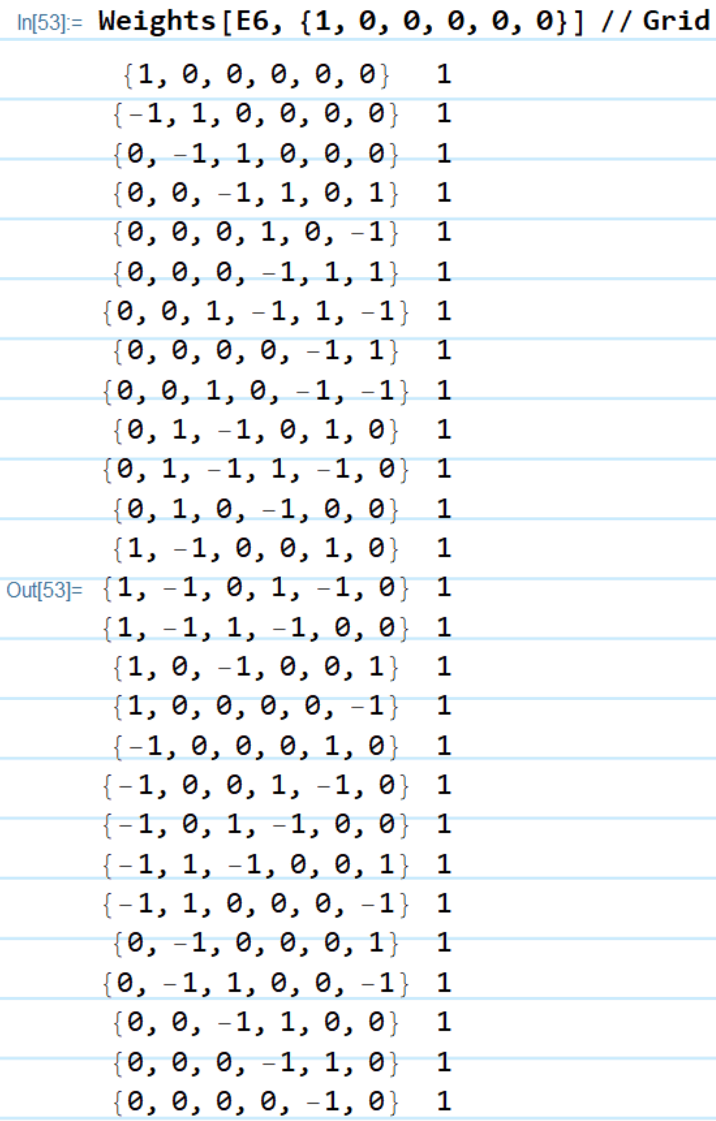
```
In[55]:= Weights[SU2, {4}] // Grid

          {4}   1
          {2}   1
Out[55]=  {0}   1
         {-2}   1
         {-4}   1
```



```
In[37]:= Weights[SU3, {1, 0}] // Grid

           {1, 0}   1
Out[37]=  {-1, 1}   1
          {0, -1}   1
```



```
In[38]:= Weights[SU3, {1, 1}] // Grid

            {1, 1}    1
           {2, -1}    1
           {-1, 2}    1
Out[38]=    {0, 0}    2
           {1, -2}    1
           {-2, 1}    1
          {-1, -1}    1
```



```
In[53]:= Weights[E6, {1, 0, 0, 0, 0, 0}] // Grid

            {1, 0, 0, 0, 0, 0}    1
           {-1, 1, 0, 0, 0, 0}    1
           {0, -1, 1, 0, 0, 0}    1
           {0, 0, -1, 1, 0, 1}    1
           {0, 0, 0, 1, 0, -1}    1
           {0, 0, 0, -1, 1, 1}    1
          {0, 0, 1, -1, 1, -1}    1
           {0, 0, 0, 0, -1, 1}    1
          {0, 0, 1, 0, -1, -1}    1
           {0, 1, -1, 0, 1, 0}    1
          {0, 1, -1, 1, -1, 0}    1
           {0, 1, 0, -1, 0, 0}    1
           {1, -1, 0, 0, 1, 0}    1
Out[53]=  {1, -1, 0, 1, -1, 0}    1
          {1, -1, 1, -1, 0, 0}    1
           {1, 0, -1, 0, 0, 1}    1
           {1, 0, 0, 0, 0, -1}    1
           {-1, 0, 0, 0, 1, 0}    1
          {-1, 0, 0, 1, -1, 0}    1
          {-1, 0, 1, -1, 0, 0}    1
          {-1, 1, -1, 0, 0, 1}    1
          {-1, 1, 0, 0, 0, -1}    1
           {0, -1, 0, 0, 0, 1}    1
          {0, -1, 1, 0, 0, -1}    1
           {0, 0, -1, 1, 0, 0}    1
           {0, 0, 0, -1, 1, 0}    1
           {0, 0, 0, 0, -1, 0}    1
```

Weights of some irreps