Accelerator Basics Part II: Elements of Lattice Design

Stephan I. TZENOV

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Some Definitions

- The arrangement of the magnets along a beam path known also as the *reference trajectory* for guiding or focusing of charged particles is called the magnetic lattice or simply lattice. The arrangement can be **irregular array** or **repetitive regular array** of magnets.
- **The repetitive regular array is called periodic lattice**.
- In a circular accelerator, the *periodic lattice can be symmetric*. Periodic lattices are also used in linear accelerators. Usually, the lattice is constructed from unitary cells and then the cells comprise **superperiods**. A number of superperiods then complete an accelerator ring.
- The ***goal of lattice design*** is to obtain simple, reliable, flexible and high performance accelerators that meet users' requirements.
- **For modern circular accelerators, lattice design is of crucial importance in the design of the accelerators.**

Basic Steps During the Lattice Design

- 1) For a modern accelerator, lattice design work **usually takes some years** to finalize the design parameters. It is an iterative process, involving users, funding, accelerator physics, accelerator subsystems, civil engineering, etc.
- 2) It starts from *major parameters* such as energy, size, etc.
- 3) Then *linear lattice* is constructed based on the building blocks. Linear lattice should **fulfil accelerator physics criteria and provide global quantities** such as circumference, emittance, betatron tunes, magnet strengths, and some other machine parameters.
- 4) **Design codes** such as MAD are used for the lattice functions matching and parameters calculations.
- 5) Usually, a design with **periodic cells** is needed in a circular machine. The cell can be *FODO, Double Bend Achromat (DBA), Triple Bend Achromat (TBA), Quadruple Bend Achromat (QBA), or Multi-Bend Achromat (MBA or nBA) types*.

Basic Steps During the Lattice Design Continued…

A) Combined-function or separated-function magnets are selected.

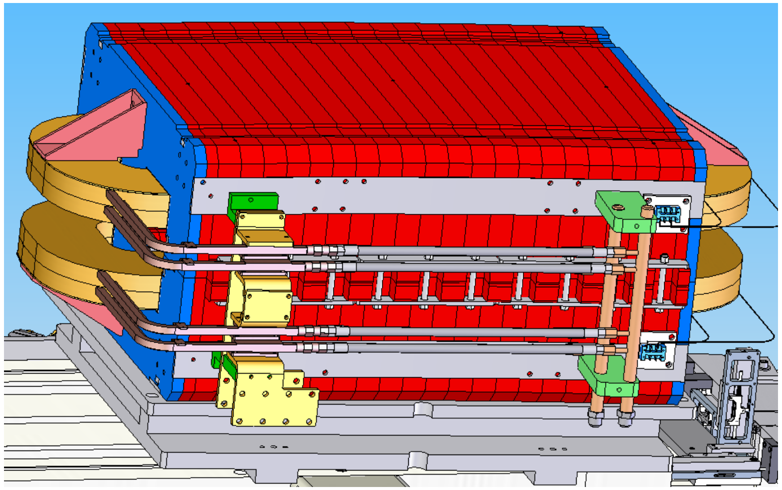
B) *Maximum magnetic field strengths* are constrained. (room-temperature or superconducting magnets, bore radius or chamber profile, etc.)

C) *Matching or insertion sections* are matched to get desired machine functions.

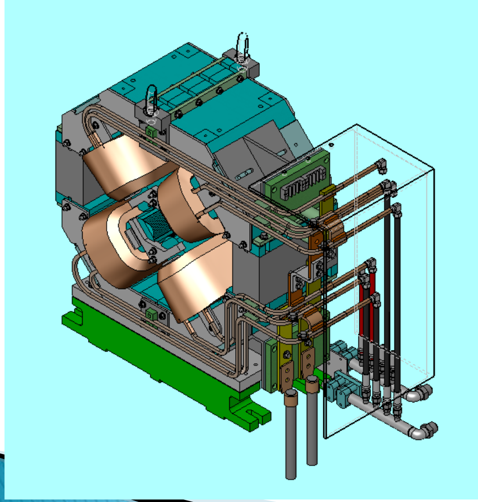
- 6) To get stable solution for the off-momentum particle, we need to put *sextupole magnets and RF cavities* in the lattice beam line. Such nonlinear elements excite nonlinear beam dynamics effects. Therefore, the dynamic acceptances in the transverse and longitudinal planes need to be carefully studied in order to get sufficient acceptances (for long beam current lifetime and high injection efficiency).
- 7) *For the modern high performance machines, strong sextupole fields to correct high chromaticity will have large impact on the nonlinear beam dynamics and it is the most challenging and laborious work at this stage.*
- 8) Study of machine imperfections, tolerances and Touschek lifetime.

Magnet Types

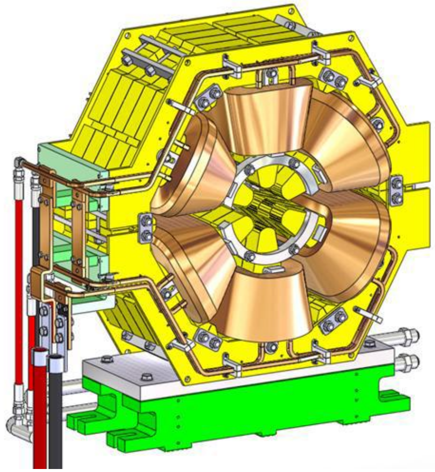
Magnetic Dipole or Bending Magnet is used for particle confinement on a circular trajectory



Magnetic Quadrupole is used for *particle focusing*

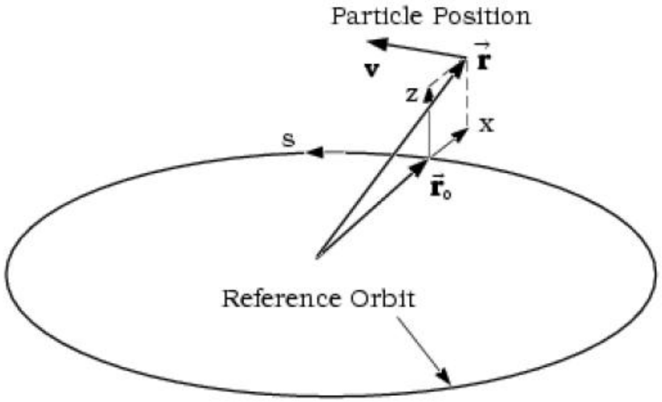


Magnetic Sextupole used for *parameter correction beam extraction, etc.*



Curvilinear Coordinate System and Equations of Motion

First step is to specify the reference trajectory. Coordinate system following the motion of a synchronous particle on the reference trajectory.



Particle position $\Rightarrow \ \mathbf{r}_0(s)$. Differential geometry formalism : $(\mathbf{n}, \mathbf{b}, \tau)$ triple.

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = K(s)\tau(s) + \kappa(s)\mathbf{b}(s), \quad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = -\kappa(s)\mathbf{n}(s), \quad \frac{\mathrm{d}\tau}{\mathrm{d}s} = -K(s)\mathbf{n}(s),$$

The tangent vector: $\tau = \mathrm{d}\mathbf{r}_0/\mathrm{d}s$. Deviation from the reference orbit:

$\mathbf{r}(x, z, s) = \mathbf{r}_0(s) + x\mathbf{n}(s) + z\mathbf{b}(s)$. *Components* of an arbitrary vector:

$$a_s = \mathbf{a} \cdot [-\kappa z\mathbf{n} + \kappa x\mathbf{b} + (1 + Kx)\tau] \qquad a_x = \mathbf{a} \cdot \mathbf{n}, \qquad a_z = \mathbf{a} \cdot \mathbf{b}.$$

Relativistic Hamiltonian in Cartesian coordinates:

$$\mathcal{H} = c\sqrt{m_0^2 c^2 + (\mathbf{p} - q\mathbf{A})^2} + q\varphi.$$

It must be transformed to the natural curvilinear coordinate system.

Hamiltonian Formalism and Equations of Motion

Approximations valid for large high-energy machines are usually used:

- *Small transverse orbit deviations* – longitudinal momentum $p_s \gg p_x, p_z$.
- **Paraxial approximation** for field expansion – small transverse *x, z*.
- *Adiabatic approximation* – slow variation with time.
- Small orbit curvature $x \ll R$, or $Kx \ll 1$.
- **Planar curve** – torsion $\kappa(s) = 0$.
- *Piecewise constant fields* and no fringes: $A_x = A_z = 0$.
- Arcs and straight sections **piecewise constant or zero curvature** $K(s)$.

With the above in hand, we obtain: $H = H_0 + H_1 + H_2 + H_3 + \dots,$

$$H_0 = -\Gamma + \frac{ec\widehat{\mathcal{E}}_0(s)}{\omega \beta_s E_s} \cos\left(\frac{\omega \tau}{c\beta_s} + \phi_0\right), \qquad H_1 = -(\Gamma - 1)Kx,$$

$$H_2 = \frac{1}{2\Gamma}\left(p_x^2 + p_z^2\right) + \frac{1}{2R^2}\left(G_x x^2 + G_z z^2\right),$$

$$H_3 = K\left[\frac{p_x^2 + p_z^2}{2\Gamma} + \frac{g_0}{2R^2}\left(x^2 - z^2\right)\right]x + \frac{\lambda_0}{6R^3}\left(x^3 - 3xz^2\right).$$

Hamiltonian Formalism and Equations of Motion Continued…

Here $p_{x,z} = P_{x,z}/p_{0s}, \quad \tau = -c\beta_s t, \quad h = \mathcal{H}/\left(\beta_s^2 E_s\right),$ and

$\Gamma = \sqrt{\beta_s^2 h^2 - \frac{1}{\beta_s^2 \gamma_s^2}}.$ In addition, $G_x = g_0 + R^2 K^2, \qquad G_z = -g_0,$

$$K = \frac{q}{p_{s0}} (B_z)_{x,z=0}, \qquad g_0 = \frac{qR^2}{p_{s0}} \left(\frac{\partial B_z}{\partial x} \right)_{x,z=0}, \qquad \lambda_0 = \frac{qR^3}{p_{s0}} \left(\frac{\partial^2 B_z}{\partial x^2} \right)_{x,z=0},$$

Important quantity is the MAGNETIC RIGIDITY $(BR) = p/(Ze) \; [T \cdot m].$

Equations of motion:

1) Quadrupole

$$\frac{\mathrm{d}x}{\mathrm{d}s} = p_x, \qquad \frac{\mathrm{d}p_x}{\mathrm{d}s} = -gx, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = p_z, \qquad \frac{\mathrm{d}p_z}{\mathrm{d}s} = gz.$$

2) Dipole

$$\frac{\mathrm{d}x}{\mathrm{d}s} = p_x, \qquad \frac{\mathrm{d}p_x}{\mathrm{d}s} = -K^2 x + K \frac{\Delta p}{p_s}.$$

Here $g = \frac{eZ}{m_p A c \beta_s \gamma_s} \left(\frac{\partial B_z}{\partial x} \right)_{x,z=0},$ and $K = \frac{1}{R}.$

Transfer Matrices

The equations of motion for the quadrupole and the dipole can be written as a single equation:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + gx = 0, \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}s^2} - gx = 0, \qquad \frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + K^2 x = K\frac{\Delta p}{p_s},$$

Since the equation for the dipole is an **inhomogeneous linear differential equation**, its general solution can be represented as $x = \widehat{x} + x_D$. The ***first term*** is a solution of the homogeneous part and describes **betatron oscillations** about the reference trajectory, while the second one is a certain partial solution. It is convenient to define the ***dispersion function***

$$x_D(s) = D(s)\frac{\Delta p}{p_s}, \qquad \frac{\mathrm{d}^2 D}{\mathrm{d}s^2} + K^2 D = K.$$

The general solution for the dipole can be written in a ***matrix form***

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p_s} \end{pmatrix} = \begin{pmatrix} \cos(Ks) & \varrho\sin(Ks) & \varrho[1-\cos(Ks)] \\ -K\sin(Ks) & \cos(Ks) & \sin(Ks) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \frac{\Delta p}{p_s} \end{pmatrix}$$

Transfer Matrices Continued…

It is important to note that the transformation of the dispersion function is similar to that of the *state vector*

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(Ks) & \varrho \sin(Ks) & \varrho[1-\cos(Ks)] \\ -K\sin(Ks) & \cos(Ks) & \sin(Ks) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}.$$

The dipole magnet in vertical direction behaves like a drift space. Similarly, its solution can be written using a transfer matrix

$$\begin{pmatrix} z \\ z' \\ \dfrac{\Delta p}{p_s} \end{pmatrix} = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z'_0 \\ \dfrac{\Delta p}{p_s} \end{pmatrix}.$$

Similar arguments can be used to analyse the linear particle dynamics in the quadrupole. If the focusing strength g is positive, the quadrupole is *focusing in the horizontal direction* and *defocusing in the vertical direction*. The general solution in matrix form can be written as

Transfer Matrices Continued…

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p_s} \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{g}s\right) & \frac{1}{\sqrt{g}}\sin\left(\sqrt{g}s\right) & 0 \\ -\sqrt{g}\sin\left(\sqrt{g}s\right) & \cos\left(\sqrt{g}s\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \frac{\Delta p}{p_s} \end{pmatrix},$$

$$\begin{pmatrix} z \\ z' \\ \frac{\Delta p}{p_s} \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{g}s\right) & \frac{1}{\sqrt{g}}\sinh\left(\sqrt{g}s\right) & 0 \\ \sqrt{g}\sinh\left(\sqrt{g}s\right) & \cosh\left(\sqrt{g}s\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z'_0 \\ \frac{\Delta p}{p_s} \end{pmatrix}.$$

It is clear from the above expressions that the ***drift space*** can be obtained **in the limit** $g \to 0.$

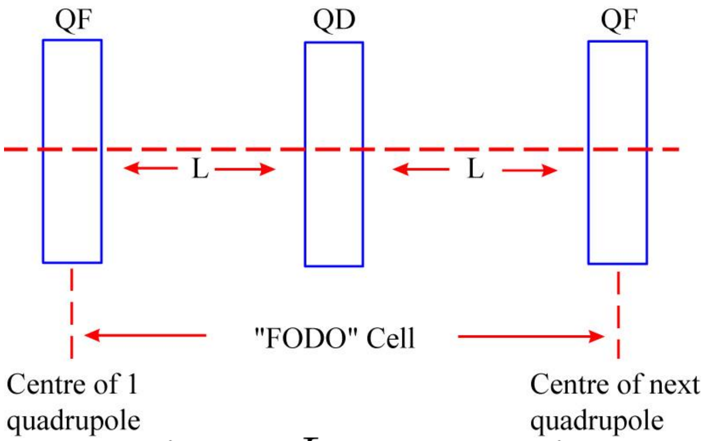
If **the length of the element is small enough but** $\lim_{s\to 0}(gs) = 1/f,$

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p_s} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ \frac{\Delta p}{p_s} \end{pmatrix}, \quad \begin{pmatrix} z \\ z' \\ \frac{\Delta p}{p_s} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z'_0 \\ \frac{\Delta p}{p_s} \end{pmatrix}.$$

THIN-LENSE APPROXIMATION!

The FODO Cell

Calculate the matrix for a half cell, starting in the middle of a focusing quadrupole. For simplicity, let the strengths of the focusing and defocusing quadrupoles and the distances between them be equal:



$$\mathcal{M}_{FODO/2} = \mathcal{M}_{QD/2} \times \mathcal{M}_L \times \mathcal{M}_{QF/2}.$$

Explicitly,

$$\mathcal{M}_{FODO/2} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L}{2f} & L \\ -\frac{L}{4f^2} & 1 + \frac{L}{2f} \end{pmatrix}.$$

The other half is obtained by the inversion $f \to -f$.

$$\mathcal{M}_{FODO} = \begin{pmatrix} 1 + \frac{L}{2f} & L \\ -\frac{L}{4f^2} & 1 - \frac{L}{2f} \end{pmatrix} \begin{pmatrix} 1 - \frac{L}{2f} & L \\ -\frac{L}{4f^2} & 1 + \frac{L}{2f} \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L + \frac{L^2}{f} \\ \frac{L^2}{4f^3} - \frac{L}{2f^2} & 1 - \frac{L^2}{2f^2} \end{pmatrix}.$$

Stability for $L < 2f$, or $gLL_Q < 2$. Basic relation for symmetric FODO!

The Dispersion Function

Standard procedure in the theory of linear betatron motion is to *cancel the first order Hamiltonian* H_1 on page 8. This is achieved by a sequence of canonical transformations. Passing to $\theta = s/R$ as a new independent variable, we can write:

$$\widehat{H}_0 = -\frac{R}{2}\left(KD - \frac{1}{\gamma_s^2}\right)\widehat{\eta}^2 + \frac{ecR\widehat{\mathcal{E}}_0(s)}{\omega\beta_s E_s}\cos\left(\frac{\omega\sigma}{c\beta_s} - k\theta + \phi_0\right),$$

$$\widehat{H}_2 = \frac{R}{2}\left(\widehat{p}_x^2 + \widehat{p}_z^2\right) + \frac{1}{2R}\left(G_x\widehat{x}^2 + G_z\widehat{z}^2\right),$$

where $\widehat{\eta} = h - 1/\beta_s^2$ and $\sigma = s - \beta_s ct.$ In addition, the "hat" variables

$$x = \widehat{x} + \widehat{\eta}D(\theta), \quad p_x = \widehat{p}_x + \frac{\widehat{\eta}}{R}\frac{\mathrm{d}D}{\mathrm{d}\theta}, \quad z = \widehat{z}, \quad p_z = \widehat{p}_z,$$

represent the linear uncoupled betatron oscillations. The function $D(\theta)$ is called the dispersion function. *It describes the deviation of off-momentum particles from the reference trajectory*. Satisfies the inhomogeneous equation

$$\frac{\mathrm{d}^2 D}{\mathrm{d}\theta^2} + G_x D = R^2 K.$$

Beam Dynamics in Periodic Lattice

The Hamilton equations of motion in the plane transverse to the reference trajectory yield

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + G_u(\theta) u = 0, \qquad u = (\widehat{x}, \widehat{z}).$$

In circular accelerators and storage rings the focusing strength $G_u(\theta)$ is a periodic function with period $2\pi/N$.

HILL'S EQUATION – assume in general $G_u(\theta + \Theta) = G_u(\theta)$. FLOQUET'S THEOREM for linear equations with periodic coefficients states that there exist two (for equations of second order) *independent solutions*

$$u_1(\theta) = f_1(\theta) e^{i\nu_u \theta}, \qquad u_2(\theta) = f_2(\theta) e^{-i\nu_u \theta}, \qquad f_{1,2}(\theta + \Theta) = f_{1,2}(\theta).$$

In order to obtain the explicit form of the Floquet solution of Hill's equation, we cast the betatron Hamiltonian in *normal form*

$$H_2 = \frac{\dot{\chi}_x(\theta)}{2}\left(P_x^2 + X^2\right) + \frac{\dot{\chi}_z(\theta)}{2}\left(P_z^2 + Z^2\right),$$

where the new and old canonical variables are related according to

$$\widehat{x} = X\sqrt{\beta}, \qquad \widehat{p} = \frac{1}{\sqrt{\beta}}(P - \alpha X).$$

Twiss Parameters

The coefficients $\alpha(\theta)$, $\beta(\theta)$ and $\gamma(\theta)$ are the TWISS PARAMETERS

$$\frac{\mathrm{d}\chi}{\mathrm{d}\theta} = \frac{R}{\beta}, \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}\theta} = \frac{G\beta}{R} - R\gamma, \qquad \frac{\mathrm{d}\beta}{\mathrm{d}\theta} = -2R\alpha, \qquad \beta\gamma - \alpha^2 = 1.$$

A single equation (β–equation) for the β–function can be derived

$$\frac{\beta}{2}\frac{\mathrm{d}^2\beta}{\mathrm{d}\theta^2} - \frac{1}{4}\left(\frac{\mathrm{d}\beta}{\mathrm{d}\theta}\right)^2 + G\beta^2 = R^2.$$

The Hamilton equations following from the normal form Hamiltonian

$$\frac{\mathrm{d}X}{\mathrm{d}\theta} = \dot{\chi}P, \qquad \frac{\mathrm{d}P}{\mathrm{d}\theta} = -\dot{\chi}X,$$

possess a simple solution

$$\mathbf{X}(\theta) = \begin{pmatrix} X(\theta) \\ P(\theta) \end{pmatrix} = \widehat{\mathcal{R}}(\theta, \theta_0)\mathbf{X}(\theta_0), \qquad \widehat{\mathcal{R}}(\theta, \theta_0) = \begin{pmatrix} \cos\Delta\chi & \sin\Delta\chi \\ -\sin\Delta\chi & \cos\Delta\chi \end{pmatrix},$$

where $\Delta\chi(\theta, \theta_0) = \chi(\theta) - \chi(\theta_0),$ is the phase advance. Since,

$$\mathbf{X} = \widehat{\mathcal{L}}(\theta)\begin{pmatrix} \widehat{x} \\ \widehat{p} \end{pmatrix} = \widehat{\mathcal{L}}(\theta)\widehat{\mathbf{x}}, \qquad \widehat{\mathcal{L}}(\theta) = \begin{pmatrix} 1/\sqrt{\beta(\theta)} & 0 \\ \alpha(\theta)/\sqrt{\beta(\theta)} & \sqrt{\beta(\theta)} \end{pmatrix},$$

we obtain

$$\widehat{\mathbf{x}}(\theta) = \widehat{\mathcal{L}}^{-1}(\theta)\widehat{\mathcal{R}}(\theta, \theta_0)\widehat{\mathcal{L}}(\theta_0)\widehat{\mathbf{x}}(\theta_0).$$

Transfer Matrix and Betatron Tunes

Finally, for the transfer matrix $\widehat{\mathcal{M}}(\theta, \theta_0) = \widehat{\mathcal{L}}^{-1}(\theta)\widehat{\mathcal{R}}(\theta, \theta_0)\widehat{\mathcal{L}}(\theta_0)$, or

$$\widehat{\mathcal{M}}(\theta, \theta_0) = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\Delta\chi + \alpha_0 \sin\Delta\chi) & \sqrt{\beta\beta_0}\sin\Delta\chi \\ \frac{1}{\sqrt{\beta\beta_0}}[(\alpha - \alpha_0)\cos\Delta\chi - (1 + \alpha\alpha_0)\sin\Delta\chi] & \sqrt{\frac{\beta_0}{\beta}}(\cos\Delta\chi - \alpha\sin\Delta\chi) \end{pmatrix}.$$

This is the most general expression for the transfer matrix characterizing the passage through an arbitrary interval between two points θ and θ_0 .

According to the Floquet's theorem, the Twiss parameters are *PERIODIC functions* of θ with the same period 2π/N as the focusing strength. Therefore, for one period we have:

$$\widehat{\mathcal{M}}(\theta) = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}.$$

Here, $$\mu = R \int\limits_{\theta}^{\theta+2\pi/N} \frac{\mathrm{d}\lambda}{\beta(\lambda)},$$ is called the phase advance per period.

The betatron tune is $$\nu = \frac{N\mu}{2\pi} = \frac{R}{2\pi} \int\limits_{\theta}^{\theta+2\pi} \frac{\mathrm{d}\lambda}{\beta(\lambda)},$$

Courant–Snyder Invariant

The transfer matrix for one period can be written also in the form
$\widehat{\mathcal{M}}(\theta) = \widehat{\mathcal{I}}\cos\mu + \widehat{\mathcal{J}}\sin\mu,$ where $\widehat{\mathcal{I}}$ is the *unit matrix* and

$\widehat{\mathcal{J}}(\theta) = \begin{pmatrix} \alpha(\theta) & \beta(\theta) \\ -\gamma(\theta) & -\alpha(\theta) \end{pmatrix}.$ It has the following important properties:

$$\det\left(\widehat{\mathcal{J}}\right) = 1, \qquad \mathrm{Sp}\left(\widehat{\mathcal{J}}\right) = 0, \qquad \widehat{\mathcal{J}}^2 = -\widehat{\mathcal{I}}.$$

Therefore, the transfer matrix for one period has properties similar to those of complex numbers on a unit circle. For ***N*** periods:

$$\widehat{\mathcal{M}}^N(\theta) = \widehat{\mathcal{I}}\cos N\mu + \widehat{\mathcal{J}}\sin N\mu.$$

A quantity $I(X, P; \theta)$ is called invariant of motion if $\dfrac{\mathrm{d}I}{\mathrm{d}\theta} = \dfrac{\partial I}{\partial \theta} + [I, H] = 0,$
where **[* , *]** is the Poisson bracket. For the normal form Hamiltonian

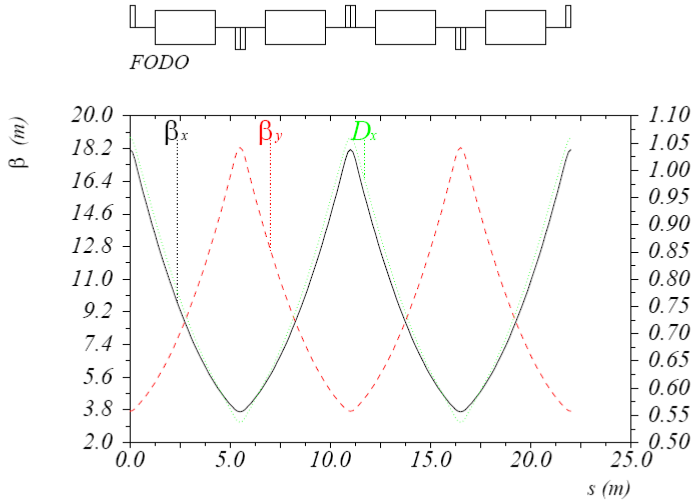
$$I(X, P; \theta) = X^2 + P^2.$$

In other words, *the invariant is proportional to the transverse energy of the particle*. Taking into account the relation between betatron an normal form coordinates:

$$I(\widehat{x}, \widehat{p}; \theta) = \gamma(\theta)\widehat{x}^2 + 2\alpha(\theta)\widehat{x}\widehat{p} + \beta(\theta)\widehat{p}^2.$$

COURANT–SNYDER INVARIANT

Example: FODO Cell



From the general expression for the transfer matrix, we have:

$$\cos\mu = \frac{1}{2}\mathrm{Sp}\left(\widehat{\mathcal{M}}\right) = 1 - \frac{\xi^2}{2}, \qquad \alpha \sin\mu = 0,$$

where $\xi = L/f$. Obviously, α = 0.

Comparing the m_{12} elements we obtain:

$$\beta = \frac{2L}{\xi}\sqrt{\frac{2+\xi}{2-\xi}}.$$

The minimum of the β–function is achieved for $\xi = \sqrt{5} - 1$. Explicitly,

$$\beta_{min} = \frac{L}{2}(1+\sqrt{5})\sqrt{\frac{1+\sqrt{5}}{3-\sqrt{5}}} \approx 3.33\ L.$$

The optimal phase advance is therefore:

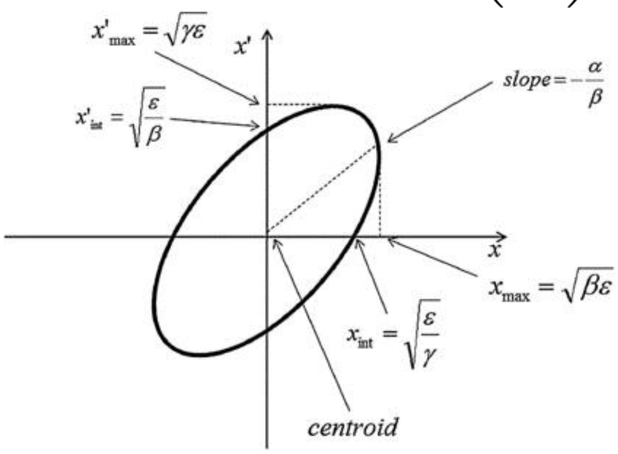
$$\mu_{opt} \approx 76.3435^o.$$

Beam Emittance and Beam Size

Many of the results in classical mechanics can be elegantly obtained by means of the action-angle variables. These can be introduced if we require that the normal form Hamiltonian acquires the form $\bar{H}_2 = \dot{\chi} J$. The action-angle variables are defined according to

$$X = \sqrt{2J}\cos\varphi, \quad P = -\sqrt{2J}\sin\varphi, \quad J = \left(X^2 + P^2\right)/2, \quad \varphi = -\arctan\left(\frac{P}{X}\right).$$

<ins>Courant-Snyder Invariant = Twice Action</ins>



The area of the ellipse $= 2\pi J$. The equilibrium particle distribution in phase space:

$$f(\widehat{x}, \widehat{p}; \theta) = \frac{1}{2\pi\epsilon}\exp\left[-\frac{I(\widehat{x}, \widehat{p}; \theta)}{2\epsilon}\right].$$

Here <ins>ε is the emittance</ins>. In configuration space:

$$\varrho(\widehat{x}; \theta) = \frac{1}{\sqrt{2\pi\epsilon\beta(\theta)}}\exp\left[-\frac{\widehat{x}^2}{2\epsilon\beta(\theta)}\right].$$

Emittance is related to RMS beam size $\sigma^2(\theta) = \epsilon\beta(\theta)$. In terms of action variable $\epsilon = \langle J \rangle$. The RMS beam size satisfies the *ENVELOPE EQUATION*

$$\frac{d^2\sigma}{d\theta^2} + G\sigma = \frac{R^2\epsilon^2}{\sigma^3}.$$

Transformation of Twiss Parameters

Recall that the equations for the evolution of Twiss parameters **are linear**. Therefore, a linear transformation relating their values between different locations along the ring must exist. We know that:

$$\gamma x^2 + 2\alpha x p + \beta p^2 = \gamma_0 x_0^2 + 2\alpha_0 x_0 p_0 + \beta_0 p_0^2,$$

and $$\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ p_0 \end{pmatrix}, \qquad \begin{pmatrix} x_0 \\ p_0 \end{pmatrix} = \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

Express the initial values of the state vector in terms of the final ones and plug the result into the right-hand-side of the Courant-Snyder invariant. Equating terms proportional to x^2 , p^2 and xp , the *transformation law* for the Twiss parameters is found to be

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} m_{11}m_{22} + m_{12}m_{21} & -m_{11}m_{21} & -m_{12}m_{22} \\ -2m_{11}m_{12} & m_{11}^2 & m_{12}^2 \\ -2m_{21}m_{22} & m_{21}^2 & m_{22}^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \gamma_0 \end{pmatrix}.$$

Chromaticity

Recall that the **betatron tunes** were defined for the *synchronous (having the prescribes energy) particle*. In real beams particles possess a certain **energy distribution** about the synchronous energy. Therefore, their betatron oscillations must differ in some sense. **The dependence of the betatron tunes on the deviation from the synchronous energy is called beam chromaticity.** The term $V_N = -R\eta\widehat{p}^2/2$ gives rise to the so-called NATURAL CHROMATICITY

$$\xi_N = -\frac{1}{4\pi R}\int_0^{2\pi} \mathrm{d}\theta G(\theta)\beta(\theta).$$

The *new betatron tune* will be $\widetilde{\nu} = \nu + \xi_N\eta.$

How to compensate chromaticity? Suppose a **sextupole magnet** is located in a place, where the *dispersion has a considerable value*. It yields a term $V_R = \lambda_0(\theta)D(\theta)\eta\widehat{x}^2/\left(2R^2\right).$ The latter induces a RESIDUAL CHROMATICITY

$$\xi_R = \frac{1}{4\pi R^2}\int_0^{2\pi} \mathrm{d}\theta\lambda_0(\theta)D(\theta)\beta(\theta).$$

which is used to **compensate** the natural chromaticity.

Acceleration

Recall the zero order (in transverse coordinates) Hamiltonian

$$\widehat{H}_0 = -\frac{R\widetilde{\mathcal{K}}(\theta)}{2}\widehat{\eta}^2 + \frac{ecR\widehat{\mathcal{E}}_0(s)}{\omega\beta_s E_s}\cos\left(\frac{\omega\sigma}{c\beta_s} - k\theta + \phi_0\right), \qquad \widetilde{\mathcal{K}} = KD - \frac{1}{\gamma_s^2}.$$

In the thin-lens approximation equations of motion are equivalent to the so-called STANDARD (Chirikov – Taylor) MAP:

$$J_{n+1} = J_n + \frac{c\Delta E_0}{\omega\beta_s E_s}\sin\psi_n, \qquad \psi_{n+1} = \psi_n + 2\pi\left(-k + \frac{k^2\mathcal{K}}{R}J_{n+1}\right).$$

Here

$$J = \frac{R\widehat{\eta}}{k}, \quad \psi = \frac{\omega\sigma}{c\beta_s} - k\theta + \phi_0 \pm \pi, \quad \mathcal{K} = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\,\widetilde{\mathcal{K}}(\theta),$$

and ΔE_0 is the maximum energy gain per turn.

The simplest analysis of the Chirikov-Taylor map consists in determining the *eigenvalues of its Jacobian according to the characteristic equation*:

$$\lambda^2 - 2(1 + K_s)\lambda + 1 = 0, \quad K_s = 2\pi^2 K_0^2\cos\psi_s, \quad K_0^2 = \frac{k\mathcal{K}\Delta E_0}{2\pi\beta_s^2 E_s}.$$

Unstable motion $\lambda_1 > 1\ (K_s > 0)$ or $\lambda_2 < -1\ (K_s < -2)$. *Large region of stochastic motion* for $K_0 \sim 1/(2\pi)$.