Bézier curves and pion PDFs with xFitter

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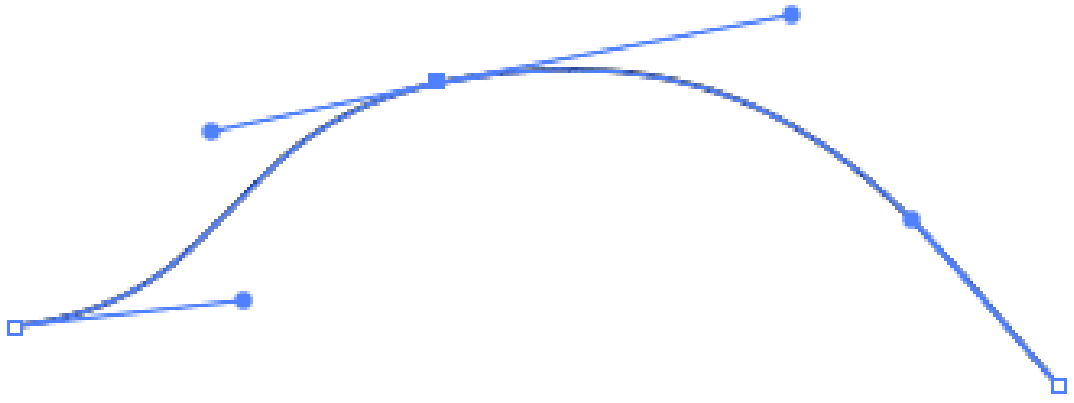
Fantômas4QCD PDF Parameterization

- **Fantômas4QCD** (Fantômas for short) is a new module implemented into xFitter.
- The parameterization we use to calculate PDFs is called a **metamorph**.
- Metamorphs are polynomial parameterizations that can approximate a variety of functional behaviors typical for PDFs and provide an alternative to neural networks.

Metamorph and Bézier Cruves

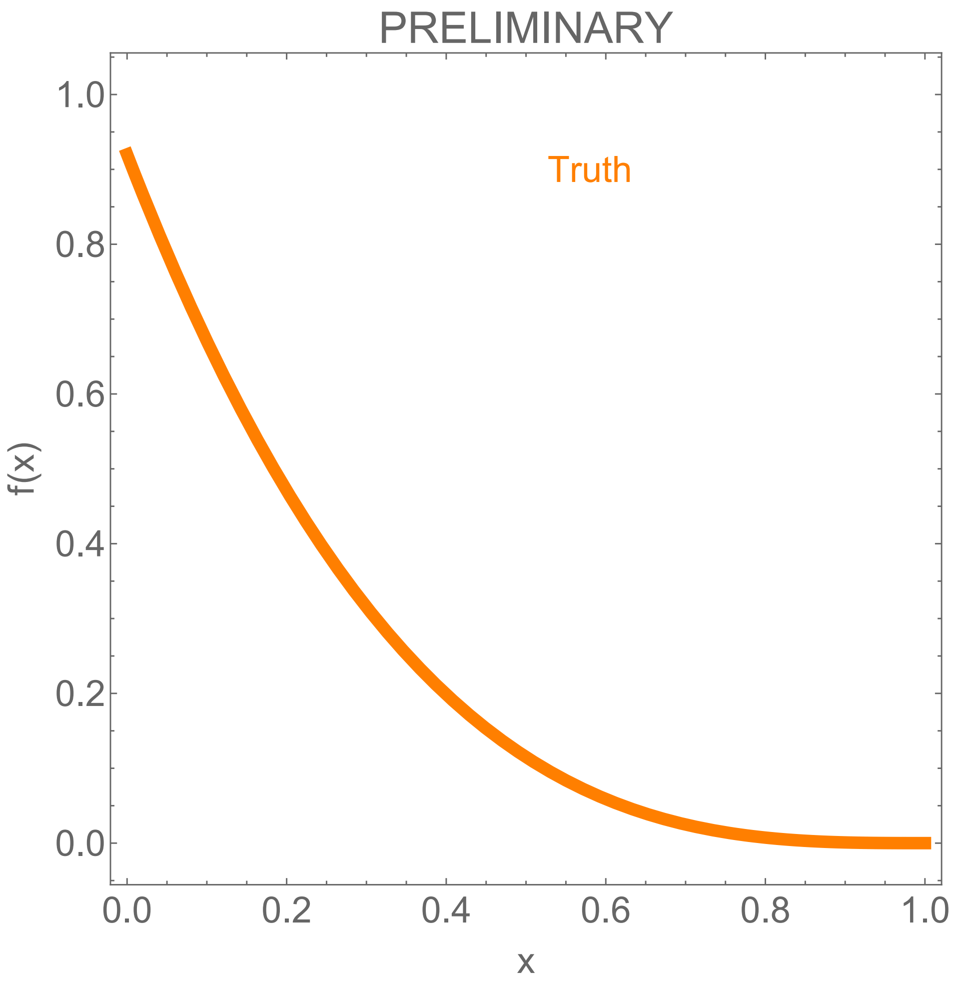
- A metamorph utilizes a Bézier curve, a polynomial of degree N_m computed from its values at **control points.**
- Flexibility of these curves allow a metamorph to approximate many PDF behaviors.



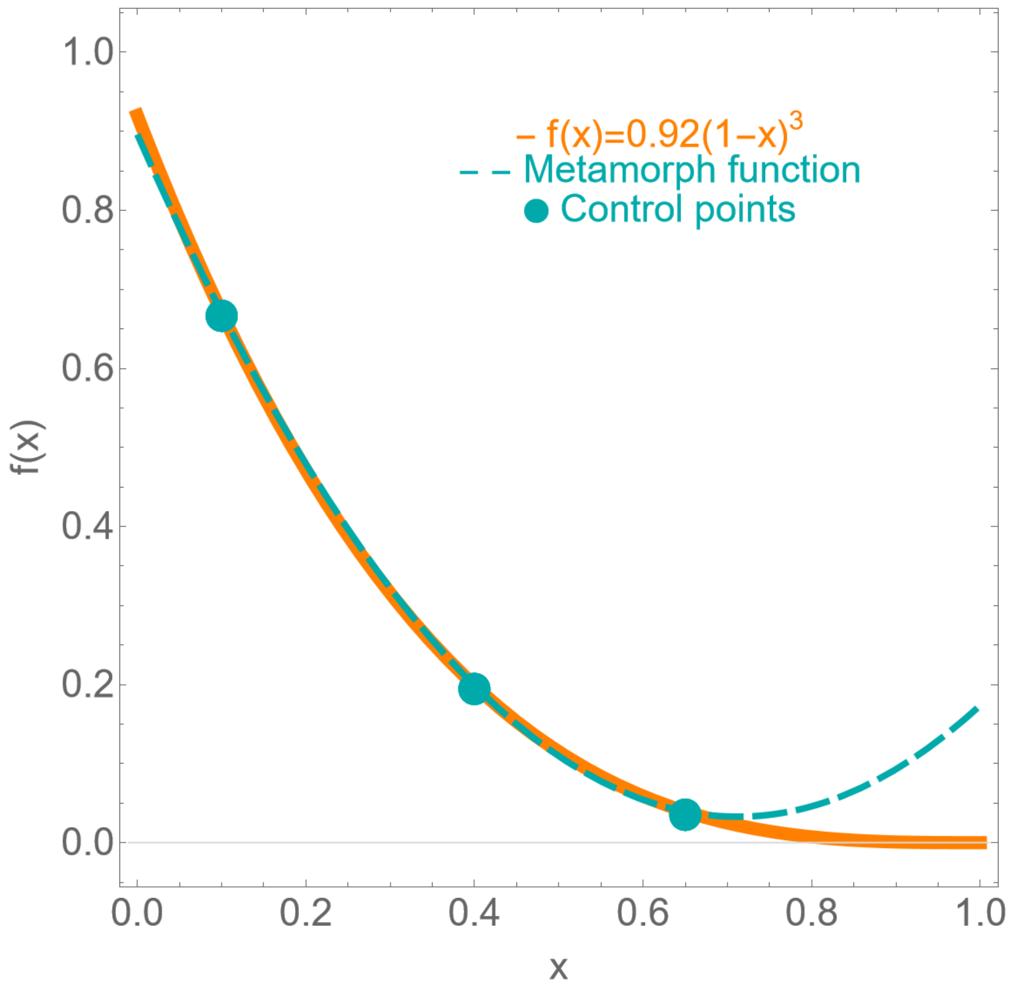
P. Nadolsky

[*A. Courtoy, P. Nadolsky*, arXiv: 2011.10078]

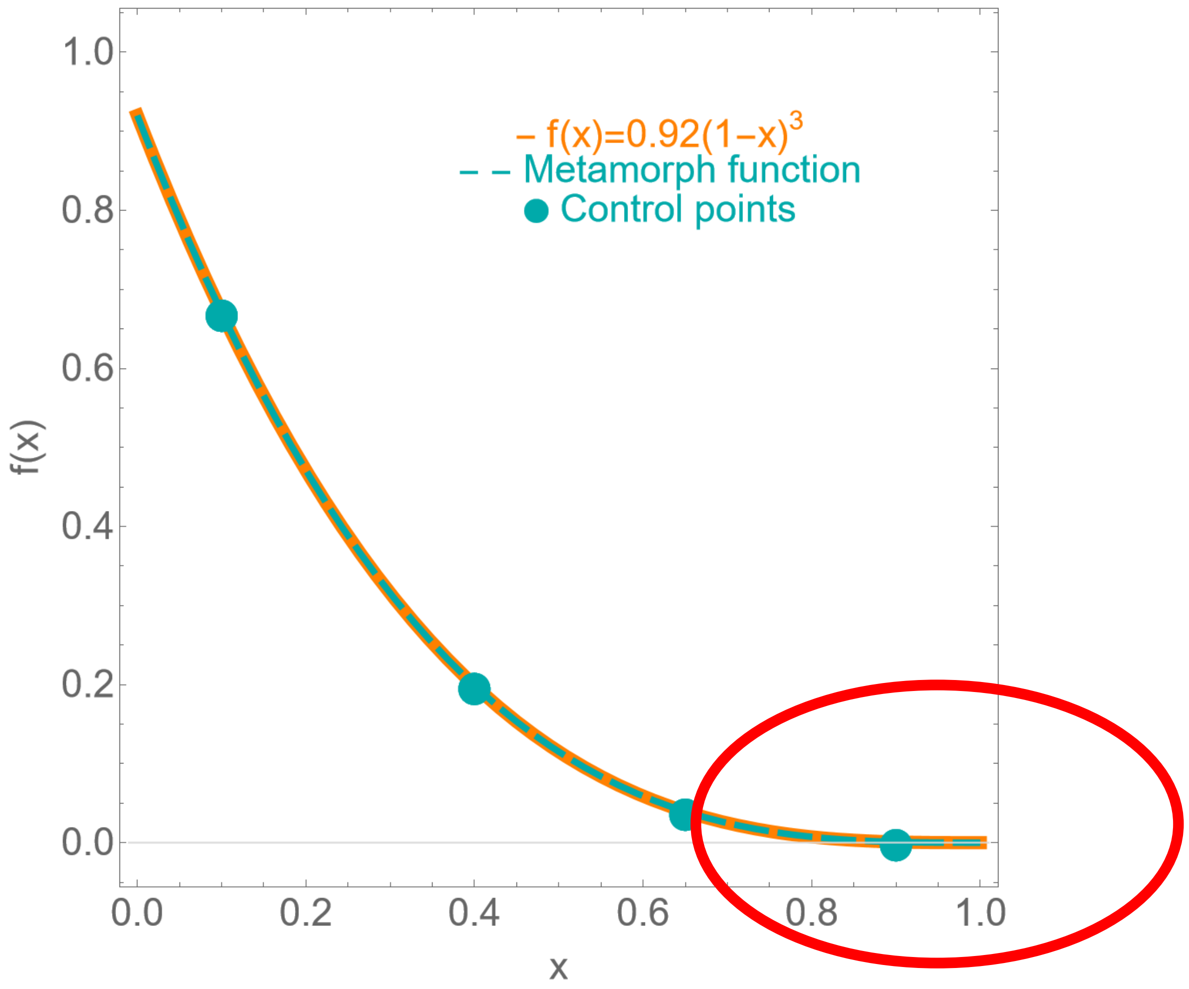
Control Points and Bézier Curves



Control Points and Bézier Curves ($N_m = 2$)



Control Points and Bézier Curves ($N_m = 3$)



Bézier Curve

$$\mathcal{B}^{(N_m)}(y) = \sum_{l=0}^{N_m} C_l B_{N_m,l}(y)$$

$$B_{N_m,l}(y) \equiv \binom{N_m}{l} y^l (1-y)^{N_m - l}$$

$$\Rightarrow \mathcal{B} = \boldsymbol{T} \cdot \boldsymbol{M} \cdot \boldsymbol{C}$$

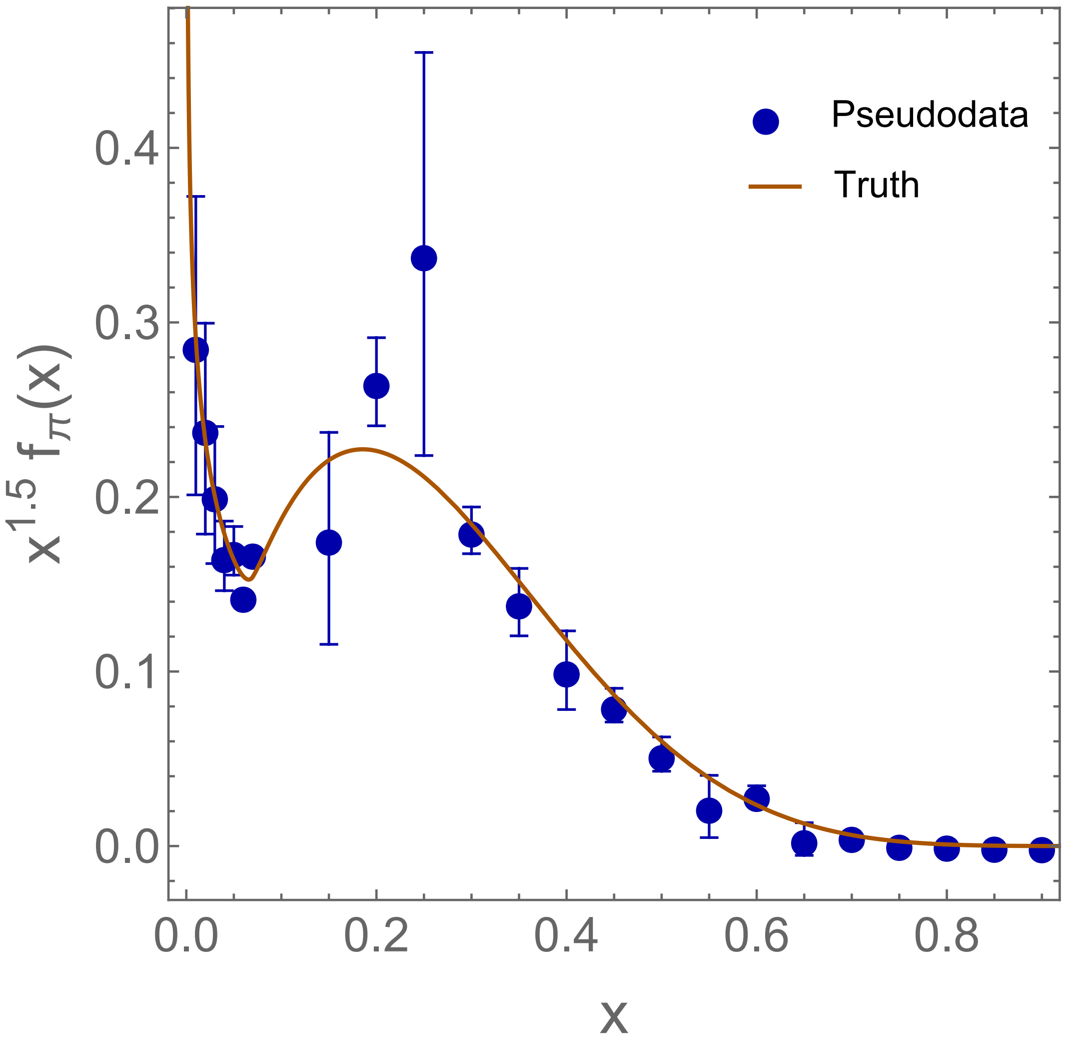
$$\text{or } \boldsymbol{C} = \boldsymbol{M}^{-1} \cdot \boldsymbol{T}^{-1} \cdot \boldsymbol{P}$$

- $\mathcal{B}^{(N_m)}(y)$: Bézier function of N_m^{th}-degree.
- $\boldsymbol{C}$: $N_m + 1$ vector containing Bézier coefficients.
- $B_{N_m,l}(y)$: Bernstein basis polynomial.
- $\boldsymbol{M}$: A fixed $N_m + 1 \times N_m + 1$ matrix containing binomial coefficients. Determined by N_m.
- $\boldsymbol{T}$: A fixed $N_m + 1 \times N_m + 1$ matrix. Determined by the positions of control points.
- $\boldsymbol{P}$: $N_m + 1$ vector containing the values at the control points.

G. Farin (2001)
Kamermans, Mike Pomax: https://pomax.github.io/bezierinfo

Performing Fits with a Metamorph

- Pseudodata are constructed by Gaussian fluctuations around the "truth" function.
- The metamorph function is fitted to the pseudodata.



Performing Fits with a Metamorph

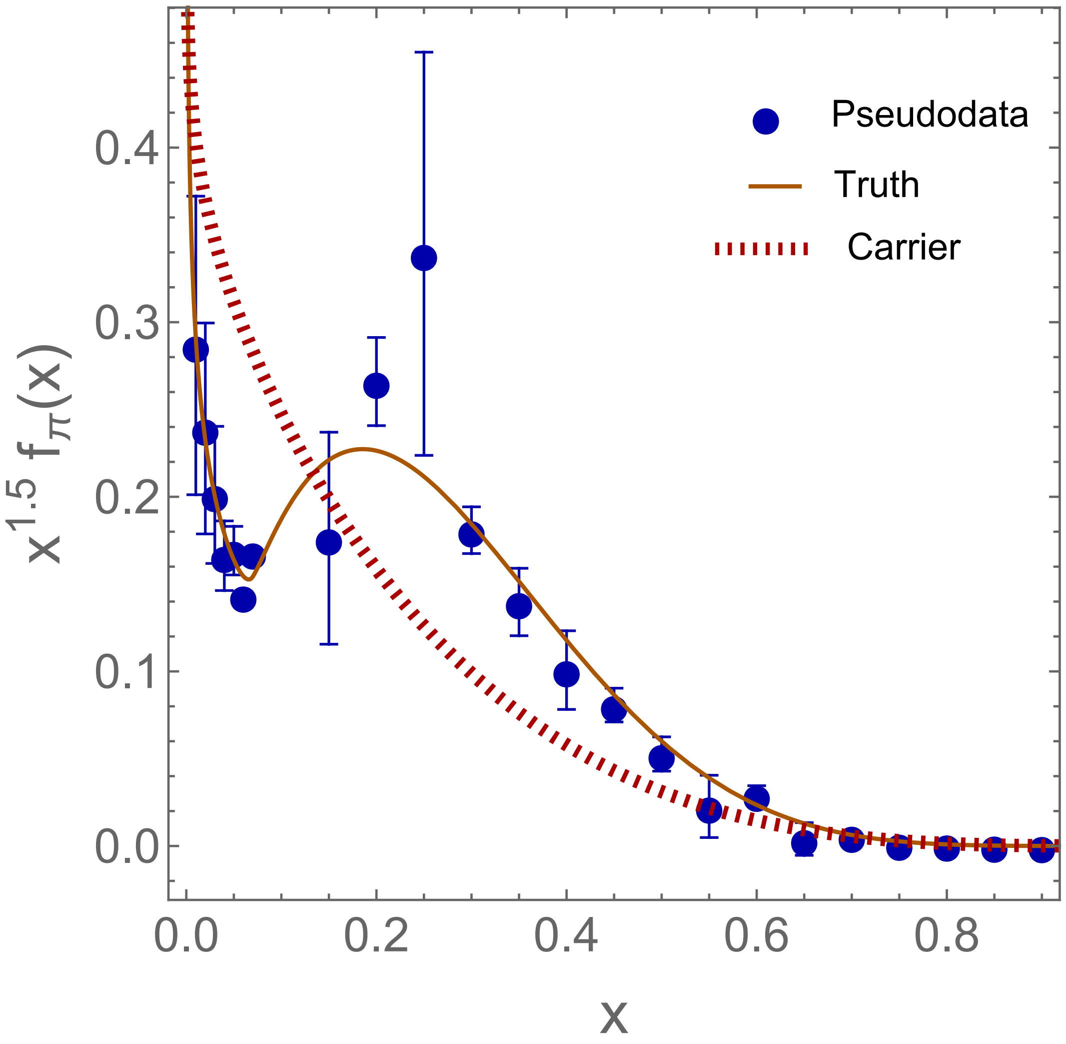
- The functional form of the Fantômas4QCD parameterization is

$$xf(x, Q_0^2) = f_{\text{Carrier}}(x) * f_{\text{Modulator}}(x^{\alpha_x})$$

where

$$f_{\text{Carrier}}(x) \equiv A_f x^{B_f} (1-x)^{C_f}.$$

- The Carrier specifies asymptotic limits of $xf(x, Q_0^2)$ at $x \to 0$ or 1.



Performing Fits with a metamorph

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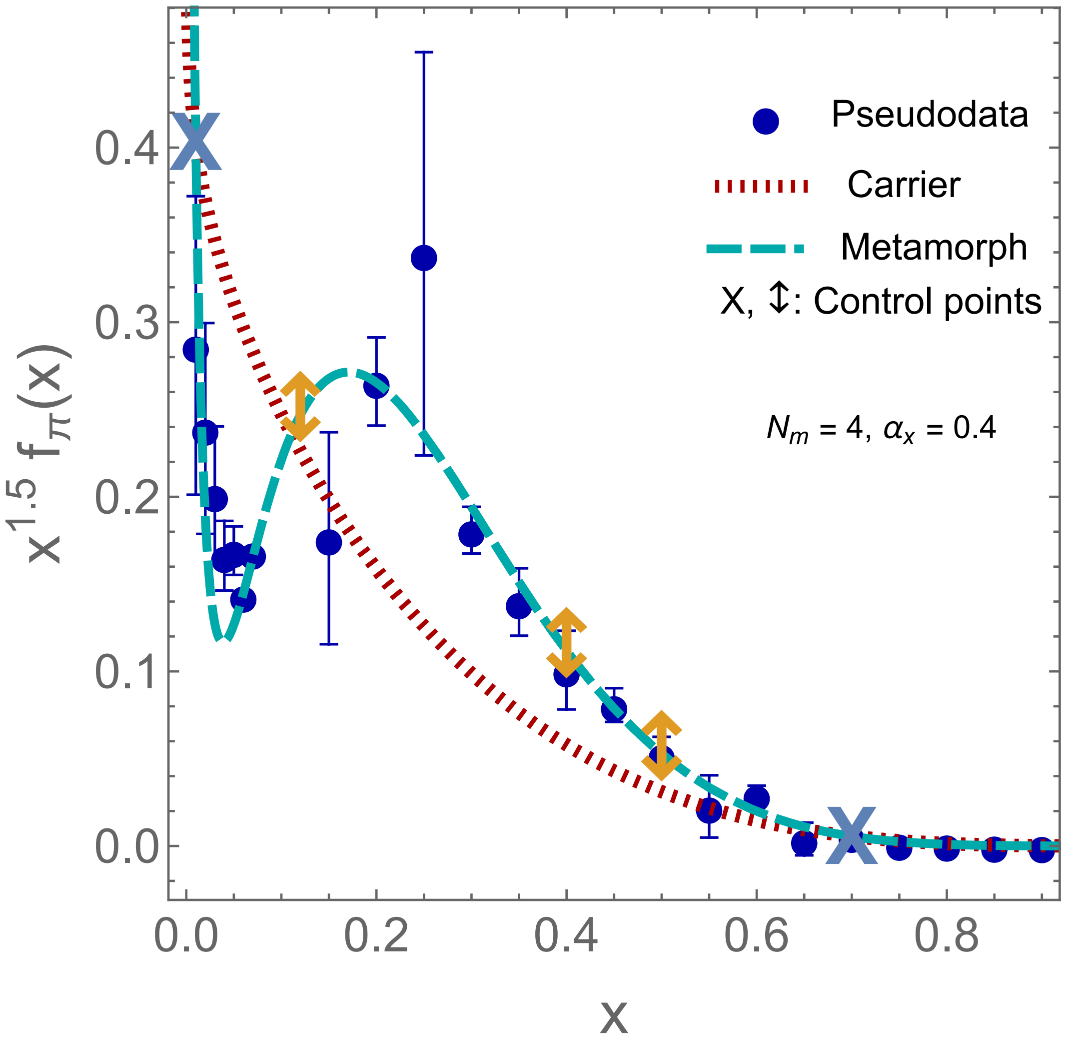
$$xf(x, Q_0^2) = f_{\text{Carrier}}(x) * f_{\text{Modulator}}(x^{\alpha_x})$$

where we choose

$$f_{\text{Modulator}}(x^{\alpha_x}) = \mathcal{B}^{(N_m)}(x^{\alpha_x}).$$

- The Modulator modifies $xf(x, Q_0^2)$ at $0 < x < 1$. α_x is an x-stretching power between 0 and 1.



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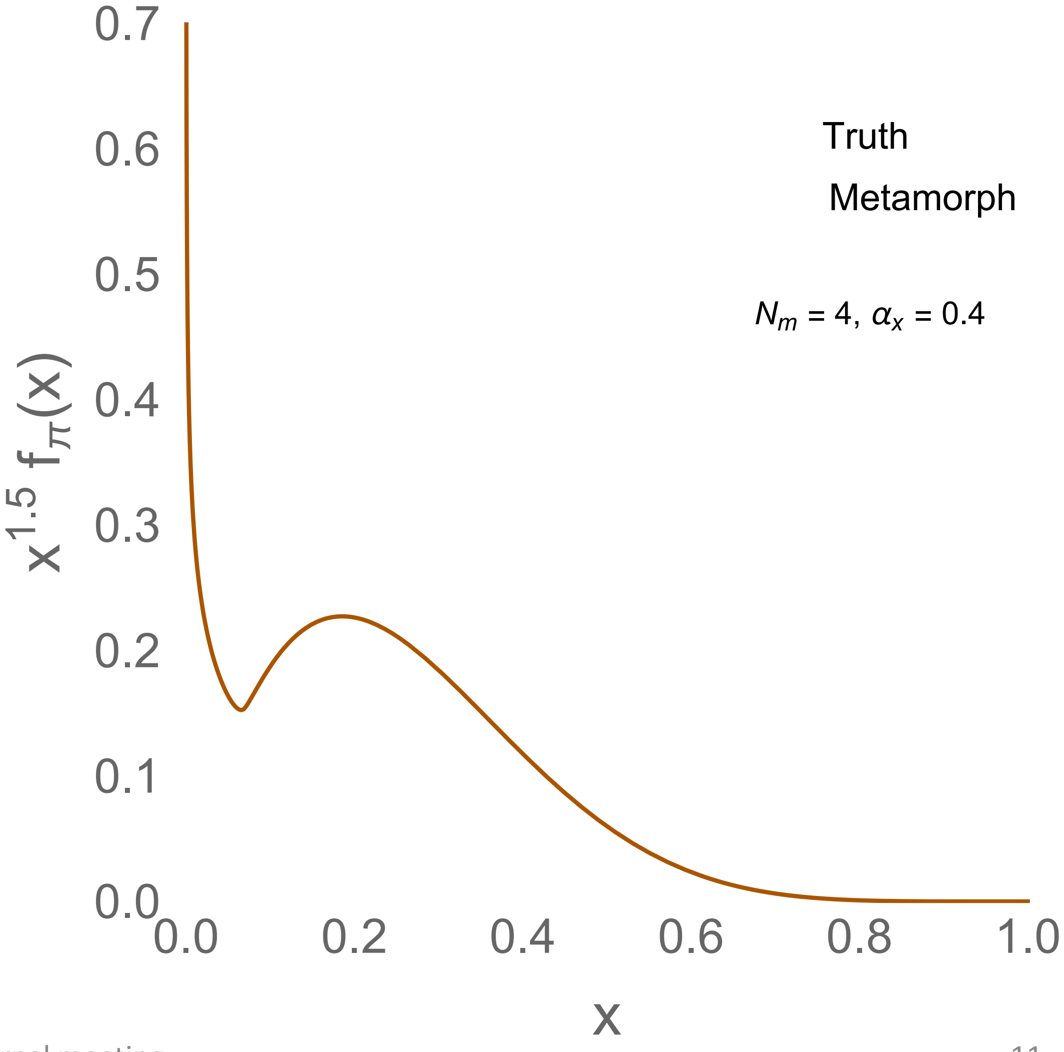


Performing Fits in xFitter

- The carrier can be fixed or free to vary within xFitter.
- Fixing the carrier requires an $N_m = 0$ fit to find the parameters to base the fit.
- A free carrier allows for further flexibility to search for the absolute best fit with no constraints.
- The control points are distributed by the user.
  - At a fixed control point, the modulator is constant. Fixed control points are used e.g. to reproduce the asymptotic power laws at x-> 0 or x->1.

Implementing Fantômas into xFitter

- Written in C++.
- Implemented in xFitter-2.2.0 Future Freeze
- Implementation was easy thanks to the streamlined structure of this version.

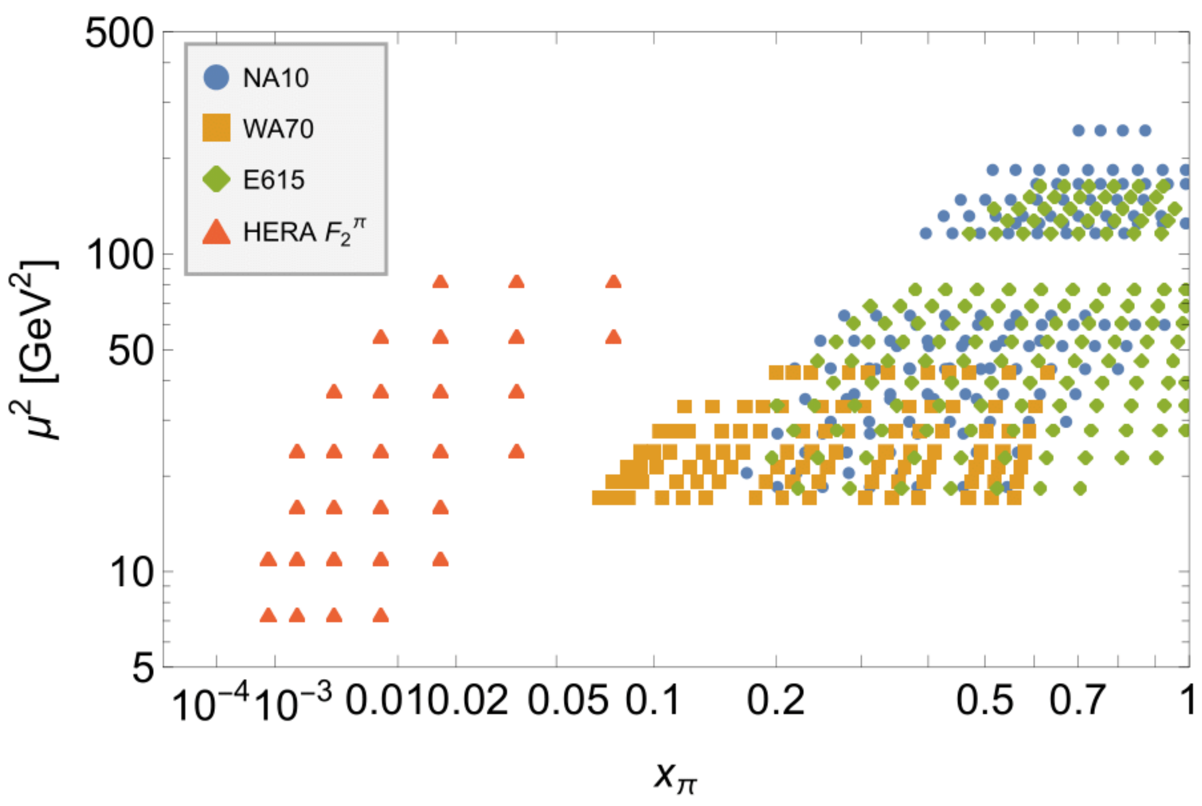
```
[lkotz@login03 pdfparams]$ ls
ABMPgluon  ABMPsea  ABMPvalence  Expression  Factor  Fantomas  HERAPDF  NegativeGluon  Normalized  Normalized_exp  PolySqrt
[lkotz@login03 pdfparams]$ ls Fantomas/
adxmoment.h     fantomas.cc    fantomas.h~             Fantomas_PdfParam.h  LUPinverse.h              metamorph.h
cl2DArray.h     fantomas.cc~   Fantomas_PdfParam.cc    integral.h           MetamorphCollection.h     metamorph.h~
CMakeLists.txt  fantomas.h     Fantomas_PdfParam.cc~   isNumber.h           MetamorphCollection.h~
```

Pion PDF Uncertainties

- The pion structure is related to properties of QCD at low energy. Non-perturbative methods can be used to describe it in terms of quarks and gluons.
- On the phenomenological point of view, the pion PDF has been extracted from data.
- Pion-nucleus Drell-Yan data is already implemented in xFitter. We use the pion PDF fit by Novikov et al., arXiv:2002.02902 as the baseline.
- The modulator is chosen to be $1 + \mathcal{B}^{(N_m)}(x)$.

Datapoints used in fits

- NA10 & E615: Covers the main kinematic region of $x > 0.2$ and $Q^2 > 10\ \mathrm{GeV}^2$. Constrains valence very well.
- WA70: Provides some sensitivity to the gluon PDFs that the DY data could not provide.
- HERAF_2^{π}: Constrains the sea and gluon PDFs at low x.



Drell-Yan only Analysis with $N_m = 0$

- The normalization differs between the two parameterizations.
  - xFitter: $xf(x) = \frac{A_f}{\mathrm{B}(B_f+1, C_f+1)} x^{B_f}(1-x)^{C_f}$
  - Fantômas: $xf_{\text{meta}}(x) = A'_f x^{B_f+\delta B_f}(1-x)^{C_f+\delta C_f}$
- The Fantômas normalization is an independent parameter that is either fitted or computed using the sum rules.
- Free B_s is compatible with zero (0.5 ± 0.8). Fix $Bs = 0.47$ to achieve stable Hessian matrix diagonalization. [*xFitter Developers' team (2020),* arXiv: 2002.02902].

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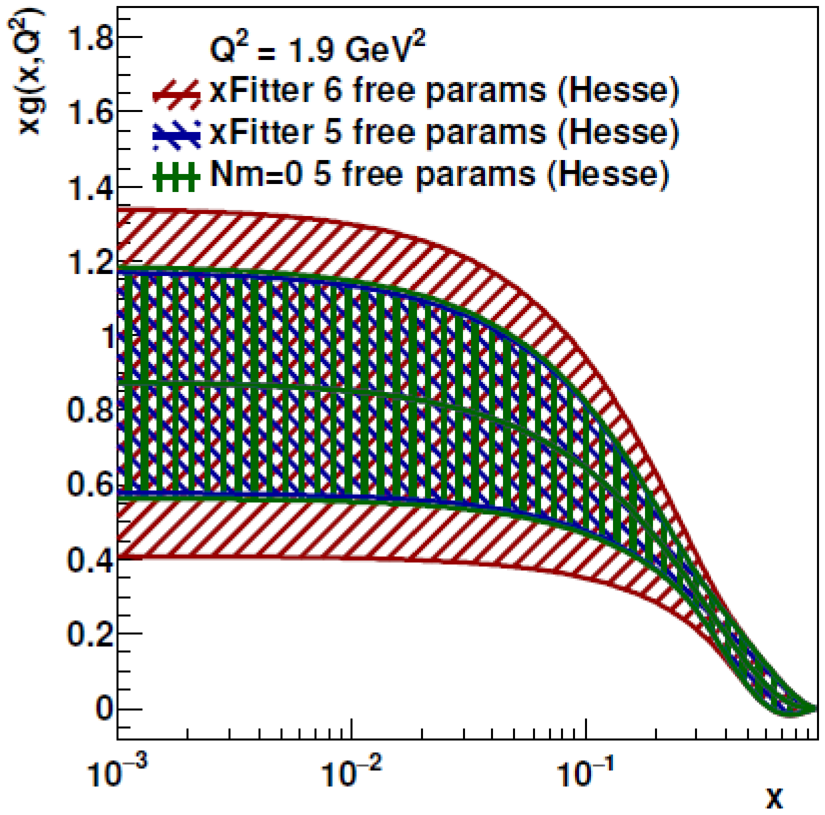
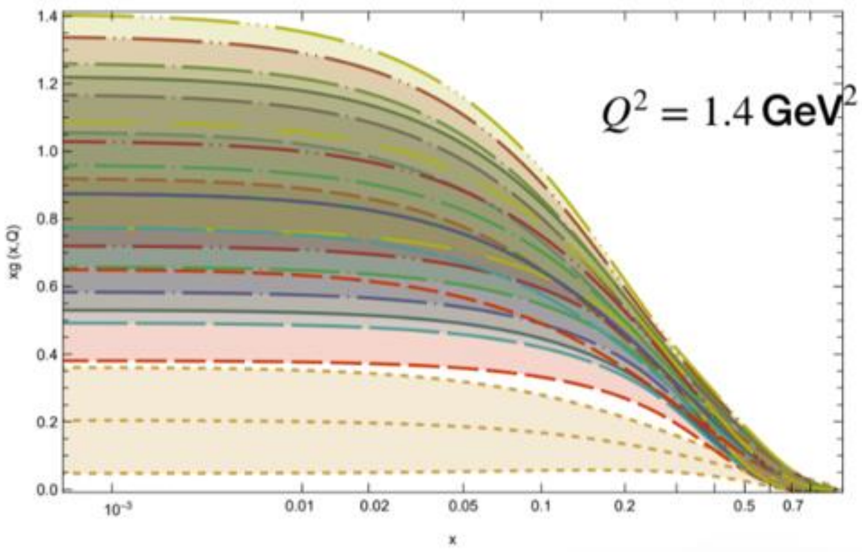


Drell-Yan only Analysis

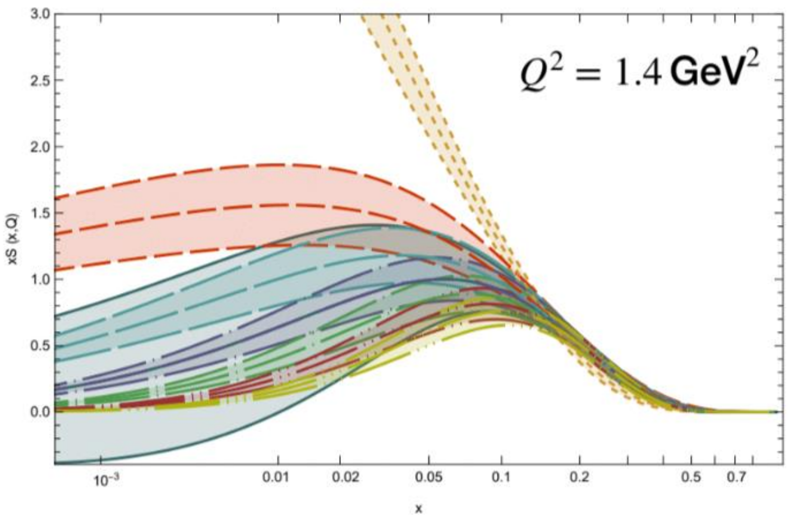
- The sea and gluon PDFs are not well determined even with the Nm=0 parametrization. Need to add data sets to resolve the gluon -- quark sea degeneracy at $x < 0.1$.

PRELIMINARY

π gluon PDF ($N_m = 0$, fix carrier)

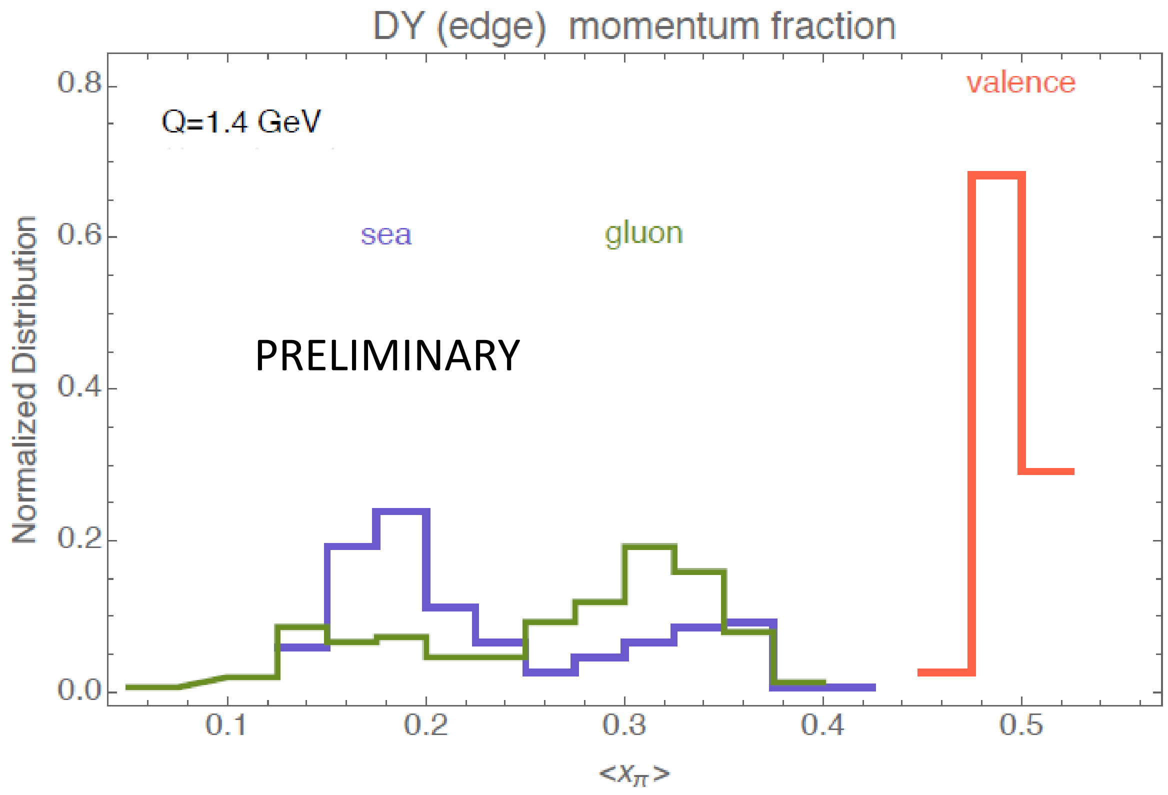


π sea PDF ($N_m = 0$, fix carrier)

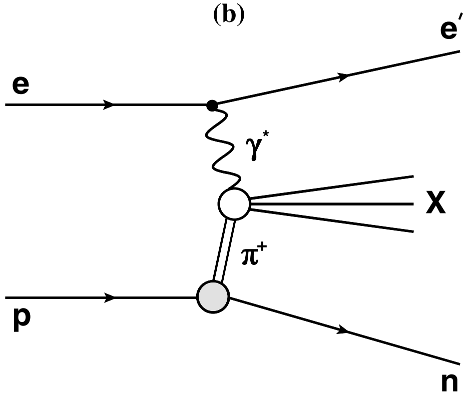


Drell-Yan only Analysis – Momentum Fractions

- The momentum fractions for sea and gluon span a broad range.



Leading-neutron data in DIS

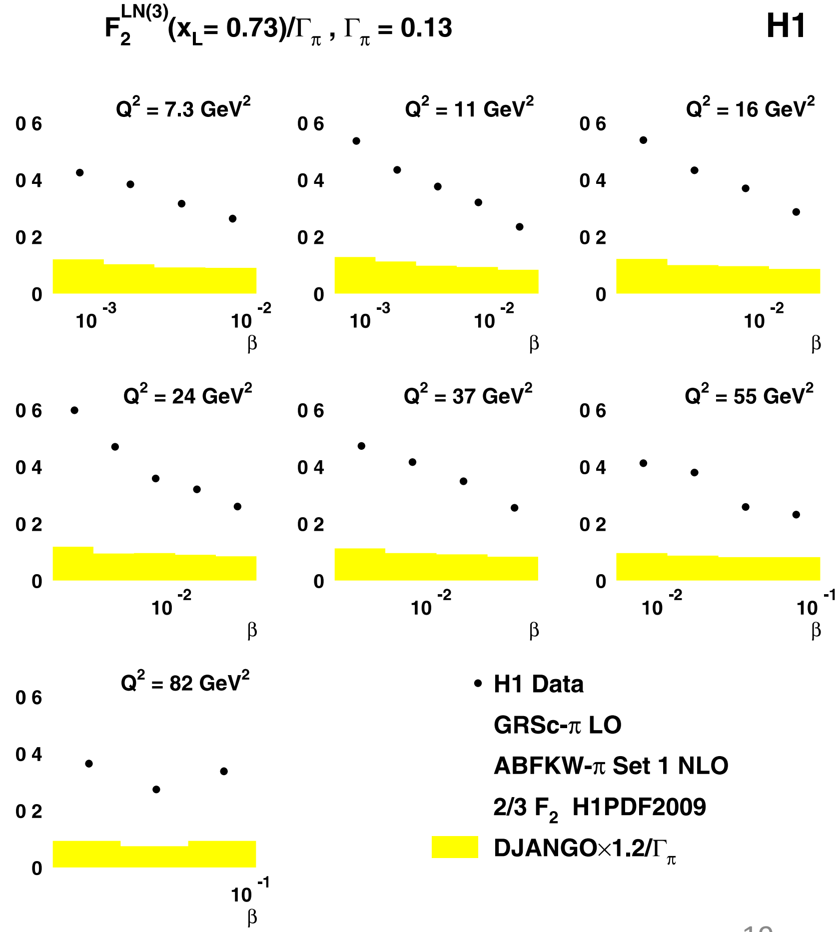


- H1 analysis [Aaron et al, Eur. Phys. J. C, 68, 2010] identifies the single-pion production to be valid around the range $0.68 < x_L < 0.77$ at low p_T of order $p_T = 0.2$ GeV -- LN production could be used to extract the pion PDF in that range.

$$F_2^{LN(3)}(Q^2, x, x_L) \quad = \quad 2\, f_{\pi N}(1 - x_L)\, F_2^{\pi}(x_\pi, Q^2)$$

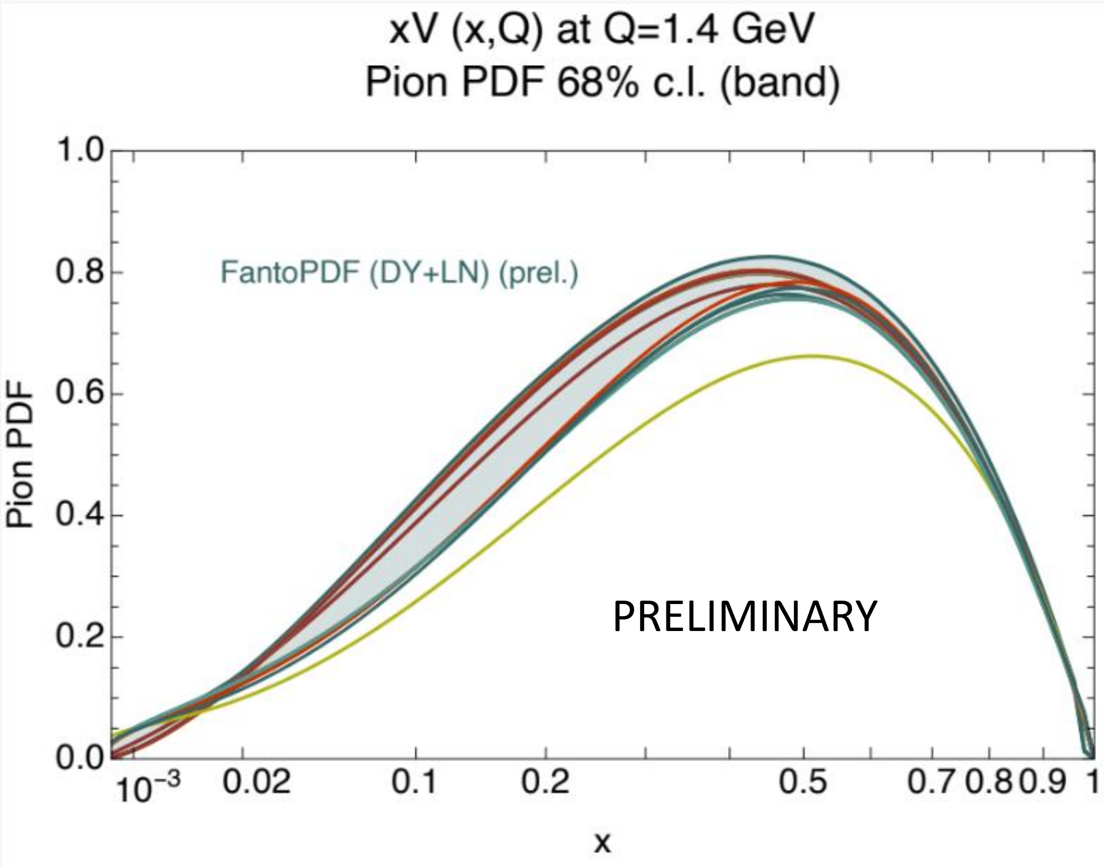
- We implement $F_2^{\pi}(x_\pi, Q^2)$ according to the flux prescription based on the light-cone representation of H.Holtmann et al., Phys.Lett.B338, 363(1994)

$$f_{\pi N}(x_L = 0.73) \simeq 0.13 \pm 0.04$$



Fantômas Pion PDFs (DY+LN)

- We performed several fits with varying N_m values and control points.
- The resulting error sets were combined using the META PDF method (J. Gao, P. Nadolsky, JHEP 07 2014).
- All fits were performed at the NLO in α_s.
- Valence is well determined at large x.



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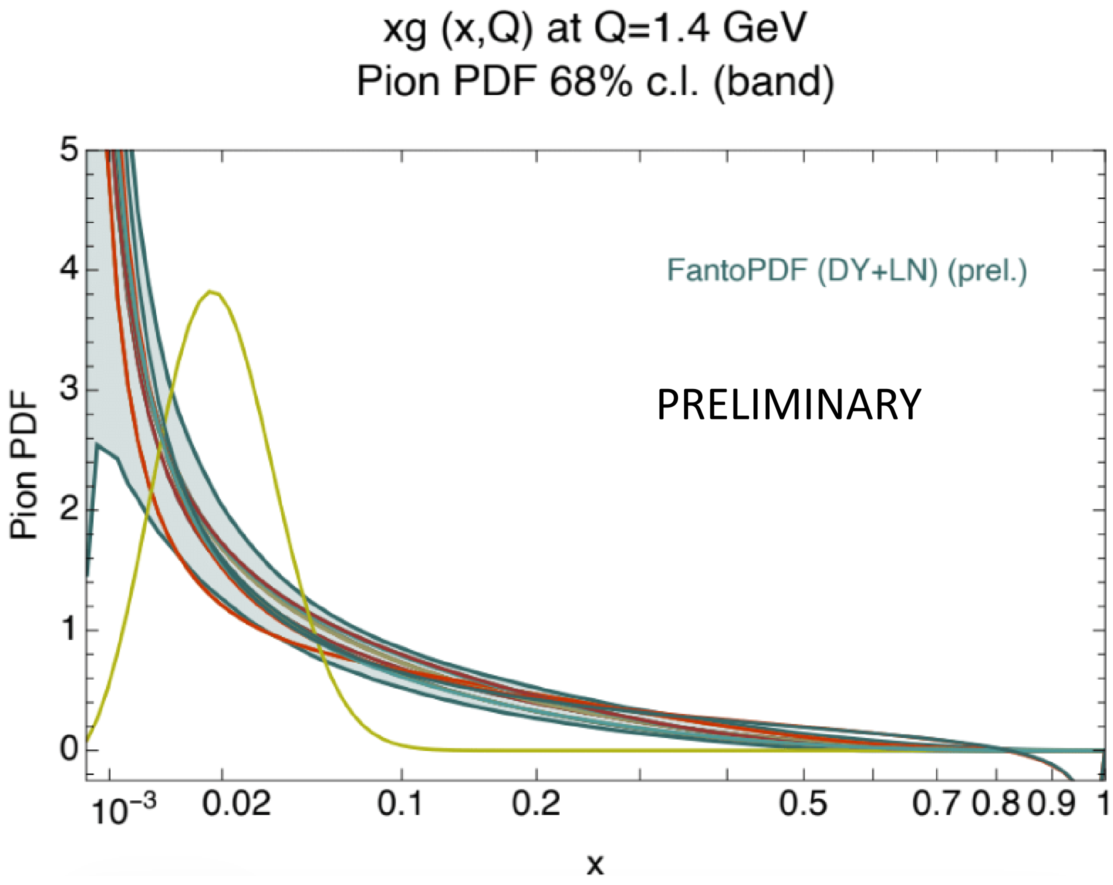
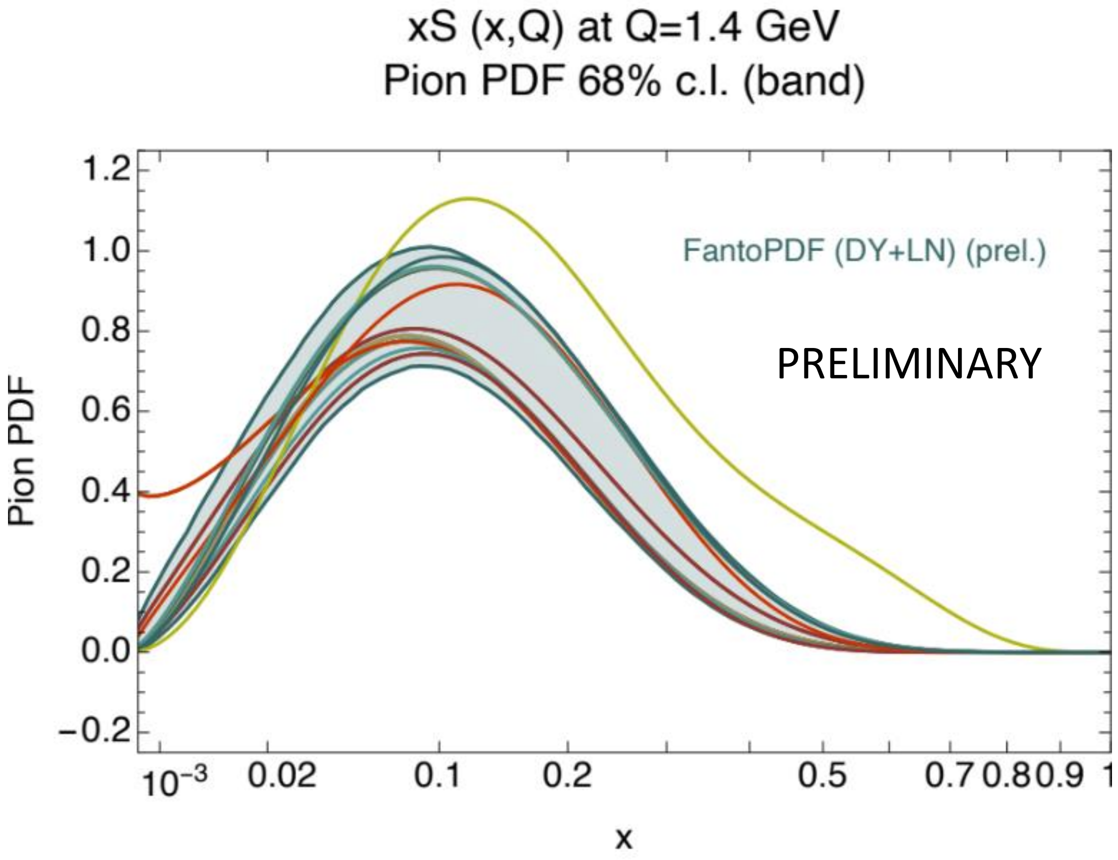
- We performed several fits with varying N_m values and control points.
- The resulting error sets were combined using the META PDF method (J. Gao, P. Nadolsky, JHEP 07 2014).
- All fits were performed at the NLO in α_s.
- The gluon PDF is not well determined at $x < 0.2$ with just DY data.
- Inclusion of the LN data helps the lack of data from DY for $x < 0.05$.



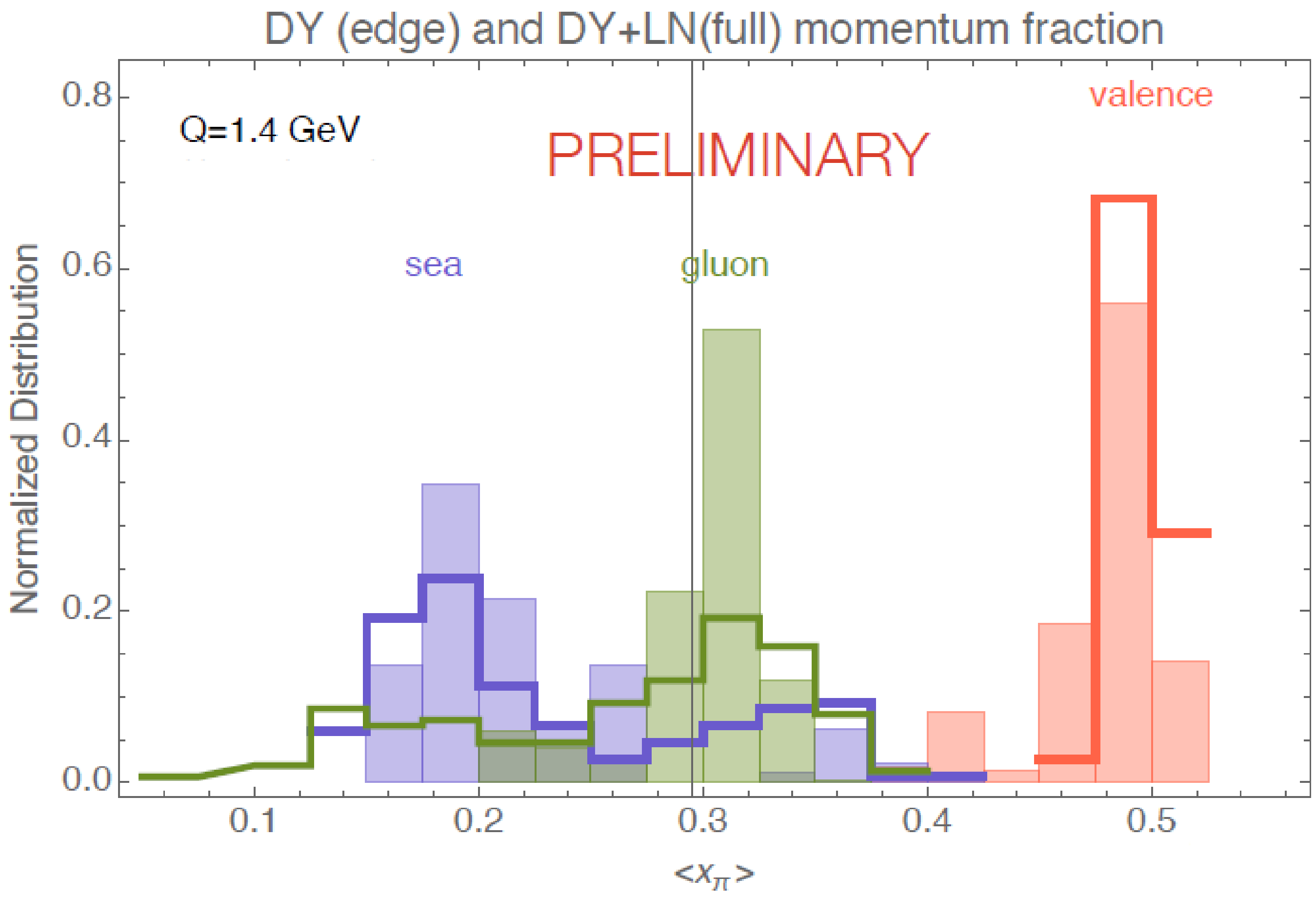
Fantômas Pion PDFs (DY+LN)

- We performed several fits with varying N_m values and control points.
- The resulting error sets were combined using the META PDF method (J. Gao, P. Nadolsky, JHEP 07 2014).
- All fits were performed at the NLO in α_s.
- Similarly, the sea is not well determined at $x < 0.2$ with just DY data.
- Inclusion of the LN data helps the lack of data from DY for $x < 0.05$.



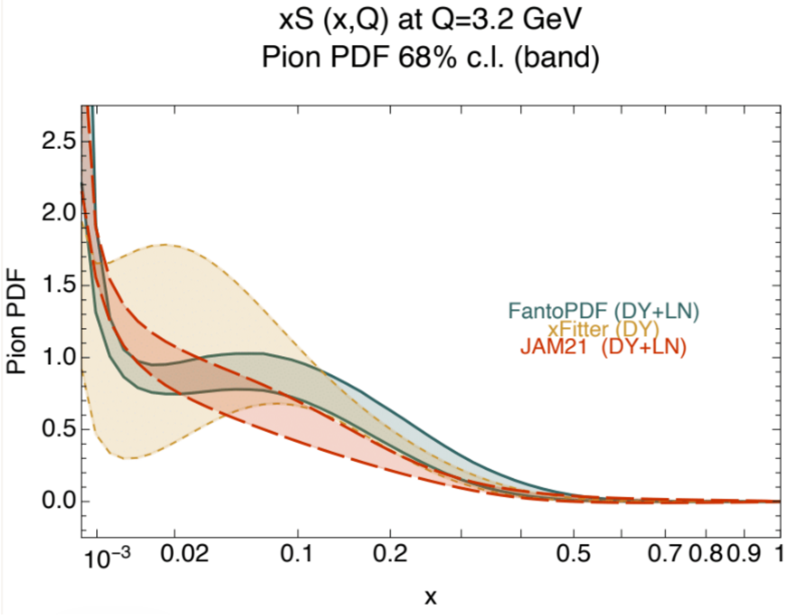
Fantômas Pion PDFs - Momentum Fraction (DY+LN)

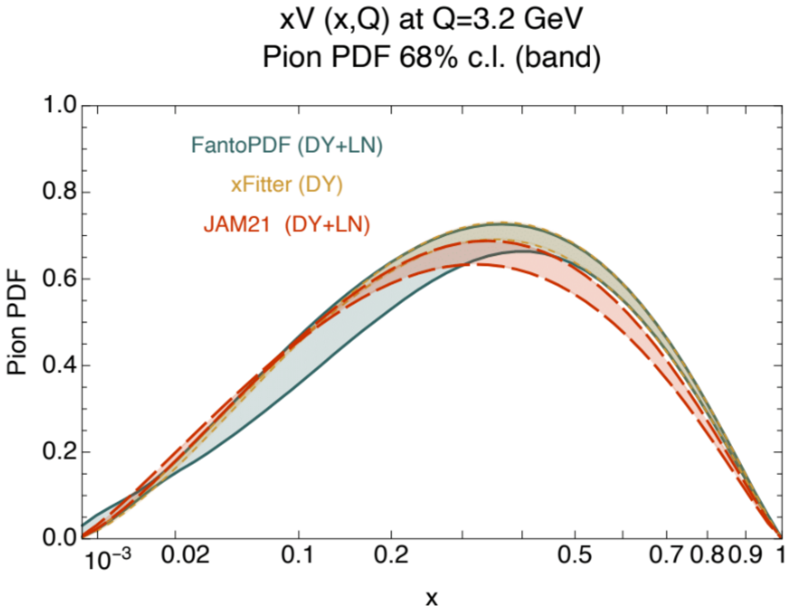
- Momentum fractions for sea and gluon narrow with the inclusion of LN data.
- The inclusion of the LN data does not change the momentum fractions much.
- Momentum fractions agree with the quoted values from xFitter and JAM21 [*JAM Collaboration (2021),* arXiv: 2108.05822].



Comparison with other pion PDFs

PRELIMINARY





JAM Collaboration (2021),
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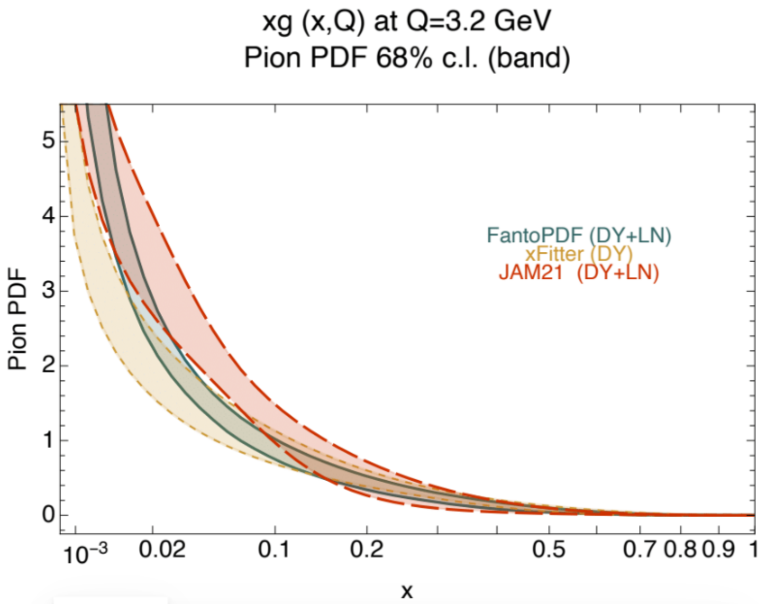
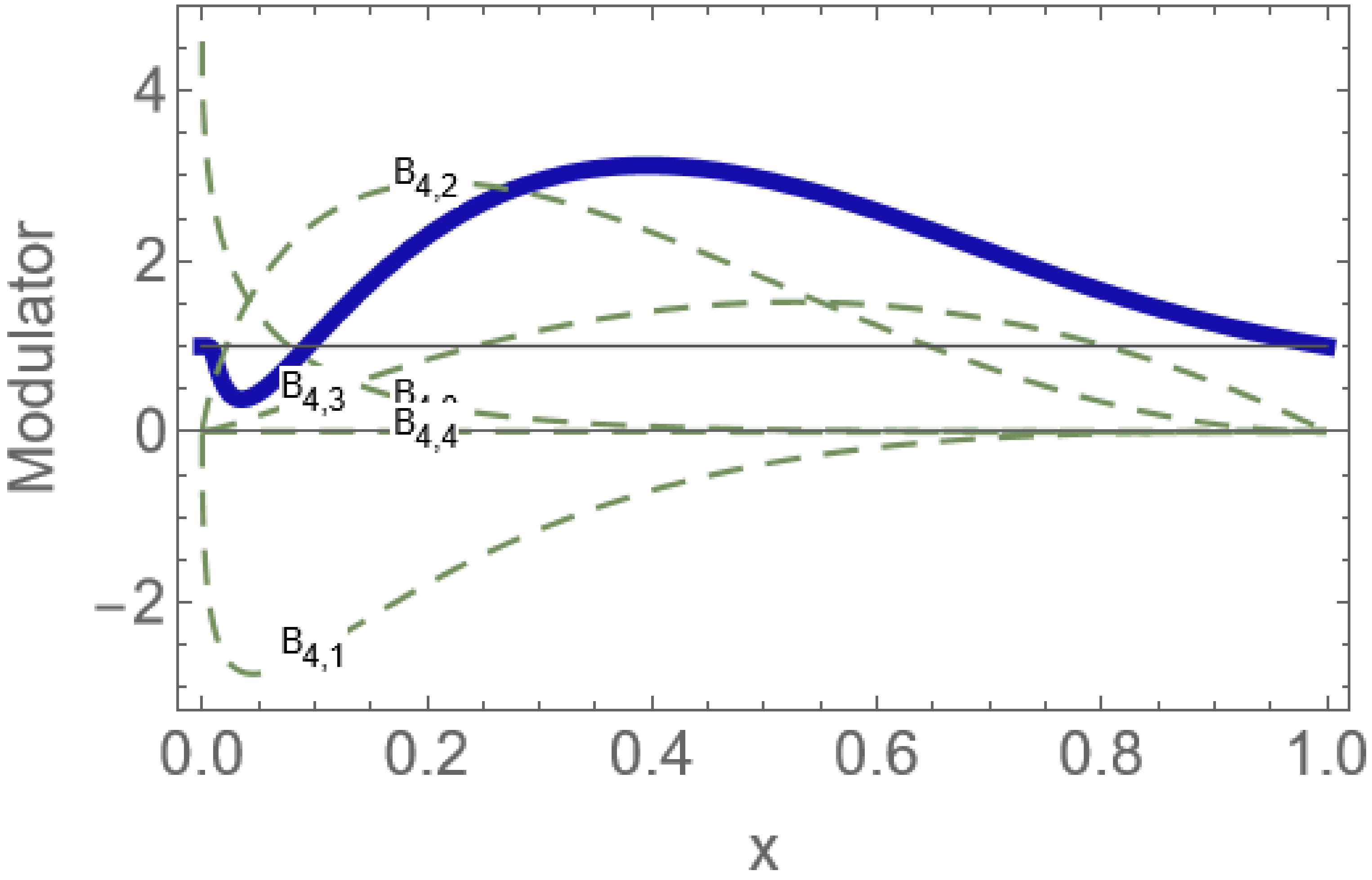


Conclusion

- Bézier curves are polynomial interpolations that approximate a variety of functional behaviors typical for PDFs.
- The flexibility of Bézier curves allow metamorph to take on a variety of functional forms, allowing them to parameterize numerous distribution functions.
- Because Bézier curves are a less rigid parameterization, it allow them to extensively explore PDF uncertainties at both high and low-x regions.

Extra Slides

Modulator



Parameter Table for Toy Fit

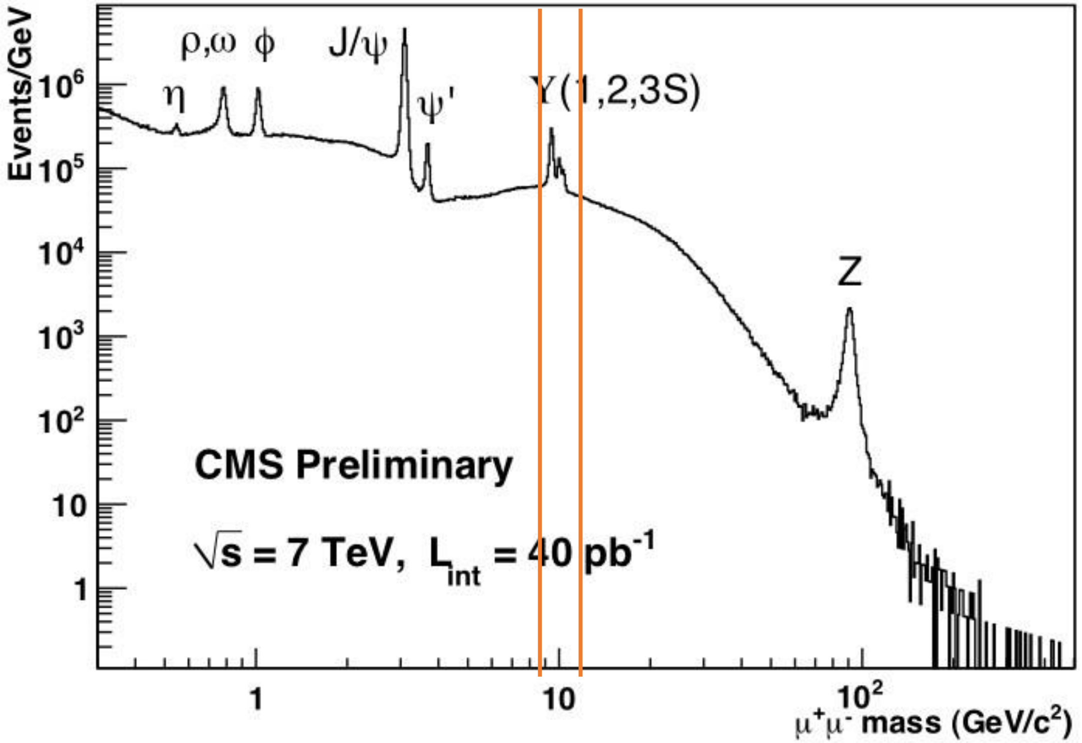
	Estimate	Standard Error	t-Statistic	P-Value
da0	−0.705153	0.156905	−4.49414	0.000319731
da1	−0.575125	0.140292	−4.0995	0.000747529
da2	0.321579	0.439633	0.731472	0.474454
db1	0.092065	0.244706	0.376228	0.711402
db2	0.930947	0.326039	2.85532	0.0109504
db3	0.698893	0.240317	2.90821	0.0097906

$$f_{\mathrm{Carrier}}(x) = (1 + \mathrm{da0})x^{0+\mathrm{da1}}(1-x)^{3+\mathrm{da2}}$$

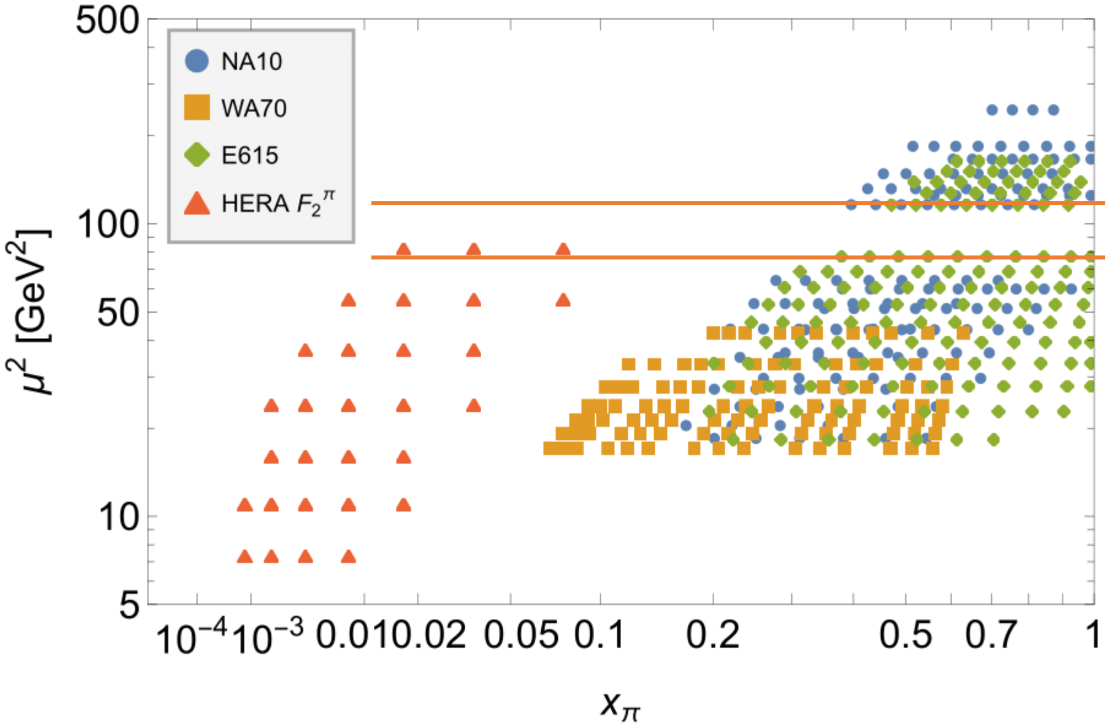
$$P_i = f_{\mathrm{Carrier}}(x_i) + \mathrm{dbi}$$

$$\chi^2 = 36.6$$

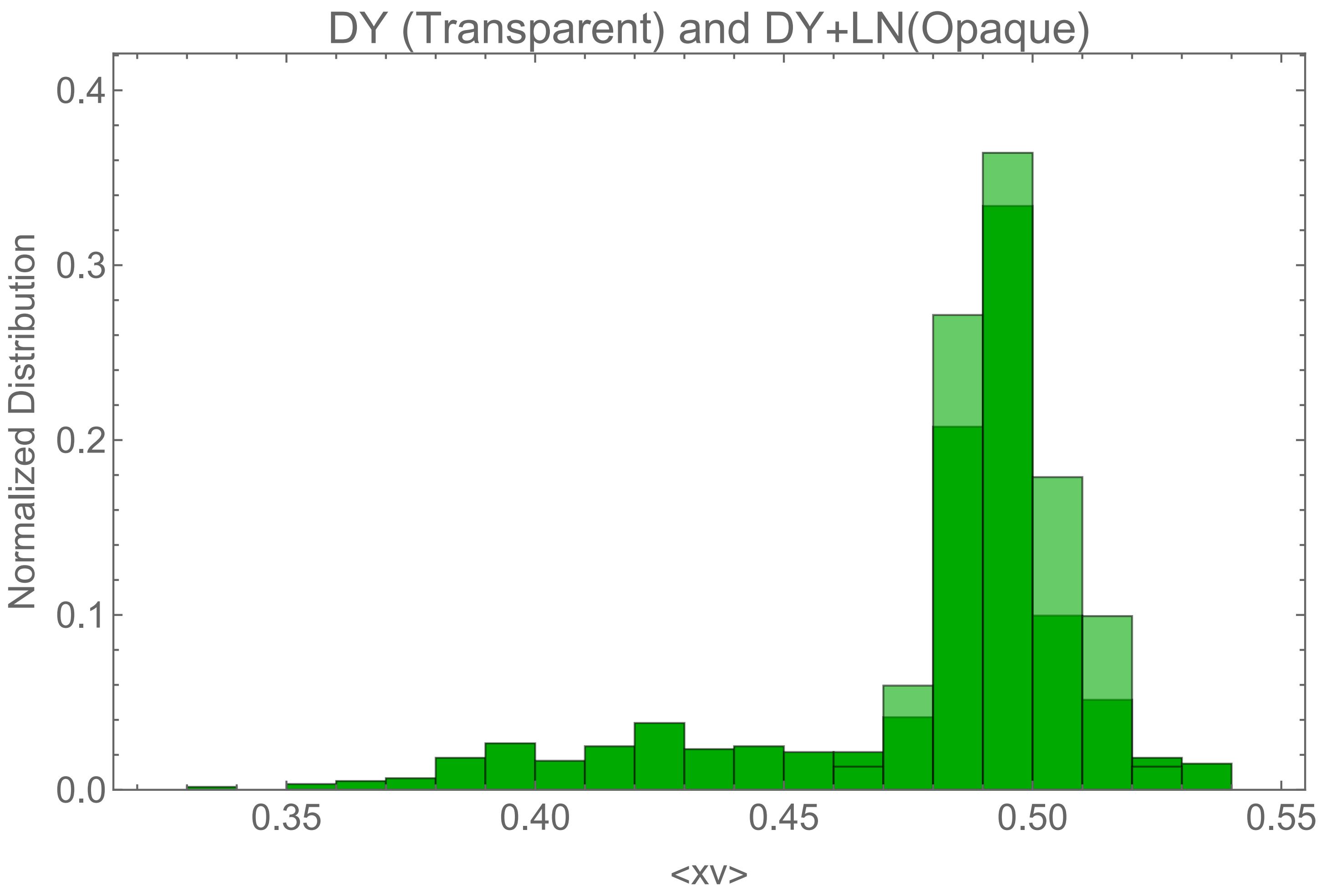
Exclusion of DY data



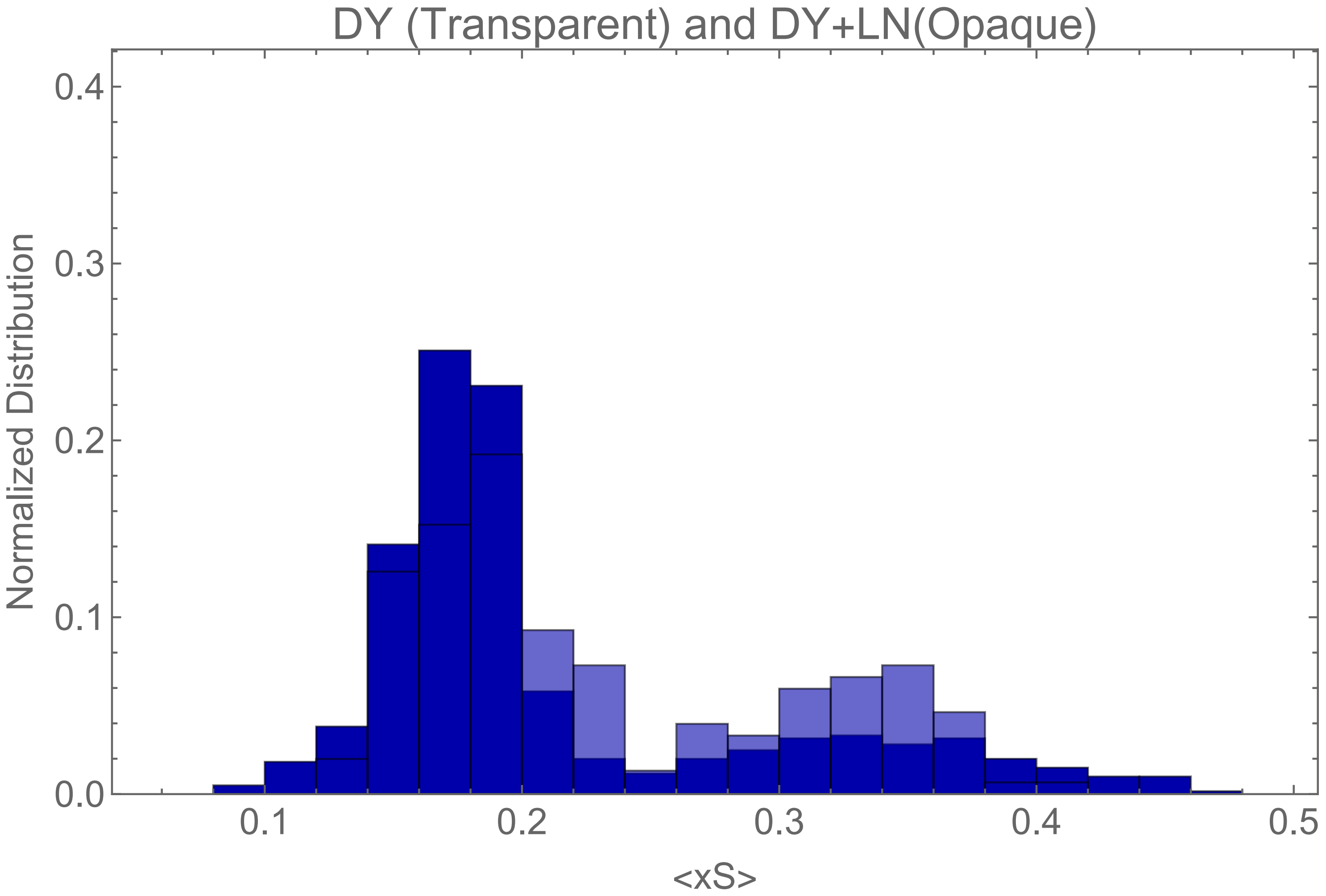
R. Ruiz, Quantum Diaries



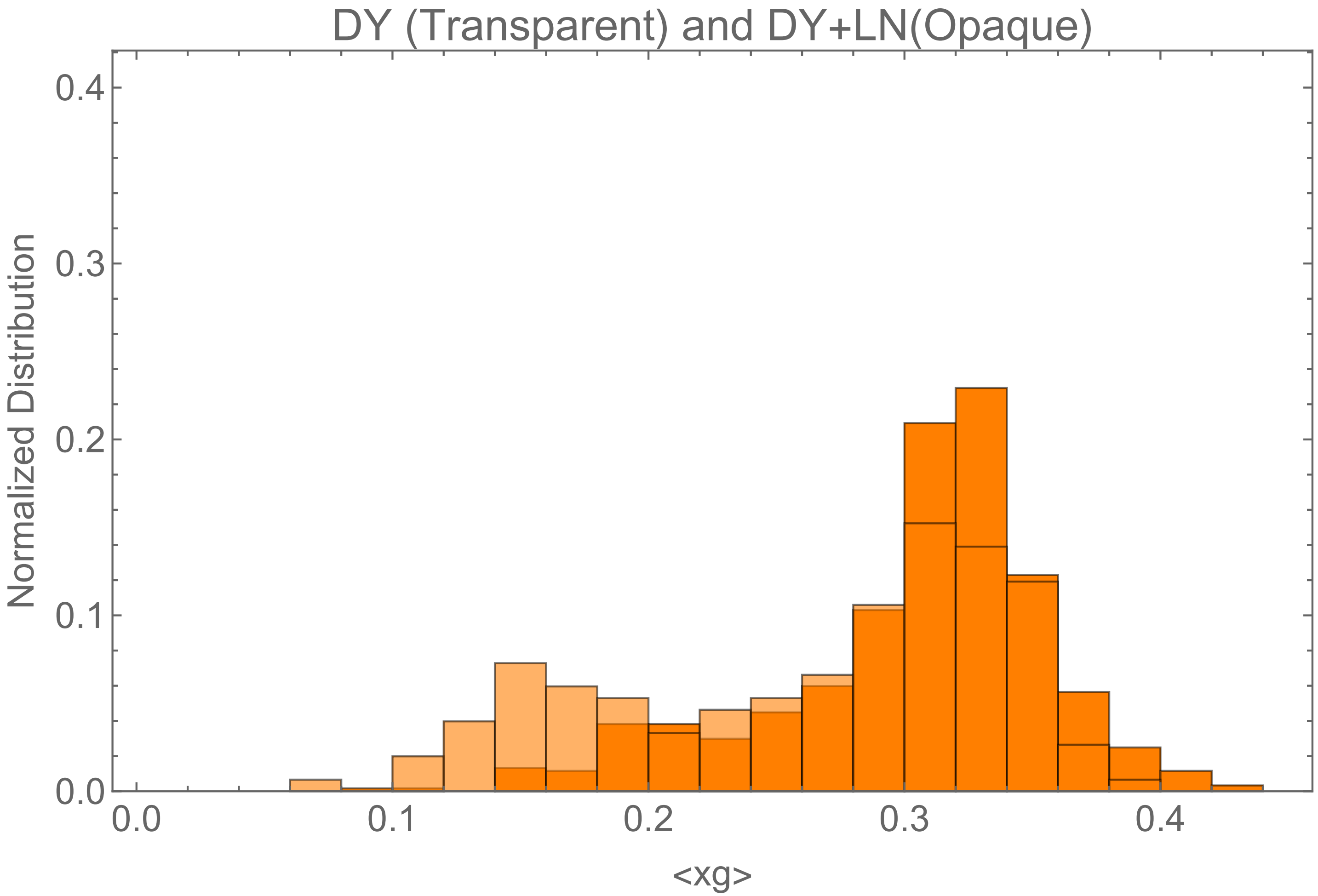
DY and DY+LN Valence momentum distribution (PRELIMINARY)



DY and DY+LN sea momentum distribution (PRELIMINARY)



DY and DY+LN Gluon momentum distribution (PRELIMINARY)



Enforcing Positivity

- A more general expression for metamorph is

$$xf(x) = f_{\text{Carrier}}(x) * F(\mathcal{B}^{(N_m)}(y)).$$

- where $F(x)$ is some function that is always positive.
  - i.e. $F\left(\mathcal{B}^{(N_m)}(y)\right) = e^{a*\mathcal{B}^{(N_m)}(y)}$ where a is a constant.