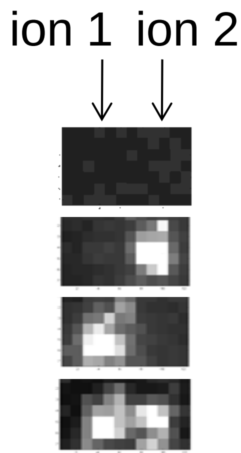
Measuring observables

Measuring one and two-qubit observables

Results of measuring 1000 copies of a two-qubit state $|\Psi\rangle$



Detected state	How often observed?
$\|\downarrow\downarrow\rangle$	500
$\|\uparrow\downarrow\rangle$	100
$\|\downarrow\uparrow\rangle$	0
$\|\downarrow\downarrow\rangle$	400

Calculate the expectation value of the observables $\sigma_z^{(1)}, \sigma_z^{(2)}, \sigma_z^{(1)}\sigma_z^{(2)}$.

We apply the single-qubit gate $U = e^{i\frac{\pi}{4}\sigma_y^{(1)}} = \frac{1}{\sqrt{2}}(I + i\sigma_y^{(1)})$ to the state $|\Psi\rangle$ before carrying out the fluorescence detection.

Which observables A can we measure in this way?

(with expectation values $\langle\Psi|A|\Psi\rangle$)

Question 1: Expectation value of observables?

For an observable $\mathcal{O}$ having eigenvalues λ_j and eigenvectors $|\phi_j\rangle$, the expectation value of the state ρ is given by $\langle\mathcal{O}\rangle = \sum \lambda_j \langle\phi_j|\rho|\phi_j\rangle$.

For $\sigma_z^{(1)}$, we therefore have

$$\begin{aligned}\langle\sigma_z^{(1)}\rangle &= \frac{1}{N}(N_{\uparrow\uparrow} + N_{\uparrow\downarrow} - N_{\downarrow\uparrow} - N_{\downarrow\downarrow}) \\ &= \tfrac{1}{1000}(500 + 100 - 0 - 400) = 0.2\end{aligned}$$

For $\sigma_z^{(2)}$, we obtain

$$\langle\sigma_z^{(2)}\rangle = \frac{1}{N}(N_{\uparrow\uparrow} - N_{\uparrow\downarrow} + N_{\downarrow\uparrow} - N_{\downarrow\downarrow}) = 0$$

and similarly

$$\langle\sigma_z^{(1)}\sigma_z^{(2)}\rangle = \frac{1}{N}(N_{\uparrow\uparrow} - N_{\uparrow\downarrow} - N_{\downarrow\uparrow} + N_{\downarrow\downarrow}) = 0.8$$

Question 2: Which observables A,B,C are measured?

The fluorescence measurement of ion 1 after applying the unitary U to the state $|\Psi\rangle$ measures the expectation value $\langle U\Psi|\sigma_z^{(1)}|U\Psi\rangle = \langle\Psi|U^\dagger\sigma_z^{(1)}U|\Psi\rangle$

Therefore, we measure the observable

$$A = U^\dagger\sigma_z^{(1)}U = \frac{1}{2}(I - i\sigma_y^{(1)})\sigma_z^{(1)}(I + i\sigma_y^{(1)})$$

$$= \frac{1}{2}(\sigma_z^{(1)} + \underbrace{\sigma_y^{(1)}\sigma_z^{(1)}}_{\sigma_x^{(1)}}\sigma_y^{(1)} - i\underbrace{[\sigma_y^{(1)}, \sigma_z^{(1)}]}_{2i\sigma_x^{(1)}}) = \sigma_x^{(1)}$$

Similarly, the observable $\sigma_x^{(1)}\sigma_z^{(2)}$ is transformed into

$$B = \sigma_x^{(1)}\sigma_z^{(2)}$$

by the unitary operation whereas

$$C = \sigma_z^{(2)}$$

is not affected by the unitary operating on qubit 1.