Entangling quantum gates

In the following, I will discuss the mechanism that creates entanglement between a pair of qubits that are off-resonantly coupled by a state-dependent force exciting the joint vibrational motion of the ions.

1 Off-resonantly excited harmonic oscillator

An off-resonantly excited classical oscillator will return to its initial state after a time $\tau = \frac{2\pi}{\delta}$ where δ is the detuning from resonance. As shown below, the same is true for a quantum oscillator but its wave function $|\psi\rangle$ gets multiplied by a phase factor: $|\psi\rangle \rightarrow |\psi\rangle e^{i\phi}$

The quantum harmonic oscillator with frequency ν and excitation frequency ω is described by the Hamiltonian

$$H = \underbrace{\hbar\nu a^{+}a}_{= H_0} + D\hat{x}\cos\omega t \quad , \quad \text{where } \delta = \omega - \nu$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\nu}}\,(a + a^{+})$$

Change into an interaction picture with respect to H_0 and you will get

$$H_I(t) = \hbar\Omega\,(a e^{-i\delta t} + a^{+} e^{+i\delta t}) \qquad (0)$$

. Here, Ω describes the strength of the coupling ($\hbar\Omega = \frac{D}{2}\sqrt{\frac{\hbar}{2m\nu}}$) and we have neglected the rapidly rotating terms evolving at frequency $\omega + \nu$ because we assume that $\delta \ll \nu$ (rotating wave approximation)

To calculate how the state of the oscillator evolves, we make use of the Baker-Campbell-Haussdorff formula

$$e^{A} e^{B} = e^{A+B} e^{\frac{1}{2}[A,B]} \quad (1)$$

for operators A, B that both commute with their commutator $[A,B]$:

$[A,[A,B]] = [B,[A,B]] = 0$

This allows us to calculate the dynamics induced by $H_I(t)$:

$$U(t_0 \to t_{final}) = \lim_{\Delta\tau \to 0} e^{-i/\hbar H(t_{final})\Delta\tau} \cdot \ldots \, e^{-i/\hbar H(t_0+\Delta t)\Delta t} e^{-i/\hbar H(t_0)\Delta t} \quad (2)$$

Here I will not carry out the complete calculation but just illustrate what needs to be done:

For two Hamiltonians $H_1 = \gamma_1 a + \gamma_1^* a^+$, $H_2 = \gamma_2 a + \gamma_2^* a^+$,

$$[H_1, H_2] = \gamma_1\gamma_2 \underbrace{[a,a]}_{=0} + \gamma_1^*\gamma_2^* \underbrace{[a^+,a^+]}_{=0} + (\gamma_1\gamma_2^* - \gamma_1^*\gamma_2) \underbrace{[a^+,a]}_{=1} = (\gamma_1\gamma_2^* - \gamma_1^*\gamma_2)\mathbb{1}$$

the commutator is simply the identity multiplied by a complex number.

Therefore, we can use (1) for calculating (2).

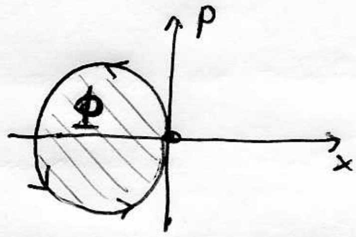
$$e^{-\frac{i}{\hbar}H_2\Delta\tau} e^{-\frac{i}{\hbar}H_1\Delta\tau} = e^{-\frac{i}{\hbar}(H_1+H_2)\Delta\tau} e^{i\phi_{12}}$$

and H_1+H_2 is once again the sum of a creation and an annihilation operator. If we multiply with the next term in (2), we get

$$e^{-\frac{i}{\hbar}H_3\Delta\tau} e^{-\frac{i}{\hbar}H_2\Delta\tau} e^{-\frac{i}{\hbar}H_1\Delta\tau} = e^{-\frac{i}{\hbar}(H_1+H_2+H_3)} e^{i(\phi_{12}+\phi_{13}+\phi_{23})}$$ and so on.

Finally, we get $U(t=0 \to t_{final} = \frac{2\pi}{\delta}) = e^{i\Phi}\mathbb{1}$ because for the choice of $t_{final} - t_0 = \frac{2\pi}{\delta}$ the first exponent $H_1 + H_2 + \ldots \to \int_{t_0}^{t_{final}} dt\, H(t) = 0$ vanishes.

This shows that the driven oscillator returns to its initial state but is multiplied by a <u>geometric phase</u> Φ that turns out to be proportional to the area in phase space enclosed by its trajectory

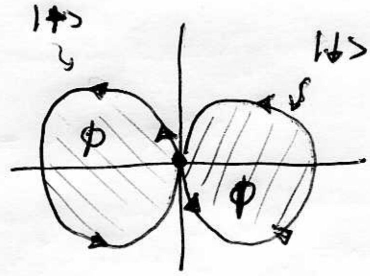


2 State-dependent force acting on one qubit

The Hamiltonian is nearly the same as in 1:

$$H_I(t) = \hbar\Omega\,(a e^{-i\delta t} + a^{+i\delta t})\,\sigma_z$$

Because of the σ_z-operator, the force becomes state-dependent. If the qubit is in state $|\uparrow\rangle$, $\sigma_z|\uparrow\rangle = +|\uparrow\rangle$, if it is in $|\downarrow\rangle$, $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$ so that the force acts in opposite directions for $|\uparrow\rangle$ and $|\downarrow\rangle$:



Because of $\sigma_z^2 = \mathbb{1}$,

$$[H_1, H_2] = [\gamma_1 a + \gamma_1^* a^\dagger, \gamma_2 a + \gamma_2^* a^\dagger]\,\sigma_z^2$$
$$= (\gamma_1\gamma_2^* - \gamma_2\gamma_1^*)\,\mathbb{1} \quad \text{as before}$$

and we obtain the same ~~commutator~~ propagator $U(t=0 \rightarrow t_{final} = \frac{2\pi}{\delta}) = e^{i\phi}\,\mathbb{1}$

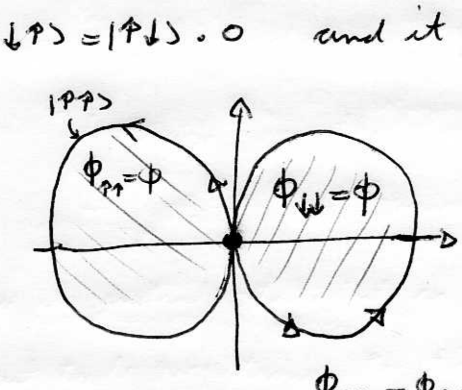
3 State-dependent force acting on two qubits

If two qubits are coupled to the same oscillator and with equal couplings, we have

$$H_I(t) = \hbar\Omega\,(a e^{-i\delta t} + a^\dagger e^{i\delta t})\,(\sigma_z^{(1)} + \sigma_z^{(2)}) \qquad (3)$$

Now, the state-dependent force vanishes for states $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ because of $(\sigma_z^{(1)} + \sigma_z^{(2)})|\uparrow\downarrow\rangle = 0\cdot|\uparrow\downarrow\rangle$ and $(\sigma_z^{(1)} + \sigma_z^{(2)})|\downarrow\uparrow\rangle = |\uparrow\downarrow\rangle\cdot 0$ and it becomes twice as strong for $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ because of $(\sigma_z^{(1)} + \sigma_z^{(2)})|\uparrow\uparrow\rangle = 2|\uparrow\uparrow\rangle$

and $(\sigma_z^{(1)} + \sigma_z^{(2)})|\downarrow\downarrow\rangle = -2|\downarrow\downarrow\rangle$



$\phi_{\downarrow\uparrow} = \phi_{\uparrow\downarrow} = 0$

For the two-qubit gate, we therefore end up with phase factors that depend on the state of both qubits:

$$U(t=0 \to t_{final} = \frac{2\pi}{\delta}) = \begin{pmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix} \qquad (4)$$

This time-evolution looks as if it was produced by an effective Hamiltonian $H_{eff} = -\frac{\hbar\delta}{4\pi}\phi\left(\sigma_z^{(1)}\sigma_z^{(2)} + \mathbb{1}\right)$ since $\sigma_z^{(1)}\sigma_z^{(2)} + \mathbb{1} = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2 \end{pmatrix}$

This effective Hamiltonian provides an accurate description of the dynamics at times $t = \frac{2\pi}{\delta}, \frac{4\pi}{\delta}, \frac{6\pi}{\delta}, \dots$ but it fails to capture the excitation of the ion motion in the times between.

If we make the coupling Ω weak compared to the detuning δ, then the motional excitation is weak at all times and H_{eff} is a good description of the quantum ~~dynmacis~~ dynamics.

The ~~unitary~~ identity operator in H_{eff} is of no importance as it produces only a global phase factor. For this reason we can drop it.

If we express the phase ϕ in terms of Ω and δ, we find $\phi = \left(\frac{\Omega}{\delta}\right)^2 2\pi\,(-1)$

and $H_{eff} = \hbar\frac{\Omega^2}{2\delta}\sigma_z^{(1)}\sigma_z^{(2)}$

If we adjust the phase ϕ in (4) such that $U = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \sim e^{i\frac{\pi}{4}\sigma_z^{(1)}\sigma_z^{(2)}}$, we obtain a maximally entangling gate. This can be seen by applying it to the product state $|\Psi_0\rangle = |+\rangle_x|+\rangle_x$ (here $\sigma_x|+\rangle_x = |+\rangle_x$ and $\sigma_x|-\rangle_x = -|x\rangle$)

$$\begin{aligned} U|+\rangle_x|+\rangle_x = e^{i\frac{\pi}{4}\sigma_z^{(1)}\sigma_z^{(2)}}|+\rangle_x|+\rangle_x &= \frac{1}{\sqrt{2}}\left(\mathbb{1} + i\,\sigma_z^{(1)}\sigma_z^{(2)}\right)|+\rangle_x|+\rangle_x \\ &= \frac{1}{\sqrt{2}}\left(|+\rangle_x|+\rangle_x + i\left(\sigma_z^{(1)}|+\rangle_x\right)\left(\sigma_z^{(2)}|+\rangle_x\right)\right) \\ &= \frac{1}{\sqrt{2}}\left(|+\rangle_x|+\rangle_x + i\,|-\rangle_x|-\rangle_x\right) \end{aligned}$$