Beam Dynamics

in storage rings

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Types of acccelerators

Machine	Ion energy E_{kin}	Relativist factor	Ion Velocity	Field	Orbit	Rotation Frequency
	MeV	$\gamma = E/E_o$	$\beta = v/c$	B	ρ	F
Electrostatic Van de Graaf & Tandems	2 - 20	1.002-1.02	$6.5 \cdot 10^{-2}$ – $2 \cdot 10^{-1}$	--	Linear	--
Isochronous Cyclotron	10 - 600	1.01 - 1.64	0.14 - 0.79	Constant $B(\rho)$	Spiral $\approx \beta$	Constant
LINACS	50 - 800	1.05 - 1.85	0.31– 0.84	--	Linear	--
Synhro Cyclotron	100 - 1000	1.1 – 2.06	0.43 - 0.87	Constant $B(\rho)$	Spiral $\approx p$	Reduced $\sim B(\rho) / \gamma E_0$
Proton/Ion Synchrotron	200 MeV- 7 TeV	1.2 - 7400	0.56 - 1	$B \sim p$	R = const	$\sim \beta$
Collider ring	1-7 TeV	1060 – 7000	~1	$B \sim p$	R = const	$\sim \beta$
Low energy Storage ring	**1-100 keV**	**~1**	**10^{-3}- 10^{-2}**	**ESD $U_{\pm} \sim E_{kin}$**	**R=const**	**$F \sim \beta$**

Total energy = rest energy + kinetic energy

$$E = E_0 + E_{kin}$$

$$E_0 = m_0 c^2 = 938 MeV (protons)$$

ACCELERATION (DECELERATION) and FOCUSING of ions are main functions of accelerators and storage rings

- **An accelerator adds or reduces kinetic energy to charged particles, hence increases or decreasing its momentum.**
- **For that one needs an electric field $\vec{E}_{II}$ in the direction of the ion momentum $\vec{p}$ or opposite to the ion momentum $\overleftarrow{p}$**

$$\frac{d\vec{p}}{dt} = e\vec{E} \qquad (2)$$

BENDING is made by a magnetic field $\vec{B}$ perpendicular to the plane of the trajectory or by an electric field $\vec{E}_{\perp}$ perpendicular to direction of ion motion. The bending radius satisfies the relation :

mgn

$$300BR = p \cdot c = \beta\gamma A W_0 \qquad (3)$$

elfld

$$|E_0 R_0| = \frac{(\gamma+1)}{\gamma}\frac{T_{kin}}{q} \qquad (4)$$

magnetic and electric fields are also used to <u>FOCUS</u> the beam i.e. to bring back, close to the nominal axis, those particles that tend to escape.

Relativistic dynamics $T_k > W_0$

$$W = W_0 + T_k \longrightarrow mc^2 = m_0 c^2 + T_k \quad (5)$$

$$W = mc^2 \qquad W_0 = m_0 c^2$$

$$m = \gamma m_0 \longrightarrow W = \gamma W_0$$

$$\gamma^2 = \frac{1}{1-\beta^2} \qquad \beta = \frac{v}{c}$$

and

$$p = mv \qquad W^2 = W_0^2 + p^2 c^2 \quad (6)$$

Ultra-relativistic case $\beta \approx 1$ $v \approx c$

$$W \approx pc$$

Classic Dynamics

$$T_k << W_0$$

The kinetic energy is

$$T_k = \frac{m_0 v^2}{2}$$

Ion momentum

$$p = m_0 v$$

Synchrotron vs cyclotron

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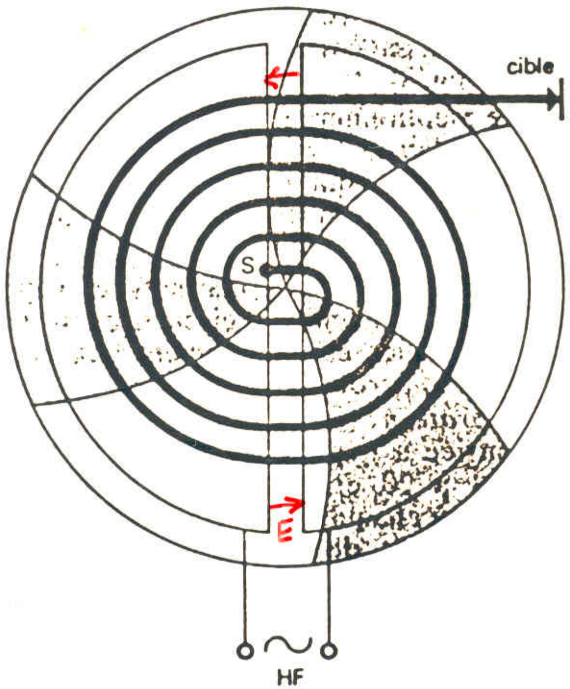

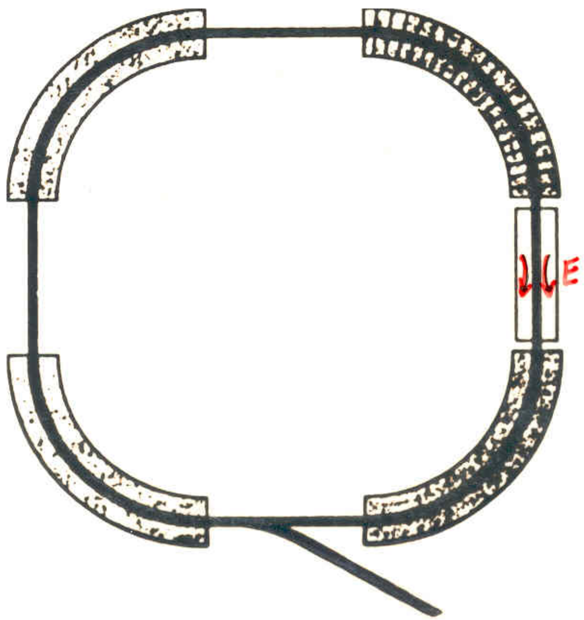

$P = e B \rho$

Cyclotron

Field ***B*** = const (in time).
Radius ***R*** variable

Synchrotron

Radius ***R*** **=** const.

Field ***B*** variable (in time)

cyclotron

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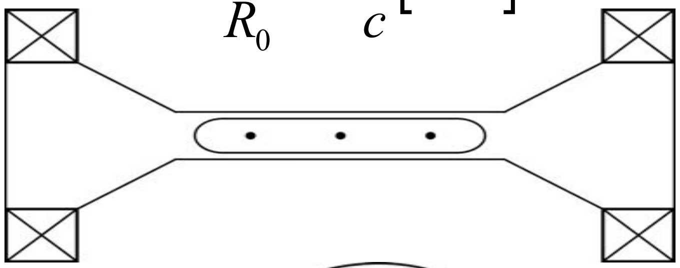
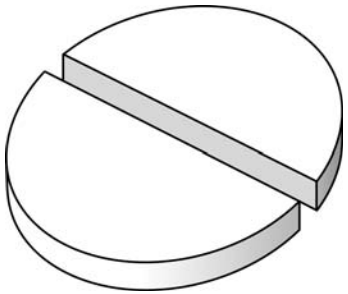
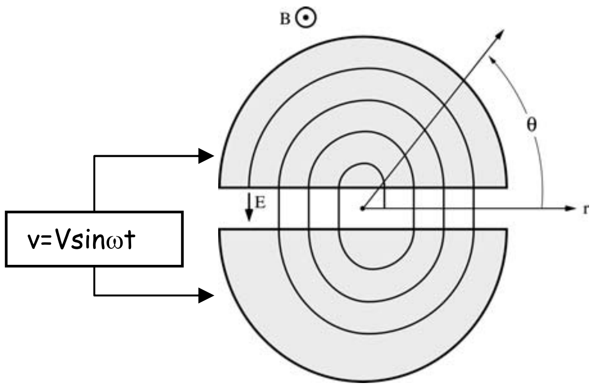
$$\frac{mV_0^2}{R_0} = \frac{e}{c}\left[\vec{B}\cdot\vec{v}\right]$$







radius ***R*** corresponds to velocity ***v*** for the accelerated particle. The circle time corresponds to a revolution period ***T***. Magnetic field ***B*** is constant in time:

$$R = \frac{p}{eB} = \frac{mv}{eB} \qquad T = \frac{2\pi\, m}{eB}$$

Rotation frequency is constant :

$$\omega_c = 2\pi f_c = \frac{2\pi}{T} = \frac{eB}{m}$$

Synchronous condition $\omega_{rf} = h_{rf}\,\omega_c$

Isochronous condition

$m = \gamma m_0 \qquad B = \gamma B_0 \qquad f = const$

Please derive equation of B vs R

Synchrotron (Mac Millan, Veksler, 1945)

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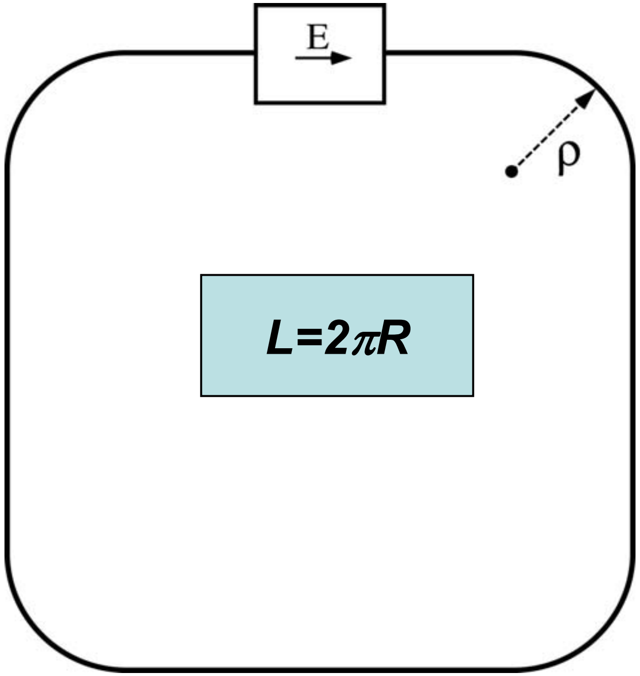
The synchrotron is a synchronous accelerator and energy gain matches the increase of the magnetic field with time (for synchronous RF phase).
Synchronous (equilibrium) particle rotates on closed orbit

$$300 B_0 \rho_0 = p \cdot c$$ (7)



$e\hat{V}\sin\Phi$ → **Energy gain per turn**

$\Phi = \Phi_s$ → **Synchronous phase**

$\omega_{RF} = h\omega_r$ → **Synchronism condition**

$\rho = const$ → **Constant orbit**

$B\rho = p/e$ → ***B ~ β·γ /c***

If v < c ω_r and ω_{RF} variable

• Synchronous condition for Circular Accelerators

- T_S – period of ion rotation in the ring , h_{RF} – integer (RF harmonic)
- $\omega_S = 2\pi / T_S$ - angular frequency of ion rotation in the Ring

$$T_S = h_{RF}\, T_{RF} \qquad\qquad \omega_{RF} = h_{RF}\, \omega_S = h_{RF}\, \beta_S\, c/R \sim W^{1/2}$$

- Lorentz force: $eB\rho = mv = \beta\gamma m_0 c$ $\qquad \omega = v/\rho = eB/\gamma m_0$

Betatron oscillations in storage rings

Motion in bending magnet

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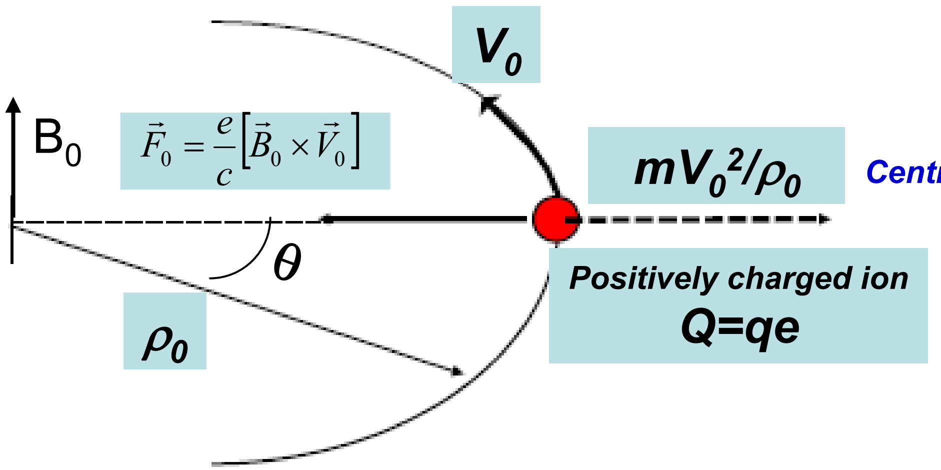
Centripetal force is produced by
Magnetic field ⊥ plane of motion
(**LORENTZ FORCE)**



The Lorentz force, $\vec{F}$ acting on a charged particle in a magnetic field $\vec{B}$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d\left(m\vec{V}\right)}{dt} = \frac{e}{c}\left[\vec{B} \times \vec{V}\right] \quad (8)$$

Cylindrical coordinates (ρ,θ,z) →

$$\begin{cases} F_\rho = \frac{d}{dt}(m\dot{\rho}) - m\rho\dot{\Theta}^2 = q\left(\rho\dot{\Theta}B_z - \dot{z}B_\Theta\right), \\ F_\Theta = \frac{1}{\rho}\frac{d}{dt}\left(m\rho^2\dot{\Theta}\right) = -q\left(\dot{\rho}B_z - \dot{z}B_\rho\right), \\ F_z = \frac{d}{dt}(m\dot{z}) = q\left(\dot{\rho}B_\Theta - \rho\dot{\Theta}B_\rho\right). \end{cases} \quad (9)$$

Field $\vec{B}$ is ortogonal to ion velocity $\vec{V}_0$ and to the the plane of ion motion

the centripetal force is equal to centrifugal force at
EQUILIBRIUM ORBIT

$$\frac{mV_0^2}{\rho_0} = -\frac{e}{c}V_0B_0 \quad (10)$$

Transverse motion magnet

Synchronous (equilibrium) particle rotates on closed orbit but

what happens if particle is displaced from equilibrium orbit ρ_0

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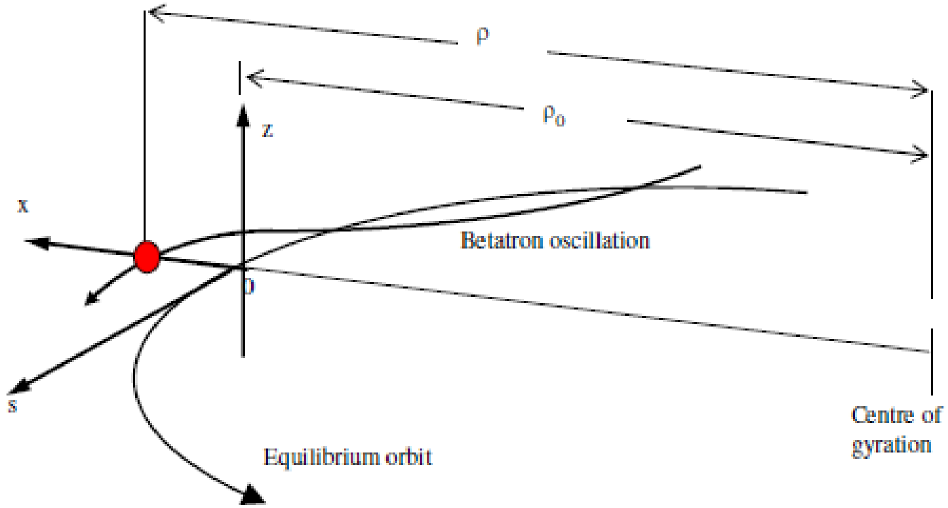
$$300 B_0 \rho_0 = p \cdot c \quad (7)$$

The centripetal force = Lorentz force at E.O.

$\rho = \rho_0 + x$



$$\frac{m V_0^2}{\rho_0} = -\frac{e}{c} V_0 B_0 \quad (10)$$

In the cylindrical coordinate system

$$\rho, \theta, z$$

force acting on particle at radius

$$\rho \neq \rho_0$$

Condition

$x \ll \rho_0$

$$F_\rho = m \frac{d^2 \rho}{dt^2} - m \frac{V_0^2}{\rho} = \frac{e}{c} V_0 B_z \quad (11)$$

magnet

Force acting on particle In radial direction

$$F_\rho = m\frac{d^2\rho}{dt^2} - m\frac{V_0^2}{\rho} = \frac{e}{c}V_0 B_z \qquad (11)$$

Let's introduce LOCAL coordinate system

$$s = \rho_0\theta,\; x = \rho - \rho_0,\; z = z \qquad (12)$$

and transform eq. (11) to *(s,x,z)* coord.system

$$\frac{d}{dt} \equiv V_0\frac{d}{ds} \qquad (13)$$

and

$$\rho = \rho_0 + x \qquad (14)$$

From $\frac{mV_0^2}{\rho_0} = \frac{e}{c}V_0 B_0$ → $\frac{e}{mc} = \frac{V_0}{B_0\rho_0}$

Equation (11) Might be written →

$$\frac{d^2x}{ds^2} - \frac{1}{\rho_0 + x} = \frac{e}{mcV_0}B_z \qquad (15)$$

magnet

at $(x \ll \rho_0)$ one may apply

$$\frac{1}{\rho_0 + x} \approx \frac{1}{\rho_0}\left(1 - \frac{x}{\rho_0}\right) \qquad (16)$$

Expand magnetic field to the first order in *x*

$$B_z(x) = B_0 + \left(\frac{\partial B_z}{\partial x}\right)_0 x + \ldots \qquad (17)$$

Introducing normalized gradient k_x

$$k_x = -\frac{1}{|B_0 \rho_0|}\left(\frac{\partial B_z}{\partial x}\right)_0 \qquad (18)$$

Equation of harmonic Oscillations around Equilibrium Obit

$$\frac{d^2 x}{ds^2} + \left(\frac{1}{\rho_0^2} - k_x\right) x = 0 \qquad (19)$$

Q. Could You repeat please?

A full derivation using Hamiltonian mechanics comes to the same result !

magnet

For <u>off-momentum</u> particles ($\Delta p/p$) one may express force acting on particle as

$$F_\rho = \frac{d}{dt}\left((m+\Delta m)\frac{d}{dt}\rho\right) - (m+\Delta m)\frac{(V_0+\Delta V)^2}{\rho} = \frac{e}{c}(V_0+\Delta V)B_z \qquad (20)$$

By transformation time variable to distance variable

$$\frac{d}{dt} \equiv (V_0+\Delta V)\frac{d}{ds}$$

and keeping only first order terms we will come to

Q. would You like to derive it?

$$\frac{d^2x}{ds^2} + \left(\frac{1}{\rho_0^2} - k_x\right)x = \frac{1}{\rho_0}\frac{\Delta p}{p_0} \qquad (21)$$

Particles oscillate around new orbit $\rho_{\Delta p} = \rho_0 + D\dfrac{\Delta p}{p}$

Magnet. Vertical plane

Taylor expansion of radial component of magnetic field →

$$B_\rho(z) \equiv B_x = \left(\frac{\partial B_x}{\partial z}\right)_0 z + \dots \qquad (22)$$

Because Field is symmetric around median plane and $B_x=0$ for $z=0$

Equation of vertical oscillations

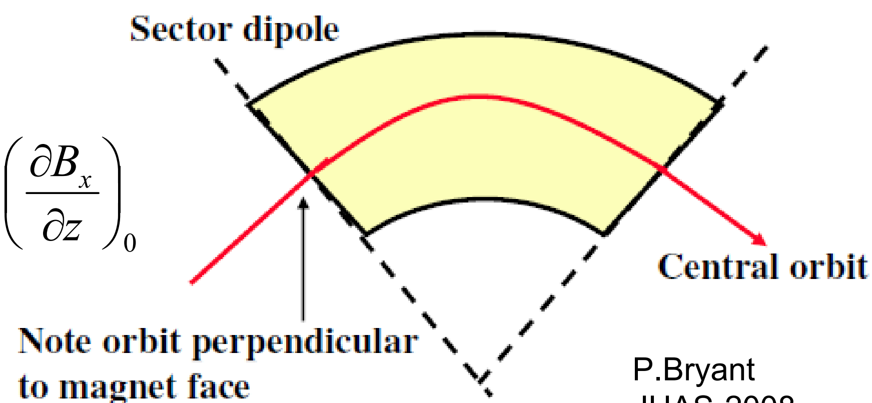
$$\frac{d^2 z}{ds^2} + k_z \cdot z = 0 \qquad (23)$$

Q. Would You like to derive it?

- Equation is valid to the first order for on- and off-momentum ions.
- The k_z *is a* gradient in combined function dipoles.

$$k_z = -\frac{1}{|B_0\rho_0|}\left(\frac{\partial B_x}{\partial z}\right)_0$$

- For a sector magnet, $k_z = 0$ and the dipole acts in vertical direction like a drift.



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Solution of differential equation for harmonic oscillator

$$y'' + K \cdot y = 0 \quad (24)$$

where $K = \frac{1}{\rho^2} - k$ **Horizontal plane**

$K = k$ **Vertical plane**

General solution: linear combination of two independent solutions

$$y(s) = a_1 \cdot \cos(\omega \cdot s) + a_2 \sin(\omega \cdot s) \quad (25)$$

derivatives

$$y'(s) = -a_1\omega \cdot \sin(\omega \cdot s) + a_2\omega\cos(\omega \cdot s)$$

$$y''(s) = -a_1\omega^2 \cdot \cos(\omega \cdot s) - a_2\omega^2 \sin(\omega \cdot s) = -\omega^2 y(s) \rightarrow \omega = \sqrt{K}$$

General solution

$$y(s) = a_1 \cdot \cos(\sqrt{K} \cdot s) + a_2 \sin(\sqrt{K} \cdot s) \quad (26)$$

Let's find initial conditions

$$y(s) = a_1 \cdot \cos(\sqrt{K} \cdot s) + a_2 \sin(\sqrt{K} \cdot s)$$

$$y'(s) = -a_1 \sqrt{K} \cdot \sin(\sqrt{K} \cdot s) + a_2 \sqrt{K} \cos(\sqrt{K} \cdot s)$$

$$s = 0 \qquad y(0) = y_0 \qquad y'(0) = y'_0 \quad \longrightarrow \quad a_1 = y_0 \qquad a_2 = \frac{y'_0}{\sqrt{K}}$$

Transformation from point S0 to the point S1 might be described by Transfer matrix formalism

$$\begin{pmatrix} y \\ y' \end{pmatrix}_{S1} = M \cdot \begin{pmatrix} y \\ y' \end{pmatrix}_{S0} = \begin{Bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{Bmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}_{S0} \qquad (27)$$

Stable (focusing)

$$M_{foc} = \begin{Bmatrix} \cos\left(\sqrt{|K|}s\right) & \frac{1}{\sqrt{|K|}} \sin\left(\sqrt{|K|}s\right) \\ -\sqrt{|K|} \sin\left(\sqrt{|K|}s\right) & \cos\left(\sqrt{|K|}s\right) \end{Bmatrix}_0 \qquad (28)$$

Unstable (defocusing)

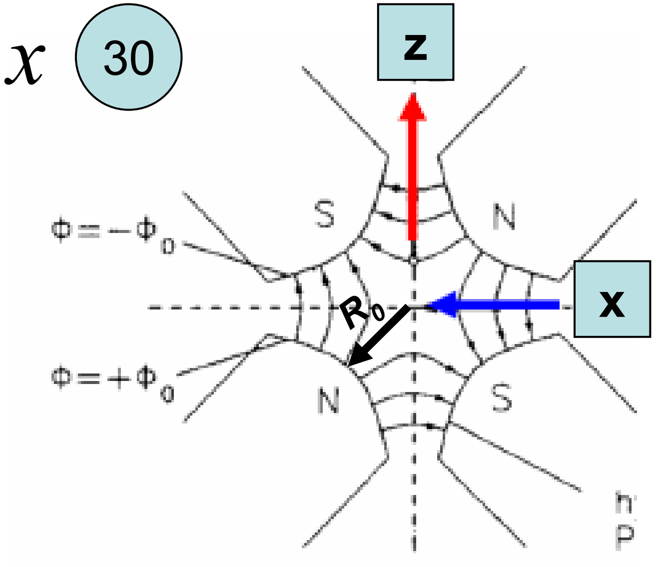
$$M_{def} = \begin{Bmatrix} \cosh\left(\sqrt{|K|}s\right) & \frac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|}s\right) \\ \sqrt{|K|} \sinh\left(\sqrt{|K|}s\right) & \cosh\left(\sqrt{|K|}s\right) \end{Bmatrix}_0 \qquad (29)$$

QUADRUPOLE

Focusing force increases proportionally to the displacement from axis

linear increasing Lorentz force
like pendulum or spring

$$F = -k \cdot x \quad (30)$$



$$B_z = \left(\frac{\partial B_z}{\partial x}\right) \cdot x = g \cdot x$$

linear increasing magnetic field

Maxwell

$$\vec{\nabla} \times \vec{B} = 0$$

$$\left(\frac{\partial B_z}{\partial x}\right) = \left(\frac{\partial B_x}{\partial z}\right)$$

g – field gradient

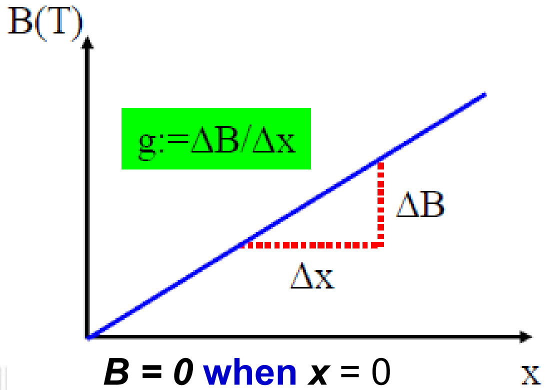
$$B_x = -\left(\frac{\partial B_x}{\partial z}\right) \cdot z = -g \cdot z$$

$$\oint H ds = N \cdot I$$

$$g = \frac{2\mu_0 n I}{R_0^2}$$

Focus in *X* and defocus in *Y*

$$k \equiv k_x = -k_z = \frac{1}{B_0 \rho_0}\left(\frac{\partial B_x}{\partial z}\right) \quad (31)$$

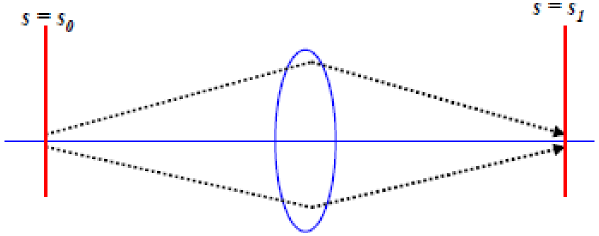


B* = *0 when ***x*** = 0

QUADRUPOLE

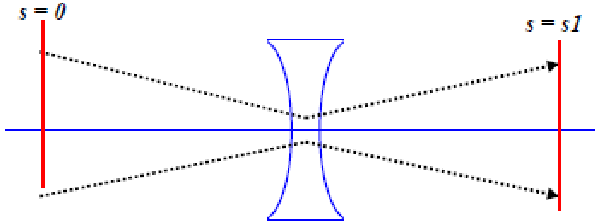
Like in bending magnet the motion in quadrupole is described by differential equations of second order which are similar to harmonic oscillator equation

$$x'' + k \cdot x = 0 \qquad (32)$$



Stable solution (focus)

$$z'' - k \cdot z = 0 \qquad (33)$$



Unstable solution (defocus)

here $\frac{d^2x}{ds^2} \equiv x''$

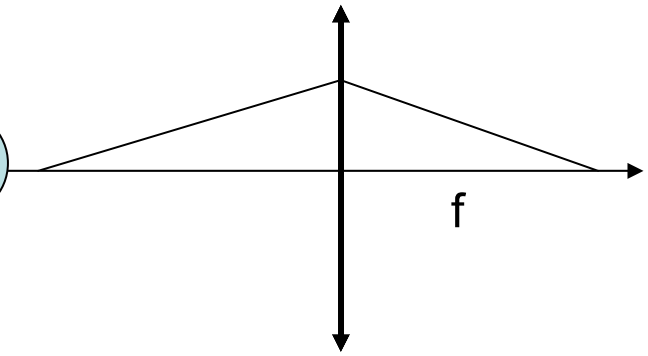
- **Quads focus in horizontal plane and defocus in vertical plane and vise versa**
- **Set of quads is required to provide focusing for both planes**

Thin lens approximation (s $\longrightarrow$ 0)

Transfer matrix of thin lens

$$y_1 = 1 \cdot y_0 + 0 \cdot y_0'$$
$$y_1' = \frac{1}{f} \cdot y_0 + 1 \cdot y_0'$$

$$\longrightarrow \quad M = \begin{Bmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{Bmatrix}_0 \qquad (34)$$

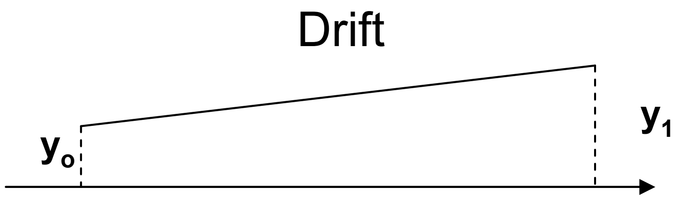


focal length of thin lens $\longrightarrow$ $f = \dfrac{1}{k \cdot l_q}$

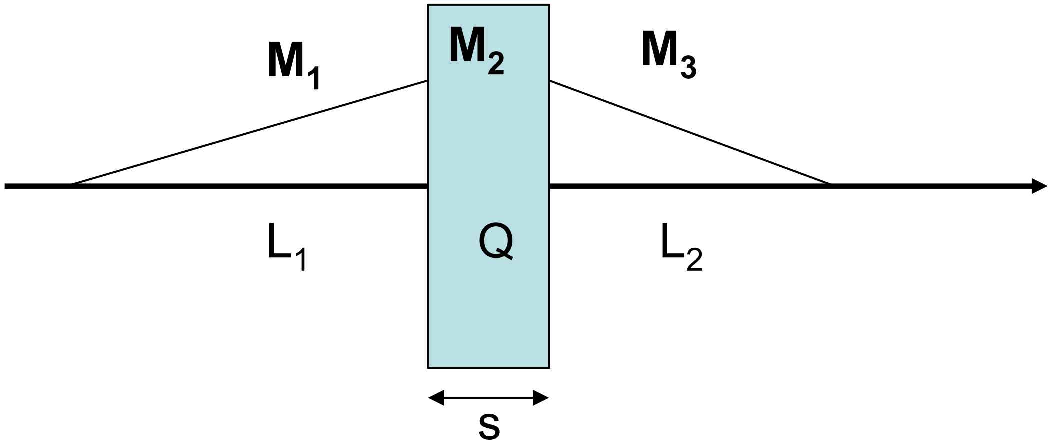
DRIFT

$$y_1 = 1 \cdot y_0 + \cdot L \cdot y_0'$$
$$y_1' = 0 \cdot y_0 + 1 \cdot y_0'$$

$$\longrightarrow \quad M = \begin{Bmatrix} 1 & L \\ 0 & 1 \end{Bmatrix} \qquad (35)$$



Matrix multiplication



Transformation through a system of lattice elements

$$M = M_3 \cdot M_2 \cdot M_1 = \begin{Bmatrix} 1 & L_2 \\ 0 & 1 \end{Bmatrix} \cdot \begin{Bmatrix} \cos\left(\sqrt{|K|}s\right) & \frac{1}{\sqrt{|K|}} \sin\left(\sqrt{|K|}s\right) \\ -\sqrt{|K|}\sin\left(\sqrt{|K|}s\right) & \cos\left(\sqrt{|K|}s\right) \end{Bmatrix} \cdot \begin{Bmatrix} 1 & L_1 \\ 0 & 1 \end{Bmatrix}$$

36

Q. Please multiply matrixes

the matrix multiplication goes from right to left.

$$y'' + K \cdot y = 0$$

where

$$K_x = \frac{1}{\rho_0^2} - k$$ **Horizontal plane**

37

$$K_z = k$$ **Vertical plane**

Condition for weak focusing in both planes

Motion is stable for both planes if $\frac{1}{\rho_0^2} > k$ **and simultaneously** $k > 0$

$$K_x = \left(\frac{1}{\rho_0^2} - k\right) > 0$$

$$K_z = k > 0$$

Conditions for STRONG focusing in both planes

Sequence of focus-defocus elements with total effect of strong focusing in both planes

$$(K_x)_i > 0 \quad (K_z)_i < 0$$

$$(K_x)_{i+1} < 0 \quad (K_z)_{i+1} > 0$$

$$i = 1,2,3,\ldots,N$$

For $K_y > 0$ solutions are harmonic functions,

For $K_y < 0$ solutions are hyperbolic and unstable

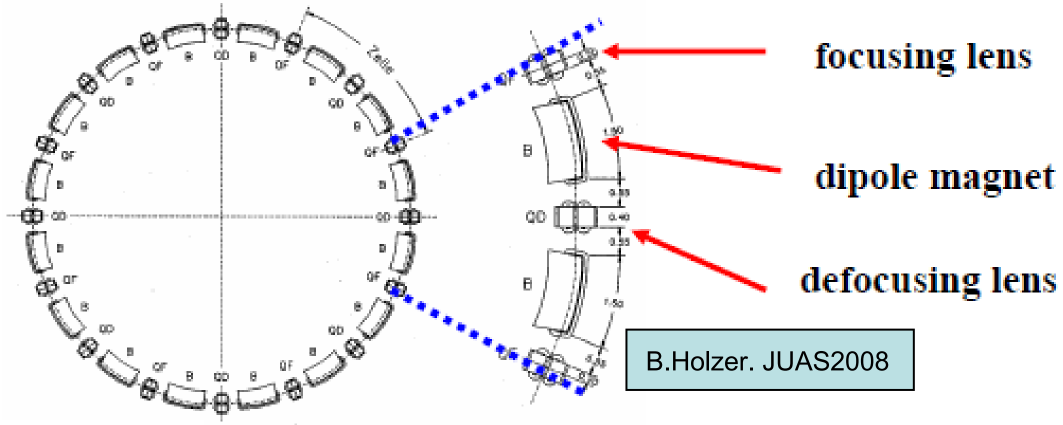
For $K_y = 0$ solutions are linear

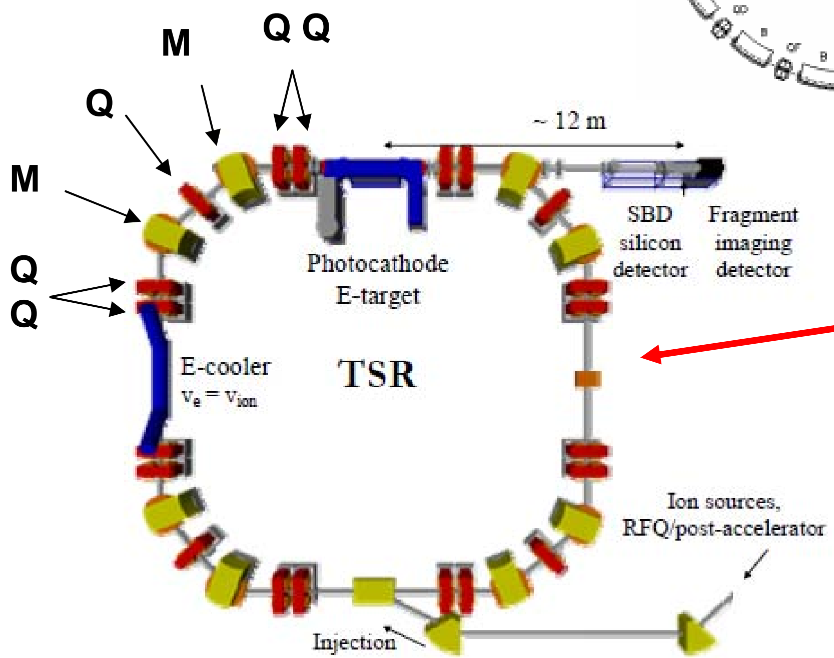
Element	K_x	K_z
Magnetic combined-function with horizontal bend	$\rho_0^{-2} - k$	k
Magnetic combined-function with vertical bend	$-k$	$\rho_0^{-2} + k$
Pure magnetic quadrupole	$-k$	k
Pure magnetic horizontal bend	ρ_0^{-2}	0
Pure magnetic vertical bend	0	ρ_0^{-2}
Drift space	0	0

Let's consider real storage rings

Ring is built of repeating focusing and bending elements to bend ions on 360°
Repeating sequence of elements is called CELL. Length of cell is L_c
In total N cells form ring circumference $C=N \cdot L_c$

Example of one **cell**
Q-d-M-d-Q-d-M-d-Q →





How many cells in the Test Storage Ring (TSR) of the Max PLanck Institute For Nuclear Physics MPI-K ?

Astronomer Hill Let‘s used equation with periodic restoring force to study motion of planets. Let‘s apply Hill equation to our case

The motion in the storage rings composed of many cells is described by differential equation of motion with periodic focusing properties

$$x''(s) + k(s)\cdot x(s) = 0 \qquad (38)$$

Where restoring force depends on the position and is periodic function of distance *s*

$$k(s + L_{cell}) = k(s) \qquad (39)$$

Solution is a Quasi-harmonic oscillations where amplitude and phase

depends on the positions *s* in the ring.

Solution is similar to harmonic oscillator

but with <u>periodic</u> amplitude and <u>periodic</u> phase function

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)}\cdot\cos(\psi(s) + \phi) \qquad (40)$$

Here ε and ϕ are integration constants and we will find it from initial conditions

$\beta(s)$ **is periodic function given by focusing properties of the ring lattice**

$$\beta\left(s + L_{cell}\right) = \beta(s) \qquad (41)$$

Substituting solution (40) into Hill equation (38) one can get expression for phase advance of the oscillations at distance *S*

$$\psi(s) = \int_0^s \frac{ds}{\beta(s)} \qquad (42)$$

For one complete turn in the ring number of oscillations per revolution is called <u>BETATRON TUNE</u>

$$Q_y = \frac{1}{2\pi}\oint \frac{ds}{\beta(s)} \qquad (43)$$

And general solution of Hill equation for trajectory and velocity is

$$x(s) = \sqrt{\varepsilon}\sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi) \qquad (44)$$

$$x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \cdot \{\alpha(s) \cdot \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi)\} \qquad (45)$$

Here we introduce <u>TWISS parameters</u>

$$\alpha(s) = -\frac{\beta'(s)}{2} \qquad \gamma(s) = \frac{1 + \alpha^2(s)}{\beta(s)} \qquad \alpha = \sqrt{\beta\gamma - 1} \qquad (46)$$

Substituting (44) into expression for velocity (45) and solving for constant ε one gets parametric representation of motion in phase space (x, x')

Equation of ellipse →

$$\varepsilon = \gamma \cdot x^2 + 2\alpha \cdot x \cdot x' + \beta \cdot x'^2 \qquad (47)$$

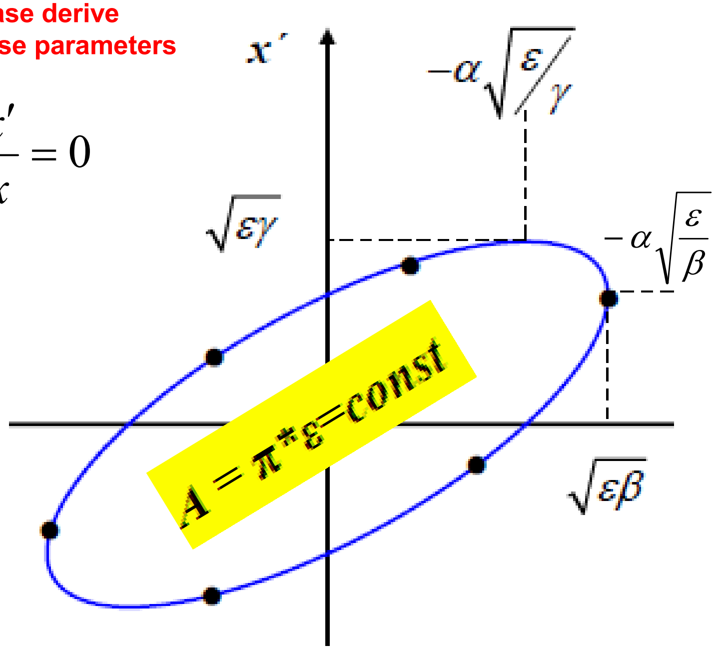
ε – constant of the motion… it is <u>independent</u> of „S“

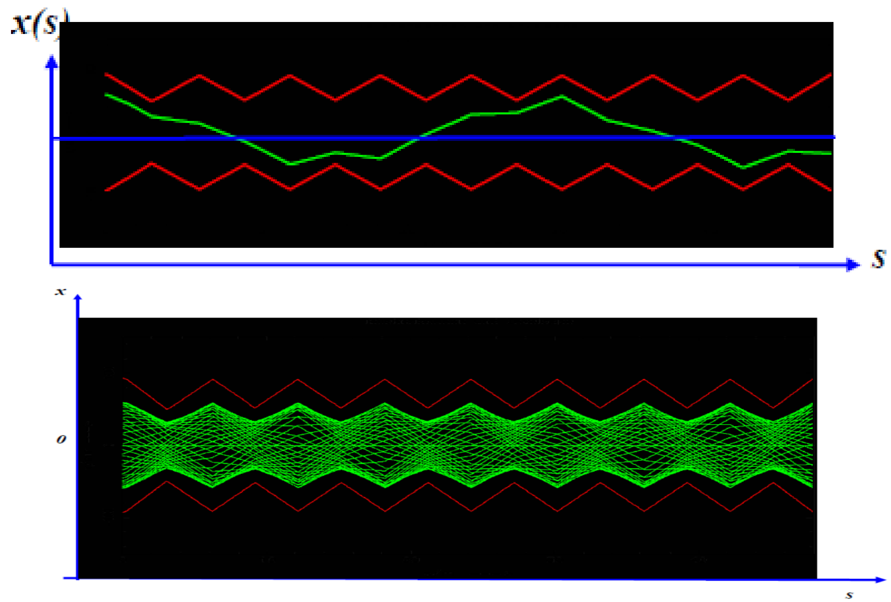
emittance → $\varepsilon = \gamma \cdot x^2 + 2\alpha \cdot x \cdot x' + \beta \cdot x'^2$

- **Parametric representation of ion motion in the (*x, x‘*) space**
- **Shape and orientation of ellipse are given by parameters *α, β, γ* while area covered by particles in phase space (*x, x‘*) is CONSTANT**
- **Density of particles in phase space is CONSTANT if no dissipation of friction forces are present i.e. Ions move in potential fields which do not depend on ion velocity (in our case ions move in external elecric and magnetic fields)**
- **The Liouville theorem directly follows from energy conservation Law (see Landau, Lifshitz „Theoretical mechanics“)**
- **emittance is intrinsic beam parameter, can not be changed by the focusing elements.**

Q. Please derive ellipse parameters

$$\frac{dx'}{dx} = 0$$





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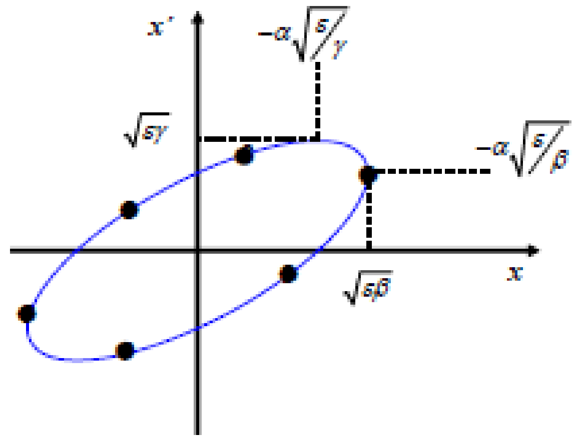
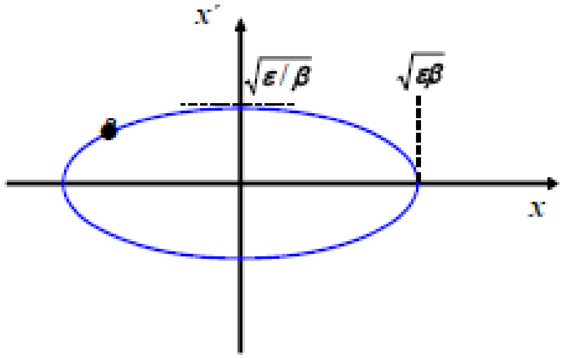
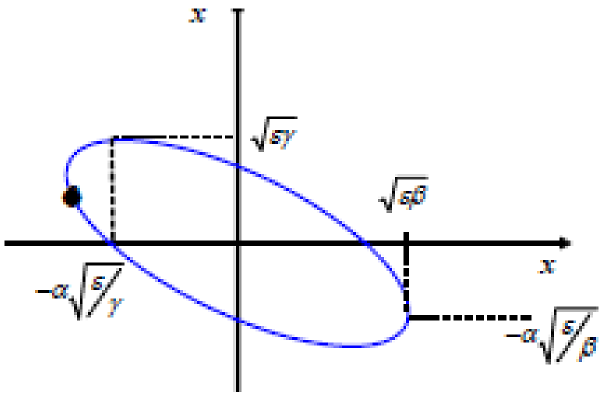


Divergent beam



Maximum envelope
Inside quadrupole



Converged beam

During acceleration

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$\varepsilon \neq const\ !$

because

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta}$$ (53)

and

$$\varepsilon = \int x' dx \propto \frac{1}{\beta\gamma}$$ (54)

The beam emittance is changed with energy as

$$\frac{1}{\beta \cdot \gamma}$$

Normalized emittance

$$\varepsilon_{norm} = \beta \cdot \gamma \cdot \varepsilon = const$$ (55)

Thanks