Lectures on Perturbative QCD

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Lecture 1: Basics of QCD at colliders

- ✓ Introduction
  - ✓ We need to know QFT, but this is *not enough* to do collider physics
  - ✓ The central role of strong interactions at modern particle colliders
- ✓ QCD: the dynamical theory of the strong interactions
- ✓ QCD versus QED: *twins that can be very far apart*
- ✓ *First attempt:* apply QCD to collider processes: surprise! It doesn't seem to work …
- ✓ Go into the deep: augment the formal theory with physics intuition
  - ✓ Infra Red singularities
  - ✓ Factorization
  - ✓ Evolution

Lecture 2: advanced topics

specific processes; collider phenomenology; advanced applications (resummation, factorization of amplitudes; calculational techniques (IBP, unitarity))…

Introduction

- I assume you know enough about formal theory:
  - S-matrix; in- and out-states
  - Perturbation theory (basic idea)
  - UV divergences and renormalization (basics)
  - Regularization techniques, especially dimensional regularization.

  - Some basics frequentist statistics and probability

  Q: do results change when we use alternative regularizations?
  Q: does it matter?

- It is enough to know the basics, i.e. have some feeling for what these things mean.

- More advanced topic that you will eventually need (beyond this meeting):

  - Gauge invariance
  - Structure of the Standard Model: QED, EW, QCD.

- It is a fact (will see why in the lectures) that knowing the above is not enough to compute a process at colliders. The reason is: we cannot solve the theory.

- One might argue about technicalities versus conceptual problems. What I mean is: we cannot solve it at the conceptual level. So additional inputs are needed.

Introduction

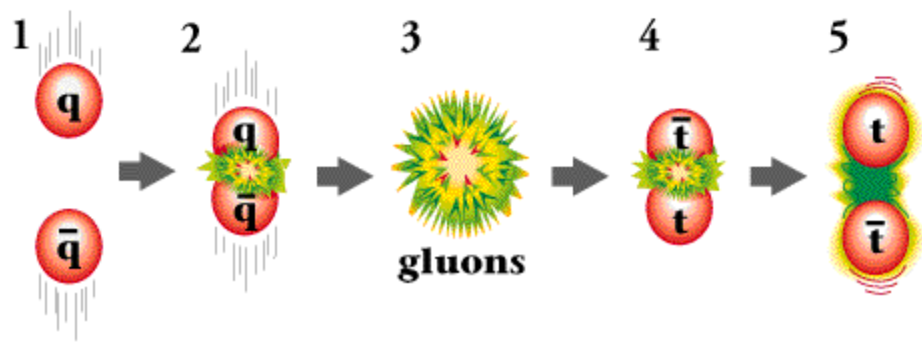
- The most important lessons you can learn:
  - What we do is never an exact science
  - We cannot solve truly complicated real problems just relying on theory and math.
  - We need physics intuition, which is indispensible.
  - So, do not be afraid from things being imprecise or seemingly incomplete. Learn to deal with it.

Any questions? Comments?

Introduction: what do we do at colliders?

- Compute scattering probabilities
  - we call them cross-sections,
  - denote them by σ,
  - measure them in [pb] (pico barns), etc.
- These are not formal objects; they are physical observables, i.e. they can be measured.
  NOTE: S-matrix elements are not observables! (will see later)
- An example:
  - Take the LHC (Large Hadron Collider).
  - It collides protons (most of the time). Or Pb (lead) nuclei.
  - The protons smash and produce a lot of stuff:
    - sometimes protons,
    - sometime electrons,
    - sometime top quarks
    - sometime Electroweak gauge bosons
    - almost always jets
    - …
    - ??? (black holes, extra dimensions, more ideas anyone?)

Here is a movie to help visualize it:



Introduction: what do we do at colliders?

- What happens is that we have a series of events (collisions) each one producing something.
- The resulting final state in each event is typically different, unique for this event.
- What kind of questions can we ask (i.e. are physical) and which cannot (i.e. are unphysical)?
  - what will happen in certain event? ☹
  - What is the probability to produce a pair of top quarks in a given event? ☺
  - What is the probability to produce a top pair with given momenta? ☹
  - What is the probability to produce a top pair with momenta in a given finite range? ☺

Do you see the difference between the allowed questions and the ones that are not?

- Let's rephrase these questions in proper terms:
  - Total inclusive cross-section: $\boxed{\sigma(pp \to t\bar{t} + X)}$
  - Differential cross-section: $\boxed{\frac{d\sigma}{d^3p}(pp \to t\bar{t} + X)}$

Introduction: what do we do at colliders?

- Let's make sure we understand what these mean (it is simple but very important):

  - Total inclusive cross-section: $\boxed{\sigma(pp \to t\bar{t} + X)}$

  What is the probability to observe two top quarks. They can have:
  - any energy
  - any momentum
  - possibly be accompanied by other particles
  - or not …

  <u>Anything is allowed as long as there are two tops present.</u>

  - Differential cross-section: $\boxed{\frac{d\sigma}{d^3p}(pp \to t\bar{t} + X)}$

  As above, however, we also require that one of the quarks has specific momentum.

Clearly, the total inclusive cross-section is obtained by integrating the differential one over the momentum (called phase space):

$$\boxed{\sigma(pp \to t\bar{t} + X) = \int d^3p \, \frac{d\sigma}{d^3p}(pp \to t\bar{t} + X)}$$

Introduction: why care about strong interactions?

- Because we mostly use hadron colliders. They collide hadrons = strongly interacting particles
- Because most of the particles produced and observed at colliders are hadrons
- Because we like perturbation theory and it really works:
  - there are 3 constants in the Standard Model:
    - the fine structure constant (it is small) $\boxed{\alpha(\mu = 0) \approx \frac{1}{137}}$
    - The Fermi constant (it is small)
    - The strong coupling constant (large) $\boxed{\alpha_S(\mu = m_Z) \approx 0.1}$

✓ The effects due to strong interactions are by far the largest and most important ones.

✓ We need to have a handle on them for any meaningful collider phenomenology.

QCD: the theory of the strong interactions

The story how QCD became the theory of the strong interactions is one of the best examples of why we should be all very proud to call ourselves physicists.

- Think 1960's. More "elementary" hadrons discovered than people you know personally.
  - **?** How do you explain this?
    - Many models were proposed.
    - Nothing worked.
    - People were abandoning QFT altogether.

- Then it became simple (Murray Gell-Mann and others early 1960's – early 1970's)

  - Realized that there is SU(3) global structure in the strong interactions. It just worked.
  - Quarks were introduced (at first, as a way of giving the hadrons their quantum numbers)
  - Then it became evident that non-abelian theories work (a parallel development)
    - Yang-Mills (1955)
    - Weinberg, Salam developed the EW interactions (1960's)
  - YM are renormalizable (t'Hooft, Veltman early 1970's)
  - QCD was proposed as a gauge theory of the strong interactions based on:
    - SU(3) gauge theory (with a single gauge coupling α_S)
    - gauge fields (gluons); dynamical fields (quarks); new quantum number (color)
  - Asymptotic freedom was discovered (Gross, Wilczek; Politzer) early 1970's

Nice reading: [M. Shifman 2001]

QCD: formal definition

- Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^A_{\alpha\beta} F^{\alpha\beta}_A + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$$

- Field strength:

$$F^A_{\alpha\beta} = \partial_\alpha \mathcal{A}^A_\beta - \partial_\beta \mathcal{A}^A_\alpha - g f^{ABC} \mathcal{A}^B_\alpha \mathcal{A}^C_\beta$$

- Covariant derivative:

$$(D_\alpha)_{ab} \quad = \quad \partial_\alpha \delta_{ab} + ig \left(t^C \mathcal{A}^C_\alpha \right)_{ab}$$

$$A, B, C = 1, \ldots, 8$$
$$a = 1, 2, 3$$

- **SU(N)** color algebra: matrix bases and Casimir operators

First Casimir of SU(3)

$$[t^A, t^B] = i f^{ABC} t^C$$

$$\sum_A t^A_{ab} t^A_{bc} = C_F \, \delta_{ac} \, , \quad C_F = \frac{N^2 - 1}{2N}$$

$$\text{Tr } T^C T^D = \sum_{A,B} f^{ABC} f^{ABD} = C_A \, \delta^{CD} \, , \quad C_A = N$$

✓ The specific form of the matrices t^A is not needed. All we ever encounter are:
- commutation relations,
- traces of products.

Further reading: [K. Ellis, QCD lectures]

QCD versus QED: twins that can be very far apart

- Perturbatively QCD and QED are very similar. In fact they are almost identical once color has been handled:

  The transition QCD → QED: $t^A \to 1$; $f^{ABC} \to 0$; $C_A \to 0$; $C_F \to 1$.

- In calculations it is often useful to compare the two, or to think of the abelian limit of QCD.

- The true differences first appear due to coupling running (not obvious in the Lagrangian):

  - In QED: coupling decreases with distance
  - In QCD: coupling increases with distance

- At large distances (or small energies) QCD becomes confining,
  i.e. the constituent particles, <u>the quarks, cannot be separated</u>.

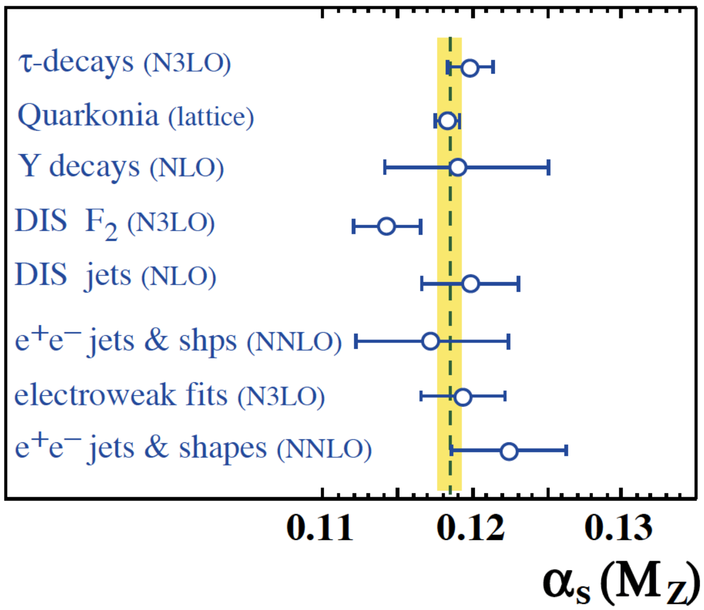
- In observables quarks always form bound states: the hadrons.

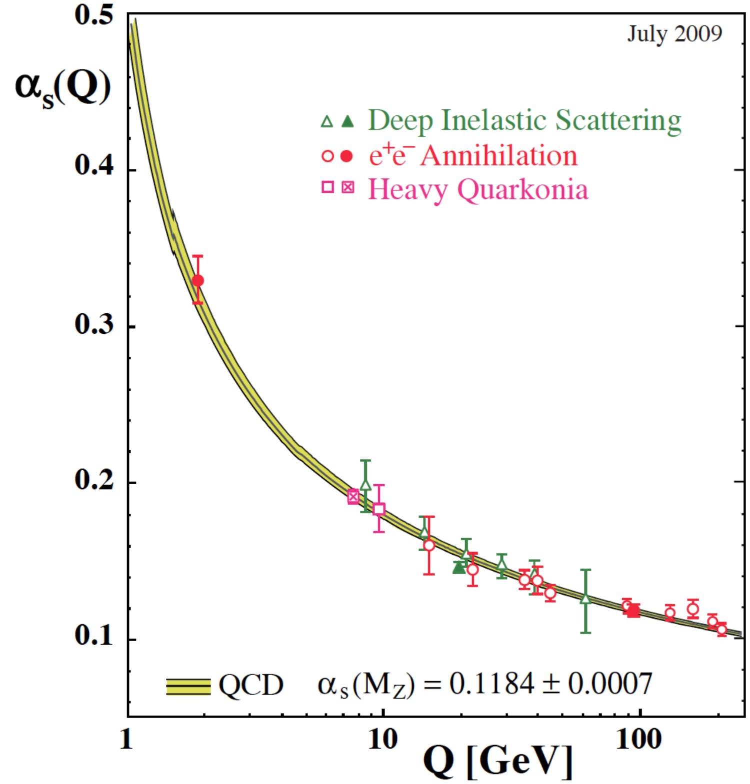
- No quark (or gluon) can be observed alone.

- Formalize:
  - both quarks and gluons carry quantum number <u>color.</u>
  - only colorless states can be observed (i.e. hadrons are always colorless).

QCD versus QED: twins that can be very far apart

- The true differences first appear due to coupling running:
  - QED: coupling decreases with distance
  - QCD: coupling increases with distance (decreases with energy)





- A firm prediction of the theory: the coupling must be process independent.
- Its running too. It is known to 4 loops!

By now plenty of <u>precision</u> data that confirms this. A triumph for QCD!

QCD: quarks or hadrons?

- We seem to face a serious problem: $\mathcal{L} = -\frac{1}{4} F^{A}_{\alpha\beta} F^{\alpha\beta}_{A} + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$

  1. QCD is formulated in terms of quarks and gluons.
  2. Yet what we measure (the observables) are hadrons.
  3. Hadrons are nowhere to be found in the QCD Lagrangian.

- So, what to make of this? How to calculate collider processes?

✓ Formally, there is no problem:

- scattering theory allows bound states
- Example: the hydrogen atom. A bound state of an electron and a proton.
- we widen the Hilbert space and include them
- solve the theory and derive all properties of the bound states (Bethe-Salpeter)

However: we cannot solve QCD

✓ This is not a mere technical complication

✓ If you are ambitious enough, and want to work for Nobel prize, this problem guarantees one! (beware: most will fail)

QCD: quarks or hadrons?

- ✓ Will see shortly how to deal with bound states in QCD (the solution is practical, not formal).
- ✓ Let's build some intuition first

  - Example 1: The hydrogen atom in QED (bound state of electron and proton). It is stable.
  - Example 2: The positronium in QED (bound state of electron and positron). It is unstable.

- ✓ The process of forming a bound state:
  - ✓ Imagine a cloud of protons, just sitting in space.
  - ✓ An electron flies in.
  - ✓ What can happen?
    - Nothing: an electron goes out,
    - A bound state is formed.

Detailed calculation required to predict the details.

- ✓ On the other side, we can answer less detailed questions without additional effort.

  - Here is a good example: what is the amount of bound states to be produced?
  - Answer: the same as the amount of electrons. This we can calculate without resort to bound states!

- ✓ This is an example of inclusive observable (i.e. we are not interested in the details…)
- ✓ Inclusive observables are encountered very often. They are very useful. Get to know them!

QCD: time to get our hands dirty!

✓ We are ready to do an inclusive calculation in QCD

Make sure you understand this well!

$$e^+ e^- \rightarrow \text{hadrons}$$

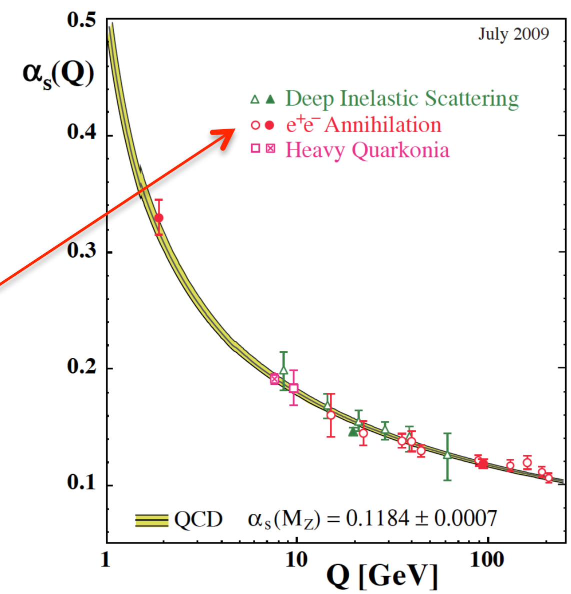
(i.e. what is the probability that by scattering $e^+ e^-$ pairs, we will produce hadrons?)

✓ This is an example of inclusive observable. We do not ask:

- what kind of hadrons are produced?
- what is their distribution
- what is their multiplicity
- etc.

✓ Essentially, this is just a counting experiment.

And an important one!



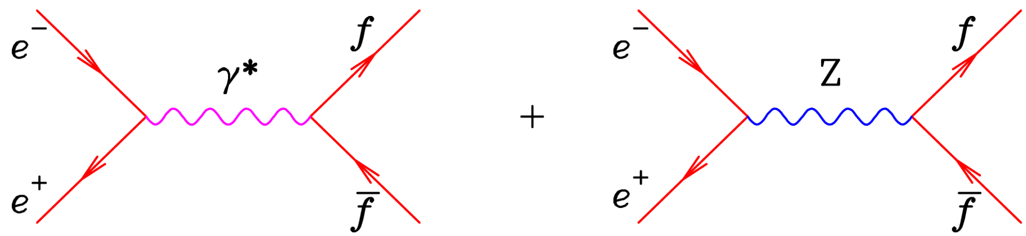
Quark – hadron duality: in very inclusive observables, quarks and hadrons are the same thing

Note this keyword

Any questions so far?

QCD: $e^+ e^- \rightarrow$ hadrons

✓ The lowest order Feynman diagrams (Born process):



✓ The R-ratio (very well measured observable; insensitive to the details of hadronization).

QCD

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \rightarrow q\bar{q})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = 3\sum_q Q_q^2$$

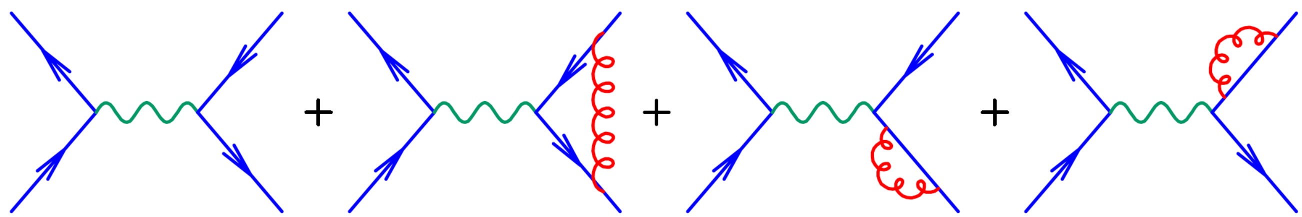
QED

From here we can experimentally determine:

- the electric charge of quarks
- the number of colors
- the number of quark generations

QCD: $e^+ e^- \rightarrow$ hadrons

- ✓ The one-loop quantum corrections:



- ✓ In dimensional regularization, i.e. in d=4-2ε dimensions, and after UV renormalization:

$$\sigma^{q\bar{q}} = 3\sigma_0 \left\{1 + \frac{2\alpha_S}{3\pi} H(\epsilon) \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon)\right]\right\}$$

✓Oooops! The result is divergent (when ε→ 0) and so is meaningless !!!

- ✓ This is not a mistake but an indication of a serious deficiency! We must be doing something very wrong.
- ✓ Let's re-analyze the whole setup.
- ✓ Any ideas? Or questions up to here?

Go into the deep: augment the formal theory with physics intuition
Infra Red (IR) singularities

- ✓ Where do these singularities come from?

$\boxed{\text{Will discuss in a moment}}$

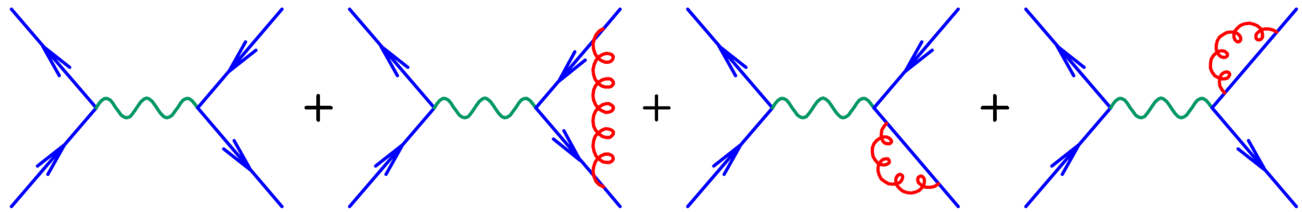
- ✓ What does their presence indicate? (i.e. what are they trying to tell us?)

$\boxed{\text{Will discuss in a moment}}$

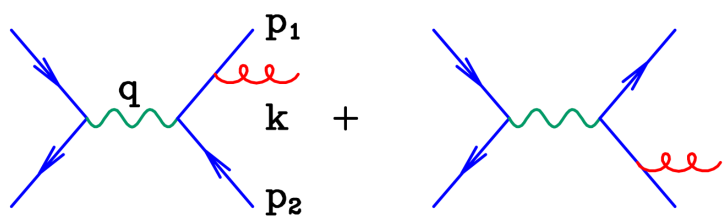
- ✓ Infrared safety: only infrared safe observables are meaningful.

QCD: back to $e^+ e^- \to$ hadrons

✓ The one-loop quantum corrections



✓ need to be supplemented by real-radiation ones:



Q: But why should this work?

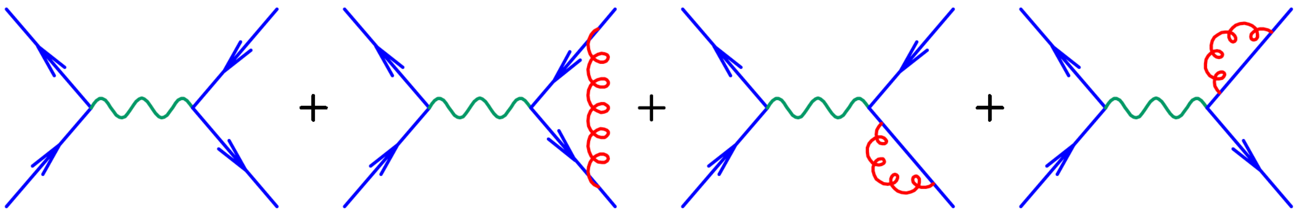
- Recall the definition of R:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{\sum_q \sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q Q_q^2$$

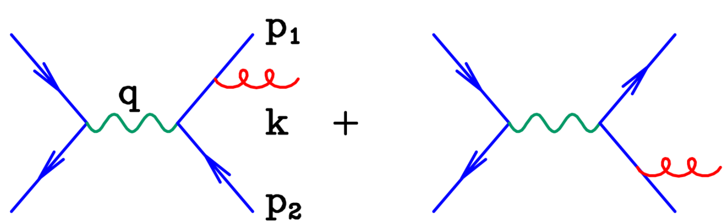
- True only at Born Level. At higher orders we have to allow for all possible combinations, not just qq!

QCD: back to $e^+ e^- \to$ hadrons

- ✓ The one-loop quantum corrections



- ✓ need to be supplemented by real-radiation ones:



$$\sigma^{q\bar{q}} = 3\sigma_0 \left\{ 1 + \frac{2\alpha_S}{3\pi} H(\epsilon) \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right] \right\}$$

$$\sigma^{q\bar{q}g} = 2\sigma_0 \frac{\alpha_S}{\pi} H(\epsilon) \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right]$$

Nice! The sum of the two is now finite ☺

$$\boxed{R = 3 \sum_q Q_q^2 \left\{ 1 + \frac{\alpha_S}{\pi} + \mathcal{O}(\alpha_S^2) \right\}}$$

More on the IR singularities

✓ There are two types of IR singularities:
- Soft
- Collinear

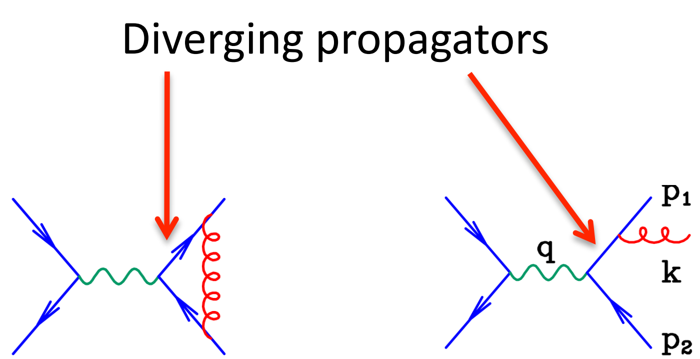
✓ They are very significant physically; appear everywhere. Let's understand them well.

- Soft singularity: due to emission (real or virtual) of a soft massless gauge boson (photon or gluon) with vanishing energy.

- Collinear singularity: due to emission (real or virtual) of a massless gauge boson (photon or gluon) with a momentum parallel to the emitting massless quark.

✓ Technically, these singularities are due to vanishing propagators.

✓ The singularities are regulated, say dimensionally, and then when you integrate over them (in the loop or over the phase space) explicit poles are generated.



Go into the deep: augment the formal theory with physics intuition
Infra Red (IR) singularities

✓ Where do these singularities come from?

$$\left.\frac{1}{(p-k)^2}\right|_{p^2=k^2=0} = \frac{1}{2p\cdot k} = \frac{1}{p^0k^0(1-\cos(\theta))}$$

$$\frac{1}{(p-k)^2}\ \text{diverges when}\ = \begin{cases} k^0 \to 0 & \text{soft singularity}\,, \\ \cos(\theta) \to 1 & \text{collinear singularity}\,. \end{cases}$$

✓ What does their presence indicate? (i.e. what are they trying to tell us?)

- A state with a soft gluon is <u>indistinguishable from</u> a state without it.
    - Note: this is conceptual limitation! Detectors have finite resolution.
    - A profound consequence: if we allow one soft gluon, we should allow arbitrary many.
    - This leads to resummation (more later)

- A state with a gluon collinear to the emitting particle is <u>indistinguishable from</u> a state without it. We can have many collinear gluons.

✓ So, a state is not a simple concept after all. It can contain many “basic” states. We always have to sum over all allowed states. This is the biggest complication in collider phenomenology!

✓ Infrared safety: only infrared safe observables are meaningful.

Definition of a final state

We noted that:

✓ So, a state is not a simple concept after all. It can contain many "basic" states. We always have to sum over all allowed states. This is the biggest complication in collider phenomenology!

1. Definition of final state depends on the definition of the observable.
2. Definition of observable must be independent of the perturbative order
3. Definition of a final state does depend on the perturbative order

Example: Higgs boson production

- Inclusive Higgs production (i.e. H+*anything*)

  Anything = (0,H,HH,g,ggggg,qqgg, etc.)

  - Born level: *anything* = 0 Leading Order (LO)
  - First correction: *anything* = g   Next-to-Leading Order (NLO)
  - Second correction: *anything* = (gg,qq) Next-to-Next-to-Leading Order (NNLO)
  - …

Note:

1. The emitted particles are not soft or collinear; they can be anything (i.e. hard).
2. When we integrate over them, there is always a region where they become soft/collinear.
3. This leads to divergences. They cancel when all states are included.
4. IR safety implies we always need to include at least soft/collinear radiation.

More on soft and collinear singularities

✓ A central property of collinear emissions is factorization

$$\sigma(\{\text{in}\} \to q + g + X)\Big|_{\vec{g}||\vec{q}} \approx \sigma(\{\text{in}\} \to q + X) \otimes P(q \to q + g)$$

- The function P(…) is called splitting amplitude
- P(…) does not depend on the process, but only on the type of splitting
- P(…) has perturbative expansion. Can be calculated once and for all.

✓ Factorization exists in the soft limit too. It is more complicated since it involves color correlations.

✓ To derive the soft limit it is enough to work in the eikonal approximation.

Collinear factorization. Partons and hadrons.

- ✓ Up to here we didn't pay any attention to the distinction quarks/gluons/hadrons.
- ✓ The reason was we *chose* to work only very inclusive observables.
- ✓ Most interesting observables are not fully inclusive. They are *differential observables*.
- ✓ For differential observables we need to distinguish between quarks/gluons and hadrons.

Some terminology:

- ✓ Parton: a quark or a gluon, i.e. an object that we can treat perturbatively
- ✓ Hadronization: the process of forming a hadron. It is initiated by an energetic parton.
- ✓ Fragmentation ≈ Hadronization

Why things change when we go to differential observables?

Idea: at the differential level we start to ask questions about the nature of the final state.

Example: what is the momentum of the parton?
But that supposes we can distinguish the partons from each other. We now know this is not IR safe.
So we expect new divergences to appear

Horror: when are we going to get rid of all these divergences?? Soon – we are almost there.

Lecture 2

Recap of Lecture 1. We learned

- ✓ The importance and relevance of QCD at colliders
- ✓ formal definition and running of the strong coupling
- ✓ Inclusive versus differential observables
- ✓ IR singularities:
  - ✓ Soft: emission of soft gluons (or photons in QED)
  - ✓ Collinear: emission of a collinear gluon (or photon in QED)
- ✓ The distinction between partons and hadrons

Continue with lecture 2:

Collinear factorization. Partons and hadrons.

✓ We saw that once real and virtual corrections to an inclusive observable are added, the IR divergences cancel.
✓ But what happens when we look at differential distributions?

If you compute the NLO corrections to the energy distribution of a quark (x=normalized energy) it is still divergent (but now only a single power of ε)

$$\frac{d\sigma}{dx}(e^+e^- \to q + X) = \frac{1}{\epsilon}\sigma_{\text{born}}\, P_{qq}^{(0)}(x) + \text{finite terms}$$

$$P_{qq}^{(0)}(x) = \frac{4}{3}\left(\frac{1+x^2}{1-x}\right)_+ \qquad \int_0^1 dx\, P_{qq}^{(0)}(x) \;=\; 0 \qquad \int_0^1 (f(x))_+\, g(x)dx = \int_0^1 \Big\{f(x) - f(1)\Big\} g(x)dx$$

- P: the Altarelli-Parisi splitting function (at 1 loop; known to 3 loops)
- Notice – their integral is zero (these are not functions but distributions)
- there are also functions for any splitting (q→q, q→g, g→q,g→g)

$$\int_0^1 dx\, \frac{d\sigma}{dx}(e^+e^- \to q + X) = \sigma(e^+e^- \to X)$$

Thus, the differential distribution is divergent, but the total inclusive one is finite:

Good! We got back where we started from ☺

Collinear factorization. Partons and hadrons.

Two equivalent questions:

- ✓ So far we discussed partons – but we measure hadrons. How to describe this?
- ✓ What to do with the remaining divergence in the differential distribution?

$$\frac{d\sigma}{dx}(e^+e^- \to q + X) = \frac{1}{\epsilon}\sigma_{\text{born}}\, P_{qq}^{(0)}(x) + \text{finite terms}$$

The important feature is: the divergence factorizes, i.e. it is proportional to the LO cross-section

$$\underbrace{\frac{d\sigma}{dx}(e^+e^- \to q + X) = \frac{d\sigma}{dx}^{(0)} + \alpha_S \frac{d\sigma}{dx}^{(1)} + \mathcal{O}(\alpha_s^2)}_{\text{Perturbative cross-section (divergent)!}} \quad , \quad \frac{d\sigma}{dx}^{(0)} = \sigma_{\text{born}}\delta(1-x) \qquad \Gamma^{(0)} = \frac{1}{\epsilon}P_{qq}^{(0)}(x)$$

Altarelli – Parisi splitting function

Perturbative cross-section (divergent)!

finite

Collinear counterterms. Scheme dependent.
In MSbar contain poles only. Process independent.

$$\begin{aligned}
\frac{d\sigma}{dx}(e^+e^- \to q + X) &= \frac{d\hat{\sigma}}{dx} \otimes \Gamma \\
&= \left\{\Gamma^{(0)} + \alpha_S \Gamma^{(1)} + \dots\right\} \otimes \left\{\frac{d\hat{\sigma}}{dx}^{(0)} + \alpha_S \frac{d\hat{\sigma}}{dx}^{(1)}\right\} \\
&= \Gamma^{(0)} \otimes \frac{d\hat{\sigma}}{dx}^{(0)} + \alpha_S\left\{\Gamma^{(0)} \otimes \frac{d\hat{\sigma}}{dx}^{(1)} + \Gamma^{(1)} \otimes \frac{d\hat{\sigma}}{dx}^{(0)}\right\} + \mathcal{O}(\alpha_s^2)
\end{aligned}$$

Do you recognize the equation on top?

Collinear factorization. Fragmentation functions.

We saw and learned that in differential observables, after all corrections have been added, all soft singularities cancel. However collinear singularities remain:

$$
\begin{aligned}
\frac{d\sigma}{dx}(e^+e^- \to q + X) &= \frac{d\hat{\sigma}}{dx} \otimes \Gamma \\
&= \left\{\Gamma^{(0)} + \alpha_S \Gamma^{(1)} + \dots\right\} \otimes \left\{\frac{d\hat{\sigma}}{dx}^{(0)} + \alpha_S \frac{d\hat{\sigma}}{dx}^{(1)}\right\} \\
&= \Gamma^{(0)} \otimes \frac{d\hat{\sigma}}{dx}^{(0)} + \alpha_S \left\{\Gamma^{(0)} \otimes \frac{d\hat{\sigma}}{dx}^{(1)} + \Gamma^{(1)} \otimes \frac{d\hat{\sigma}}{dx}^{(0)}\right\} + \mathcal{O}(\alpha_s^2)
\end{aligned}
$$

We also learned that:

- The collinear singularities factorize
- Any remaining IR singularity is not scary – it just reminds us we are forgetting something

What is it that we forget? Any ideas?

$\boxed{\frac{d\sigma}{dx}(e^+e^- \to q + X)}$ versus $\boxed{\frac{d\sigma}{dx}(e^+e^- \to H + X)}$

Collinear factorization. Fragmentation functions.

We are almost done… Let's summarize what we have so far:

- ✓ We made a complete partonic calculation; the result is still divergent (collinear singularity left)
- ✓ But we do not calculate what we measure:
  - We measure hadrons
  - We calculate partons

- ✓ So, this must be it! We just have incomplete calculation and QCD reminds us we are not done!

OK. How do we describe the hadrons then?

The only good way we know is based on factorization

- ✓ Hadrons are non-perturbative objects
- ✓ This means we cannot describe them in perturbation theory
- ✓ Non-perturbative phenomena are described by QCD, too, (we believe) but:
- ✓ We cannot solve QCD non-perturbatively (a big open question; remember the Nobel Prize?)

- ✓ Therefore we have to model it.

$$\frac{d\sigma}{dx}(e^+e^- \to H + X) = \frac{d\hat{\sigma}}{dx} \otimes D_{q\to H}(x)$$

Perturbative Non-perturbative

Collinear factorization. Fragmentation functions.

$$\frac{d\sigma}{dx}(e^+e^- \to H + X) = \frac{d\hat{\sigma}}{dx} \otimes D_{q\to H}(x)$$

$$\begin{aligned}\frac{d\sigma}{dx}(e^+e^- \to q + X) &= \frac{d\hat{\sigma}}{dx} \otimes \Gamma \\ &= \left\{\Gamma^{(0)} + \alpha_S \Gamma^{(1)} + \ldots\right\} \otimes \left\{\frac{d\hat{\sigma}}{dx}^{(0)} + \alpha_S \frac{d\hat{\sigma}}{dx}^{(1)}\right\} \\ &= \Gamma^{(0)} \otimes \frac{d\hat{\sigma}}{dx}^{(0)} + \alpha_S \left\{\Gamma^{(0)} \otimes \frac{d\hat{\sigma}}{dx}^{(1)} + \Gamma^{(1)} \otimes \frac{d\hat{\sigma}}{dx}^{(0)}\right\} + \mathcal{O}(\alpha_s^2)\end{aligned}$$

Note: the same $\frac{d\hat{\sigma}}{dx}$ appears in both places.

- What happened to the divergent counterterm?
- It was "absorbed" into the non-perturbative function.
- Why? Because it is a long distance physics and belongs into the fragmentation function.

The way to think about the collinear divergences is that they are not real divergences;
They are just artifacts of our idealized picture that the process of producing a hadron goes through an on-shell massless parton.

- Indeed, if we consider massive quarks, then the collinear divergence will not appear at all
- How can that be?? Answer: it is finite but contains ln(m) term that diverges when m→ 0

Called a "quasicollinear" singularity. Needs to be resummed

Collinear factorization. Fragmentation functions.

$$\frac{d\sigma}{dx}(e^+e^- \to H + X) = \frac{d\hat{\sigma}}{dx} \otimes D_{q\to H}(x)$$

✓ How do we model the fragmentation functions D(…)?

- They are non-perturbative
- process independent i.e. universal
- Therefore, can be extracted from experiment
- They only depend on q and H.

$$\begin{aligned}
\frac{d\sigma}{dx}(e^+e^- \to q + X) &= \frac{d\hat{\sigma}}{dx} \otimes \Gamma \\
&= \left\{\Gamma^{(0)} + \alpha_S \Gamma^{(1)} + \dots\right\} \otimes \left\{\frac{d\hat{\sigma}}{dx}^{(0)} + \alpha_S \frac{d\hat{\sigma}}{dx}^{(1)}\right\} \\
&= \Gamma^{(0)} \otimes \frac{d\hat{\sigma}}{dx}^{(0)} + \alpha_S \left\{\Gamma^{(0)} \otimes \frac{d\hat{\sigma}}{dx}^{(1)} + \Gamma^{(1)} \otimes \frac{d\hat{\sigma}}{dx}^{(0)}\right\} + \mathcal{O}(\alpha_s^2)
\end{aligned}$$

Few more things to note:

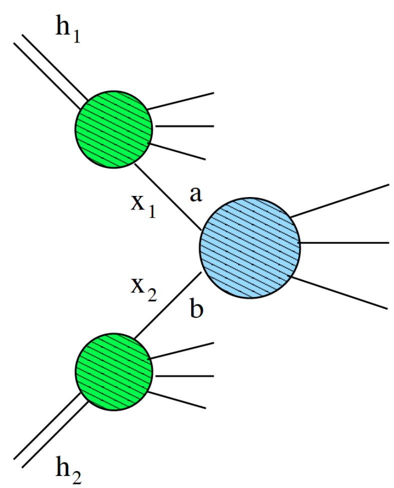
- fragmentation functions are not observables
- they are not unique: depend on the definition of the counterterms
- usually in the MS-bar scheme.

- Fragmentation of massive quarks (charm, bottom) requires additional perturbative component for resummation of ln(m) terms.
- This was the resolution of the b-production puzzle at Tevatron 10 years ago.

Collinear factorization. Parton distribution functions.

$$\frac{d\sigma}{dx}(e^+e^- \to H + X) = \frac{d\hat{\sigma}}{dx} \otimes D_{q\to H}(x)$$

✓ We encounter exactly the same story when we consider hadrons in the initial state



✓ Even for fully inclusive observables collinear singularities remain.
They are associated with the Initial hadron → parton transition.

✓ This non-perturbative transition is described by parton distribution functions.

✓ They are very similar to fragmentation functions but are not the same!

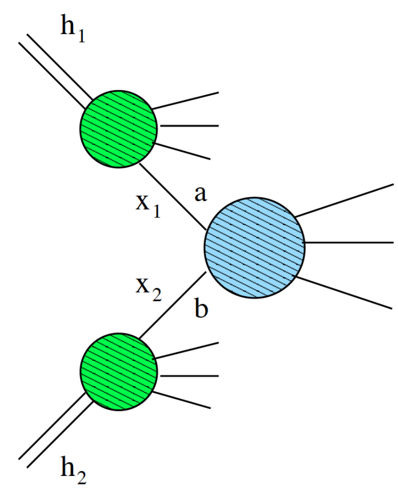
✓ Extracted from experiment

✓ Universal (i.e. process independent)

✓ Scheme dependent (typically MS-bar)

Collinear factorization in all its glory

$$\frac{d\sigma}{dp_T}(pp \to H+X) = \sum_{i,j,k=q,\bar{q},g} f_i \otimes f_j \otimes \frac{d\hat{\sigma}}{dp_T}(ij \to k+X) \otimes D_{k\to H}(x) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}\right)$$



Many things to note:

- ✓ Two pdf's for LHC collisions (1 for DIS)
- ✓ One fragmentation function for each observed final state hadron
- ✓ We sum over all possible partons initial/final state

- ✓The factorization formula does not apply to any processes
  - ✓ For some never been proven,
  - ✓For other may not apply (or remainders changes)
- ✓ It is not exact. It misses terms that are small
- ✓ Q is a "typical" scale. A hard scale which is large. That's why we can neglect the remainder.
- ✓ There is, of course, scheme dependence (this is OK – does not indicate deficiency)
- ✓ The schemed dependence only means that what is meaningful is the LHS and not any one term on the RHS

From factorization to evolution

Great – we are almost "there". Just one more thing remains: evolution.

- ✓ We know the coupling evolves with the scale. Which scale?
- ✓ We choose typical scales for the process (no better way known)
- ✓ This is simple in very inclusive observables
- ✓ Less so in more differential ones (like dijets).

- ✓ Similarly, the pdf's and F.F's also change with scale. Factorization scale.
- ✓ This scale is unphysical. The observable does not depend on it (to a given order).
- ✓ This requirement determines the scale evolution of pdf's and FF's: DGLAP

$$\widetilde{F}_2\left(N, \frac{Q^2}{m^2}, \alpha_s\right) = \widetilde{\mathcal{F}}_2\left(N, \frac{Q^2}{\mu_F^2}, \alpha_s\right) \widetilde{f}\left(N, \frac{\mu_F^2}{m^2}, \alpha_s\right)$$

$$\frac{d\widetilde{F}_2}{d\mu_F} = 0 \quad \to \quad \frac{d\log\widetilde{f}}{d\log\mu_F} = \gamma_N\left(\alpha_s\right) \ .$$

DGLAP: Evolution equations for pdf and FF

Dokshitzer-Gribov-Lipatov-Altarelli-Parisi: 1970's

$$\frac{\partial q_f(z,Q^2)}{\partial \log Q^2} = \frac{\alpha_s}{2\pi}\int_z^1 \frac{dy}{y}\left[P_{qq}\left(\frac{z}{y},\alpha_s(\mu)\right) q_f(y,Q^2) + P_{qg}\left(\frac{z}{y},\alpha_s(\mu)\right) g(y,Q^2)\right]$$

$$\frac{\partial g(z,Q^2)}{\partial \log Q^2} = \frac{\alpha_s}{2\pi}\int_z^1 \frac{dy}{y}\left[P_{gq}\left(\frac{z}{y},\alpha_s(\mu)\right) \sum_f q_f(y,Q^2) + P_{gg}\left(\frac{z}{y},\alpha_s(\mu)\right) g(y,Q^2)\right]$$

… and the LO splitting functions

$$P_{qg}^{(1)}(z) = \frac{1}{2}\left(z^2+(1-z)^2\right), \quad P_{gq}^{(1)}(z) = \frac{1}{2}\left(\frac{1+(1-z)^2}{z}\right),$$

$$P_{gg}^{(1)}(z) = 2C_A\left(\frac{z}{[1-z]_+} + \frac{1-z}{z} + z(1-z)\right) + \delta(1-x)\left(\frac{11C_A-2n_f}{6}\right).$$

✓ These equations predict the scale dependence: if you know the functions at 1 scale Q

✓ They predict it at any other scale. This ensures their portability across colliders and processes

DGLAP is:

- A system of matrix equations
- Known to 3-loops (NNLO) in analytical form
- The splitting functions for FF (timelike) and pdf(spacelike) are different starting from 3 loops

Jets at hadron colliders

✓ There is an alternative way of thinking about hadron production: Jets

✓ Extensively covered in Lecture 4 of Peter Jacobs.

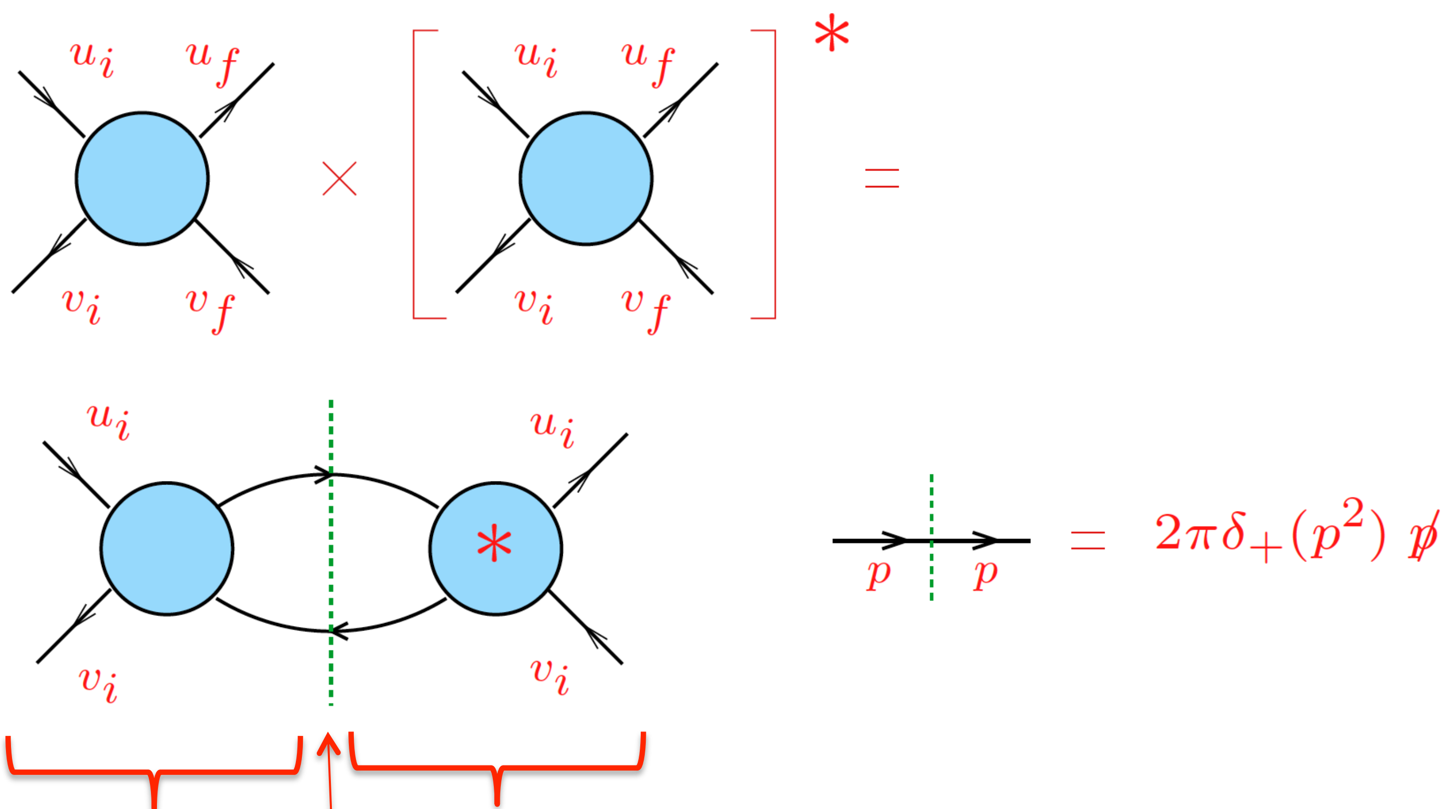
Collider applications

Cut diagrams

Drawings: thanks to L. Magnea

When we compute x-sections we need squared matrix elements.

Natural notation: cut diagrams (they put real and virtual loops on more equal footing):

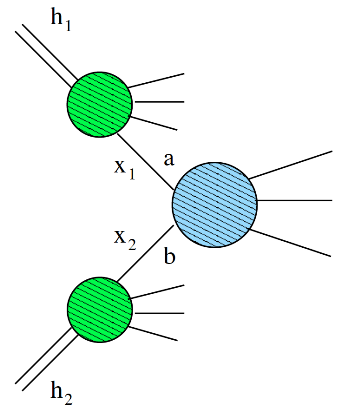


ME

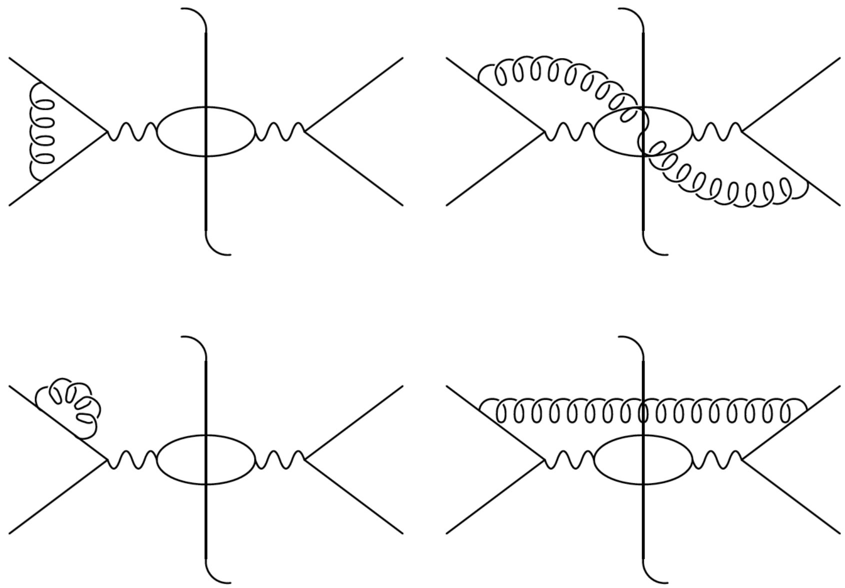
Complex conjugated ME

The "cut", i.e. the physical final state

Drell-Yan

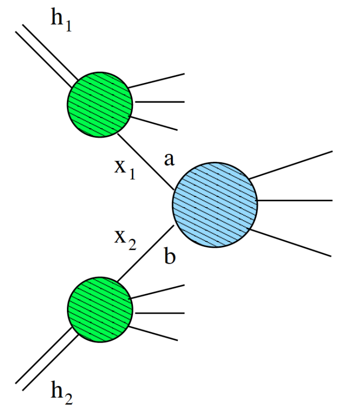


The most basic hadron collider process: $pp \to \gamma^* \to \mu^+ \mu^-$



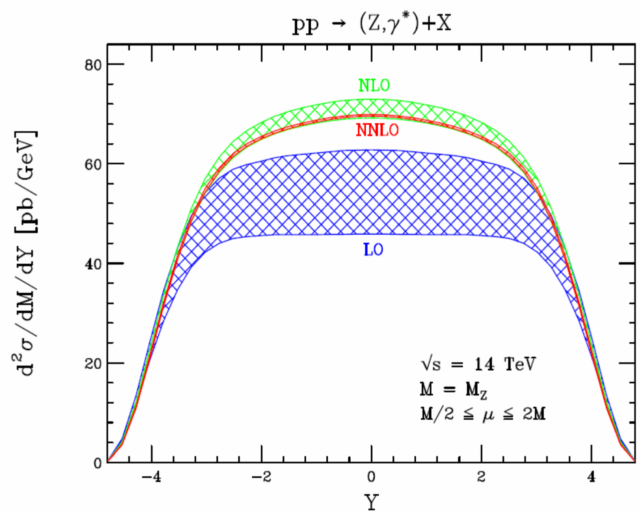
- NLO corrections shown above
- Now know through NNLO
- It is the "inverted" version of $e^+ e^- \to$ hadrons

Drell-Yan



Vector boson rapidity distribution

Anastasiou, Dixon, Melnikov, Petriello '03

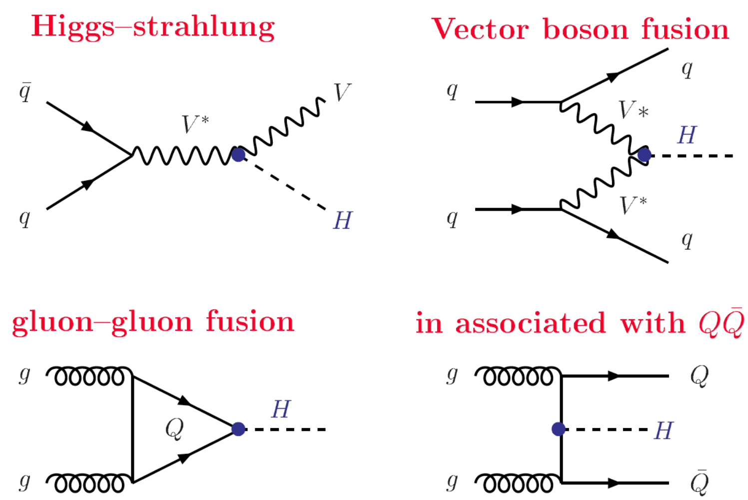


Notice the scale variation.

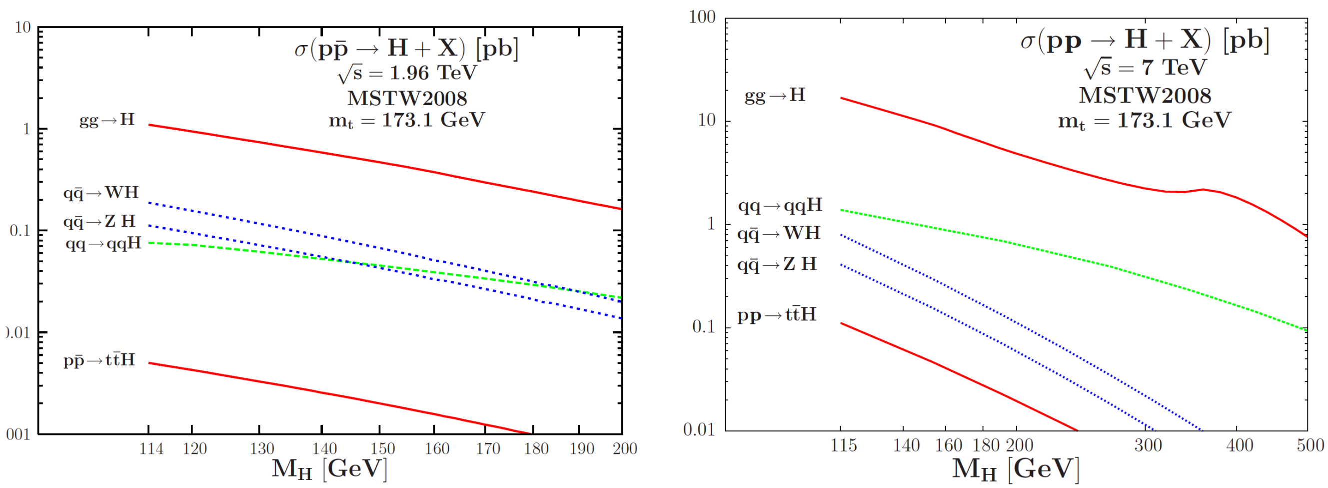
Higgs production

See review arXiv:1203.4199 Djouadi

LO Feynman diagrams:

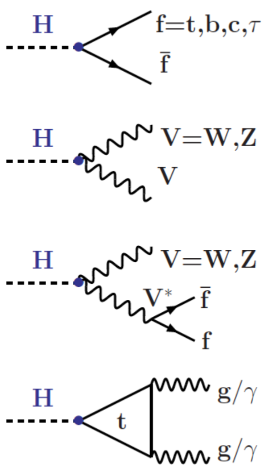


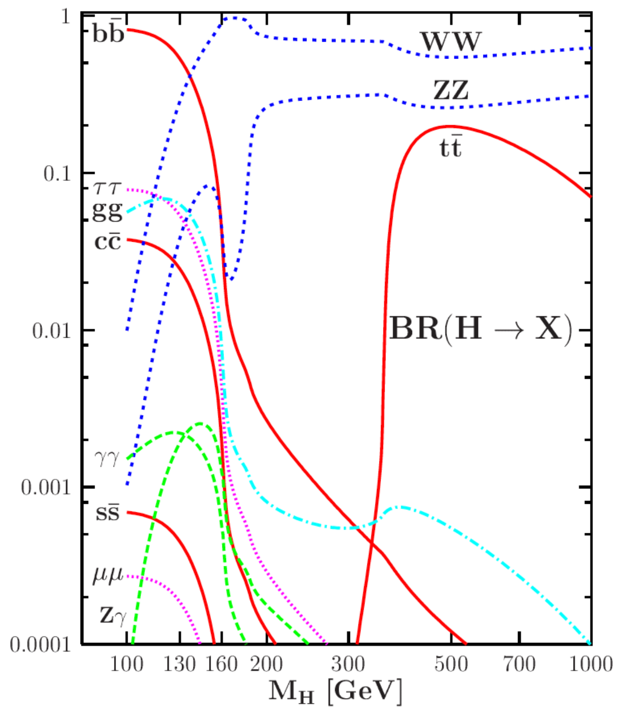
Production at Tevatron and LHC7:

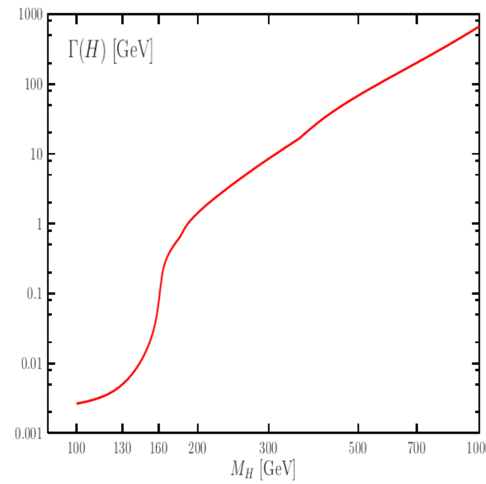


Higgs production

Higgs decay channels:

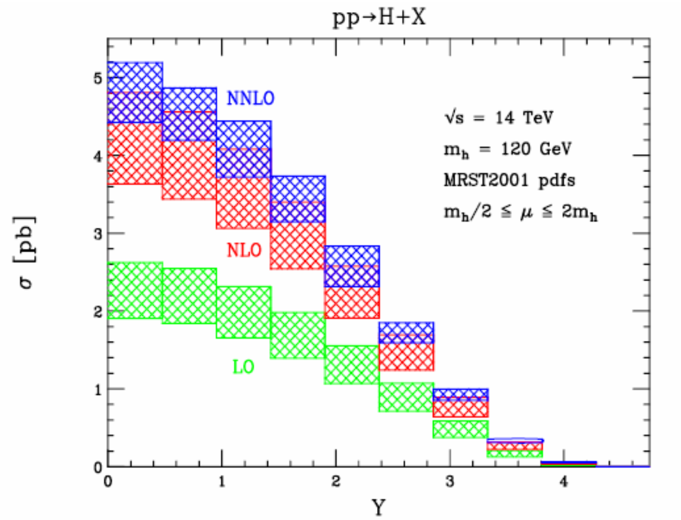






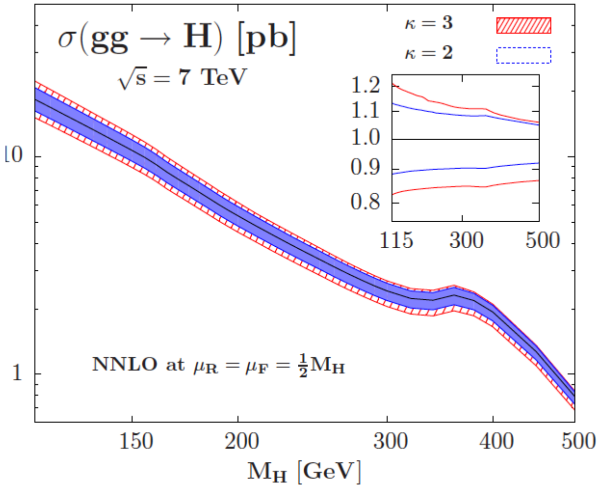
Higgs production

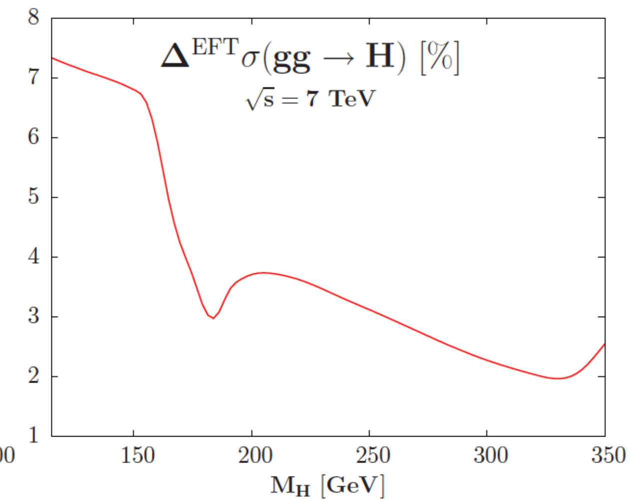
Theoretical uncertainties in Differential Higgs production:

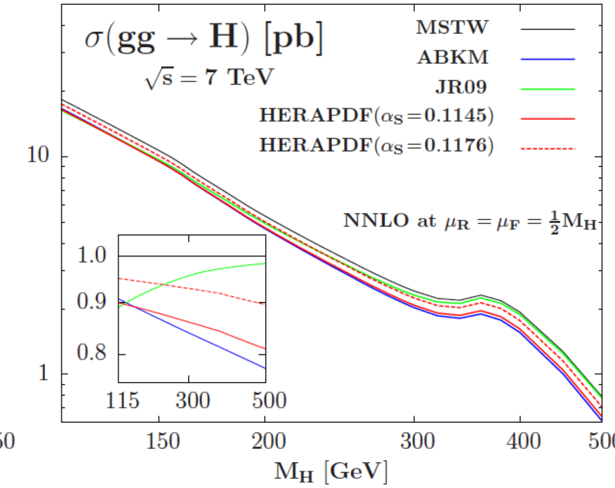


Anastasiou, Melnikov, Petriello `04

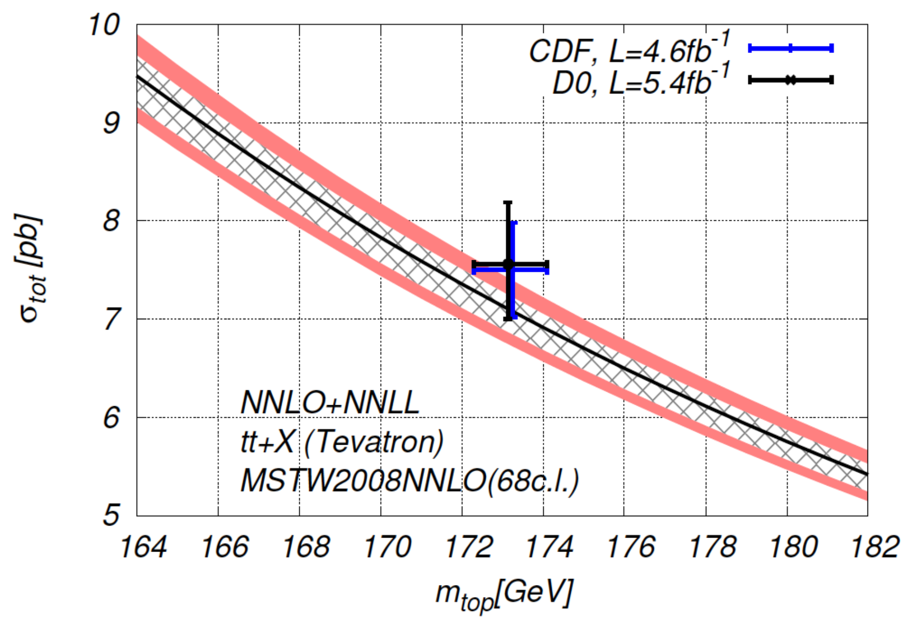
And for the total cross-section:

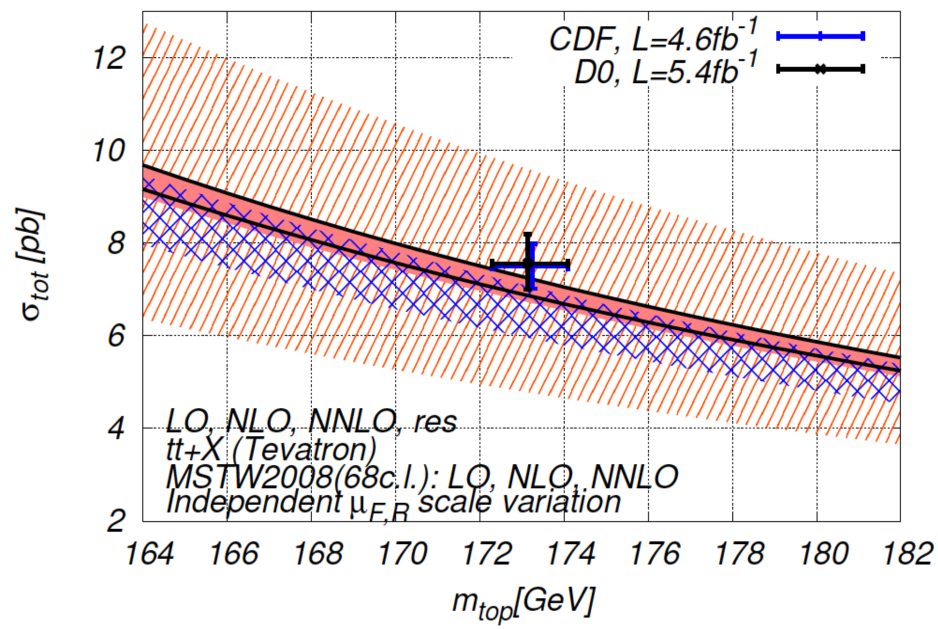






Top-pair production at Tevatron





Bärnreuther, Czakon, Mitov `12

Likely the most precisely known collider process.

Top-pair production: top-mass measurement

- **The fate of the Universe might depend on 1 GeV in M_{top}!**

Cosmological implications:

Bezrukov, Shaposhnikov '07-'08
De Simone, Hertzbergy, Wilczek'08

- Higgs Inflation: Higgs = inflaton

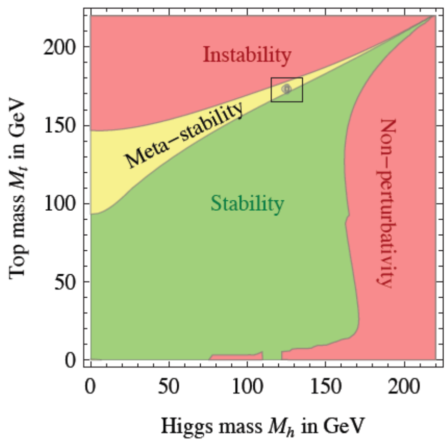
$$\mathcal{L}_h = -|\partial H|^2 + \mu^2 H^\dagger H - \lambda (H^\dagger H)^2 + \xi H^\dagger H \mathcal{R}$$

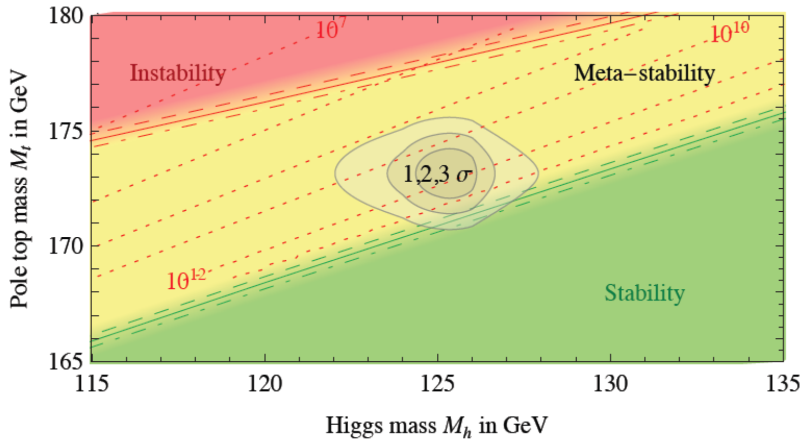
$$m_h > 125.7\,\mathrm{GeV} + 3.8\,\mathrm{GeV}\left(\frac{m_t - 171\,\mathrm{GeV}}{2\,\mathrm{GeV}}\right) - 1.4\,\mathrm{GeV}\left(\frac{\alpha_s(m_Z) - 0.1176}{0.0020}\right) \pm \delta$$

Strong dependence on the top mass!

- Higgs mass and vacuum stability in the Standard Model at NNLO.

Degrassi, Di Vita, Elias-Miro, Espinosa, Giudice, Isidori, Strumia '12

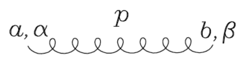




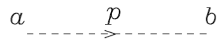
Instability scale Λ [GeV]
δM_{top} is the dominant uncertainty!

Computational aspects

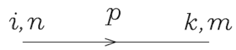
Feynman Rules for QCD



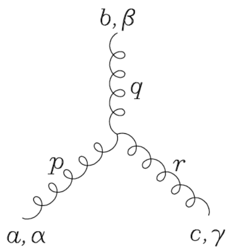
$$= \delta^{ab}\frac{-i\,g^{\alpha\beta}}{p^2+i\epsilon} \quad \text{(Feynman gauge)}$$



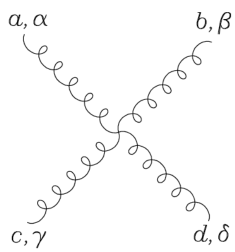
$$= \delta^{ab}\frac{i}{p^2+i\epsilon}$$



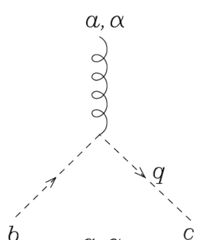
$$= \delta^{ik}\left.\frac{i}{\not p - m + i\epsilon}\right|_{mn}$$



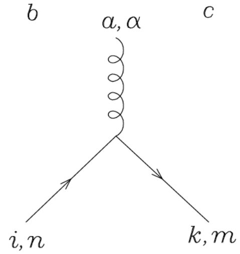
$$= gf^{abc}\left[g^{\alpha\beta}(p-q)^\gamma + g^{\beta\gamma}(q-r)^\alpha + g^{\gamma\alpha}(r-p)^\beta\right]$$



$$\begin{aligned} = & -ig^2 f^{xac} f^{xbd}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \\ & -ig^2 f^{xad} f^{xbc}\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta}\right) \\ & -ig^2 f^{xab} f^{xcd}\left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right) \end{aligned}$$



$$= -gf^{abc}\,q^\alpha$$



$$= ig\,\lambda^a_{ki}\,\gamma^\alpha_{mn}$$

Modern computational methods

✓ IBP (Integration by parts identities). Multi-loop integrals. See book by V. Smirnov.
- Best for computing Feynman integrals analytically.
- Reduces the number of integrals to a minimal basis: $10^N \rightarrow 10^1$
- Produces system of differential equations for the master integrals.
- Public program s implementing them exist

✓Unitarity: the basis for the current NLO revolution. Amazing results achieved.
See arXiv:1105.4319v3 [hep-ph] by Ellis, Kunszt, Melnikov, Zanderighi

✓ Dipole subtraction How to combine real and virtual corrections at NLO such that:
- IR singularities are cancelled in a universal way
- Finite parts are calculated with a straightforward MC integration

Catani Seymour (!996); Frixione, Kunszt, Signer (1986)

✓ Numerics (at NLO and NNLO):
- Sector decomposition: a way of numerically computing the divergent Feynman integrals
  Binoth, Heinrich (2000); Anastasiou, Melnikov, Petriello (2004)

- Mellin-Barnes approach to numerical evaluation of Feynman integrals

✓ FORM – computer algebra system [Vermaseren]: traces of gamma matrices, color matrices and algebraic manipulations.

Some references:

[Shifman 2011], M. Shifman, "Historical curiosity: How asymptotic freedom of the Yang-Mills theory could have been discovered three times before Gross, Wilczek, and Politzer, but was not" in "At the frontier of particle physics", vol. 1* 126-130

Muta, "Foundations of QCD", World Scientific (1986).
Ellis, Stirling, Webber, "QCD and Collide Physics" Cambridge University Press (1996).

R.K. Ellis: 5 lectures in QCD (online)

If you have any questions please feel free to write me:

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