QCD, Chirality and Topological Insulators

Wherein QCD is related to the Integer Quantum Hall Effect,

more exotic topological materials learn something from lattice QCD,

and such materials suggest how to regulate chiral gauge theories

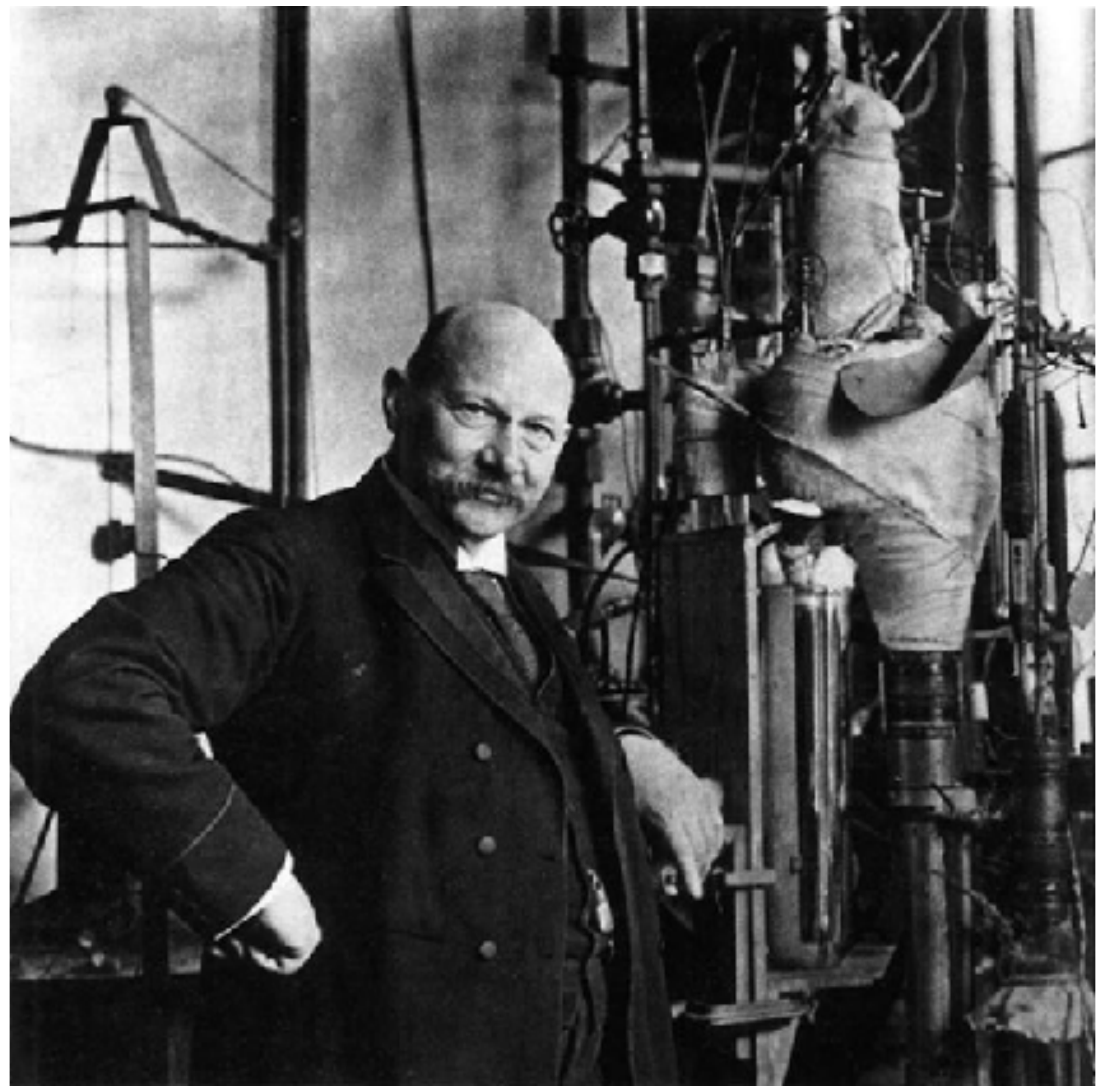
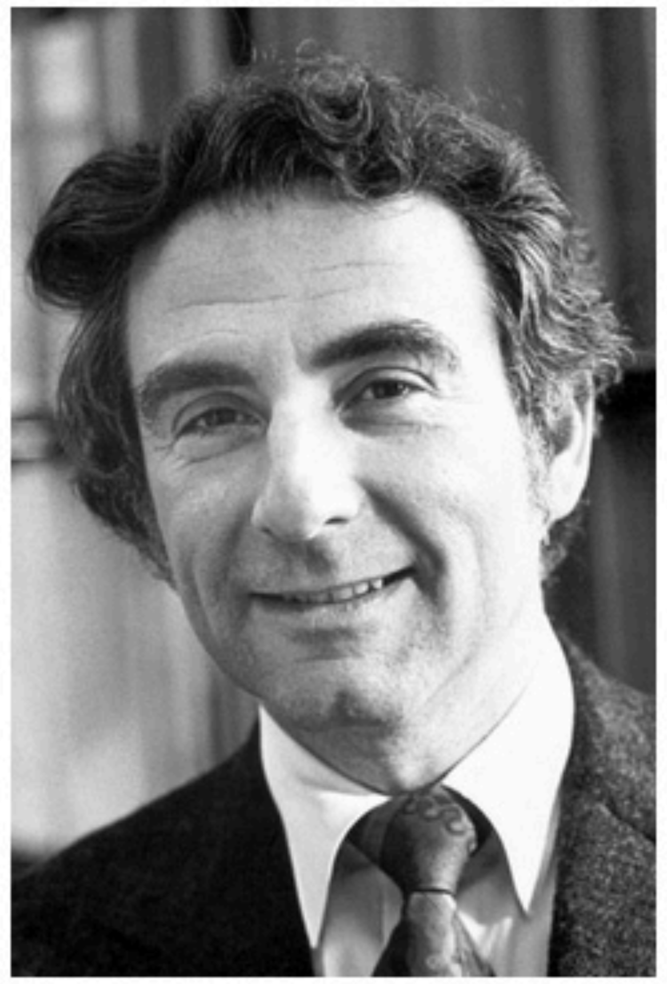


Photo from the Nobel Foundation archive.

John Bardeen

Prize share: 1/3



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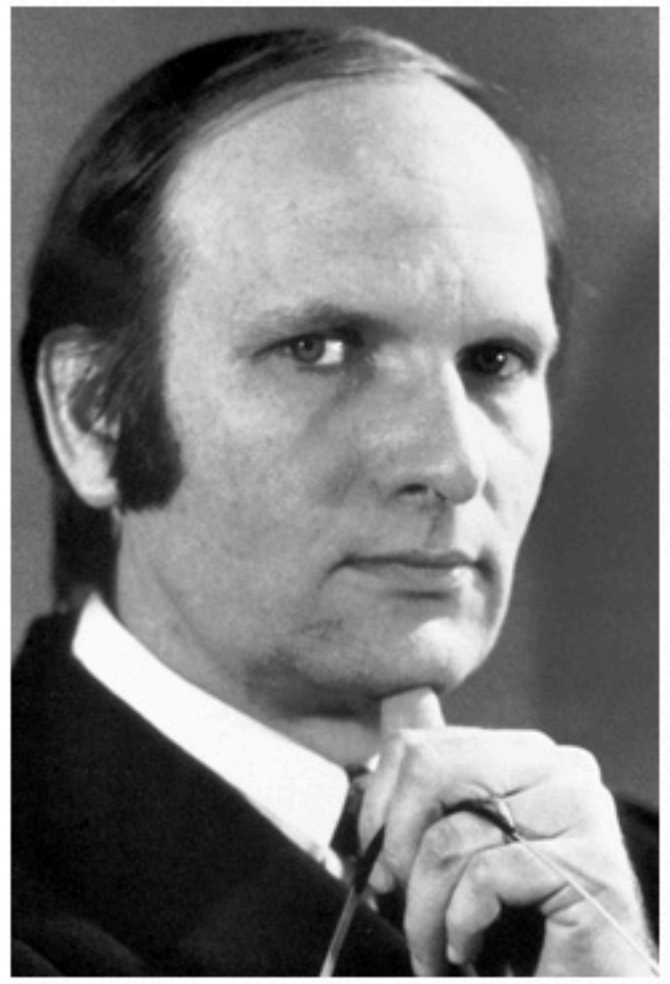
Leon Neil Cooper

Prize share: 1/3



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John Robert Schrieffer

Prize share: 1/3

Superconductivity had a huge influence on the understanding of quantum field theory and the strong interactions:

- Chiral symmetry breaking
- Higgs mechanism
- Confinement (dual Meissner effect)
- Topological defects (strings)

"The vacuum is complicated"

Topological materials have dominated the CMT conversation for the past 4 decades… can they tell us something about QCD and the Standard Model? Or vice versa?

- Virtually every type of theory of a massive fermion is an analogue to some type of topological insulator or superconductor
- Theories of massless fermions can be formulated as edge states of a higher dimension theory
- Anomalies are the common theoretical thread
- Key to understanding chirality in lattice QCD
- A promising framework for a nonperturbative regulator for chiral gauge theories, such as the Standard Model.

Two big influences on my thinking about the subject:

- A conversation with David Gross (1981): chirality on the lattice is interesting
- An influential paper by Callan and Harvey (1984): anomalies as inflow

ANOMALIES AND FERMION ZERO MODES ON STRINGS AND DOMAIN WALLS

C.G. CALLAN, Jr.[1] and J. A. HARVEY[2]

Jadwin Hall, Princeton University, Princeton, New Jersey 08544, USA

Received 10 September 1984

We show that the mathematical relation between non-abelian anomalies in $2n$ dimensions, the parity anomaly in $2n+1$ dimensions, and the Dirac index density in $2n+2$ dimensions can be understood in terms of the physics of fermion zero modes on strings and domain walls. We show

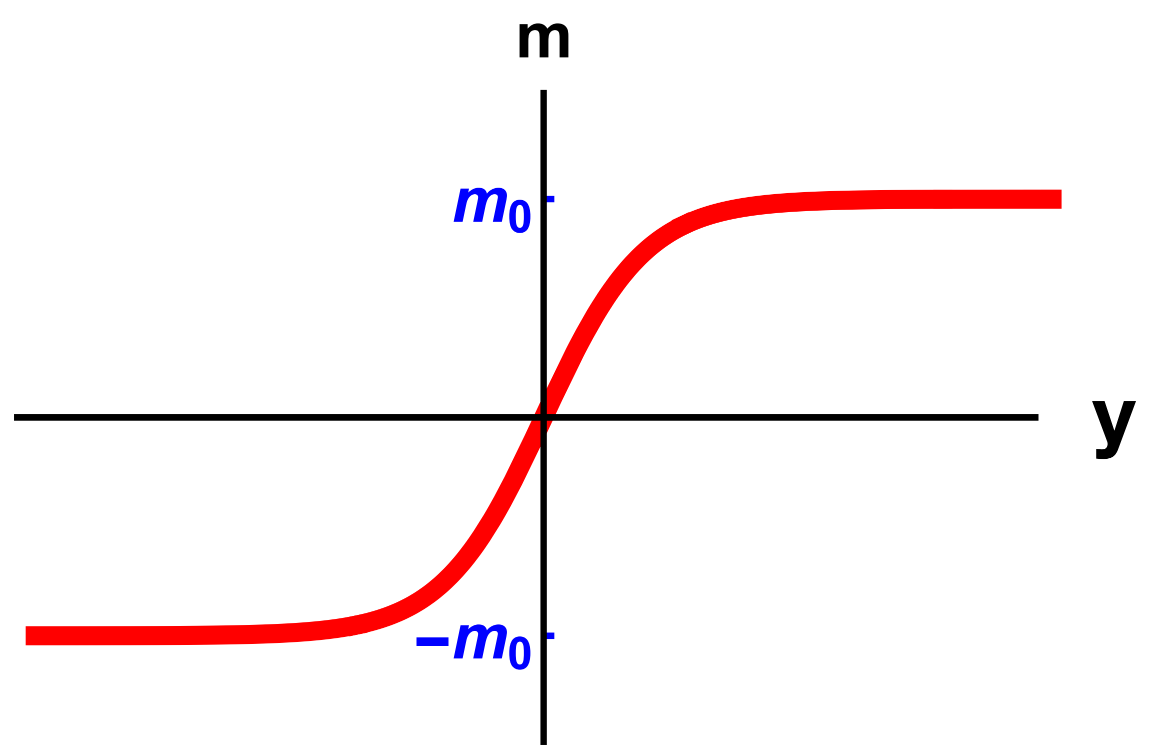
Simple models for the relationship between anomalies in different dimensions: charge flowing on and off of topological defects (boundaries)

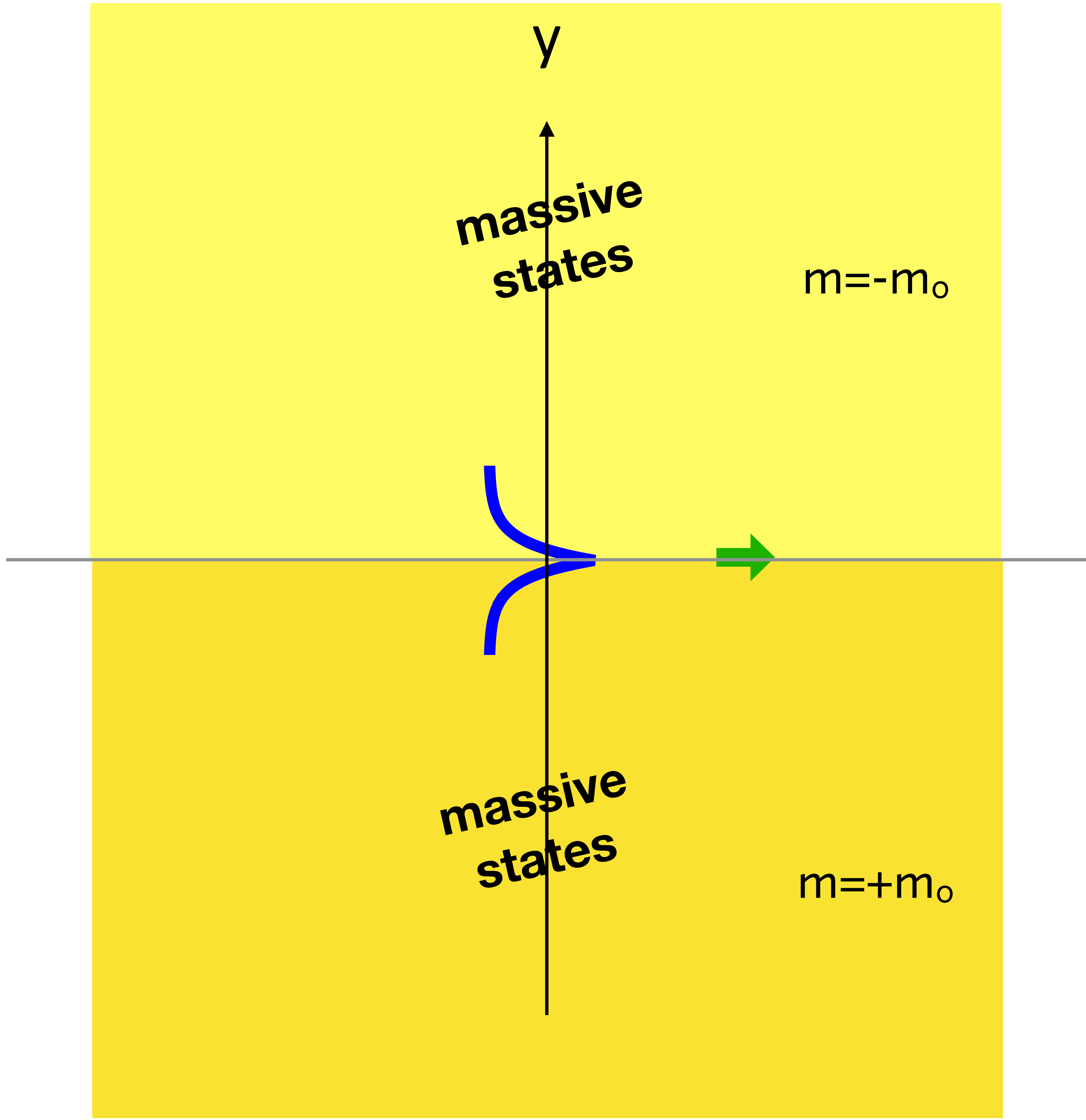
Jackiw & Rebbi's surprising discovery (1976)

Consider Dirac fermion in 2+1 dimensions with "domain wall" mass, $m = m_0\, \varepsilon(x_2)$ with $m_0 > 0$:

$$\left[i\gamma^\mu \partial_\mu - m_0 \epsilon(y)\right] \Psi = 0$$

$$\gamma_y = i\sigma_3$$

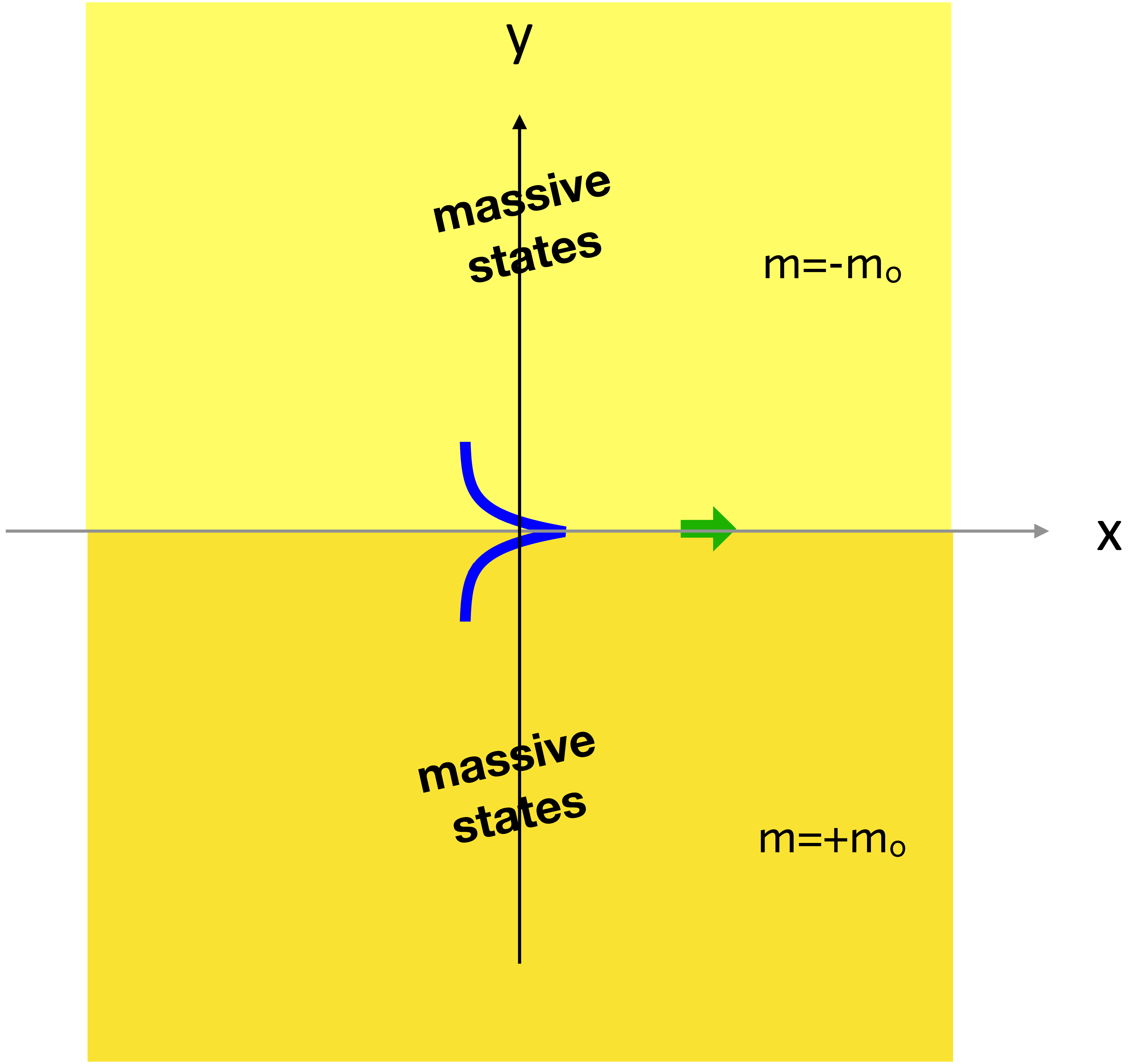


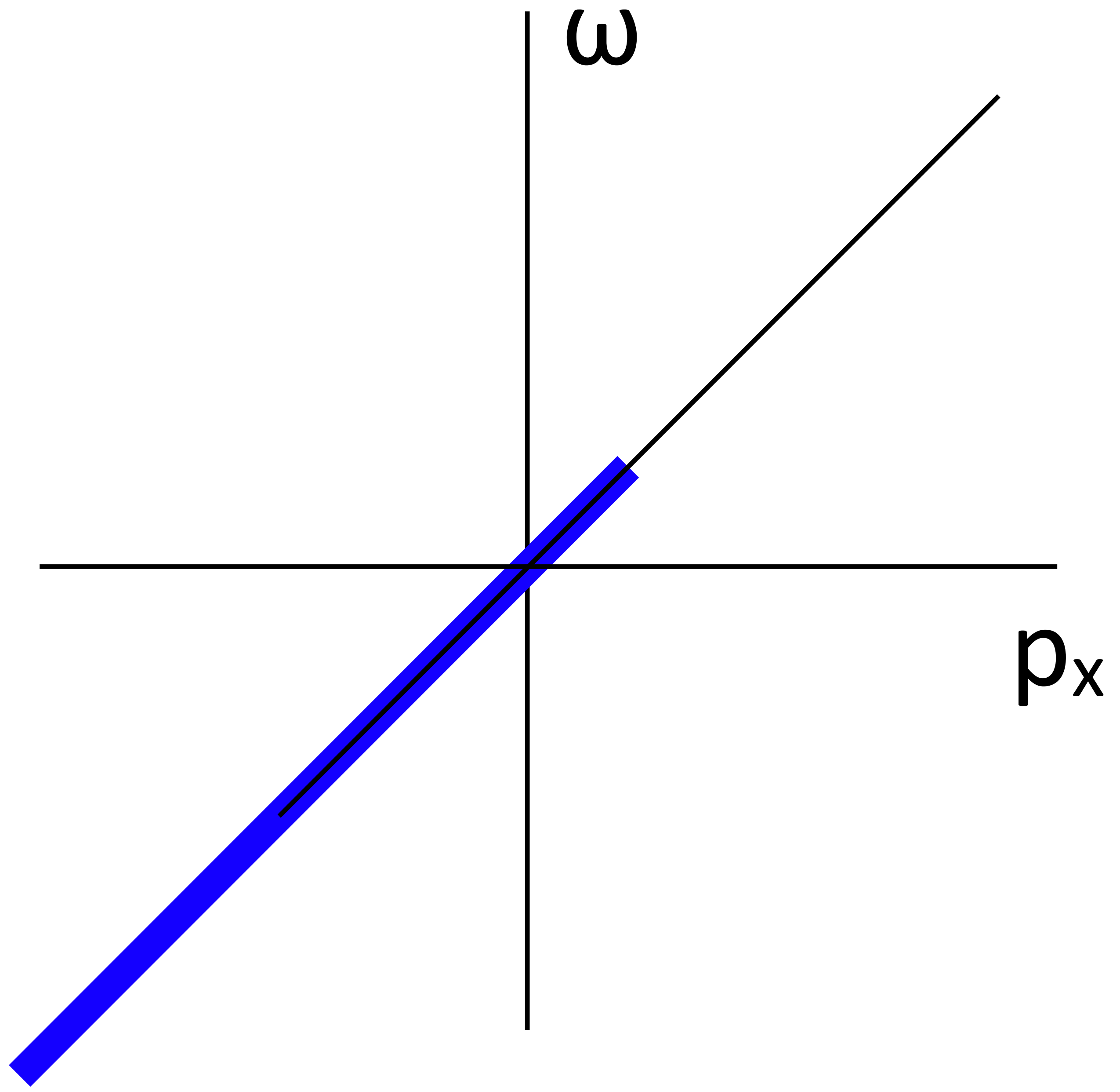


Dirac equation has a solution: $\Psi = e^{-m_0|y|} \begin{pmatrix} f(t-x) \\ 0 \end{pmatrix}$

This is a massless chiral fermion bound to the surface y = 0, moving at the speed of light… only in the positive x direction.

There is no normalizable solution with the opposite chirality moving in the -x direction… would be proportional to $e^{+m_0|y|}$





Chiral fermion on 1+1 dimension boundary

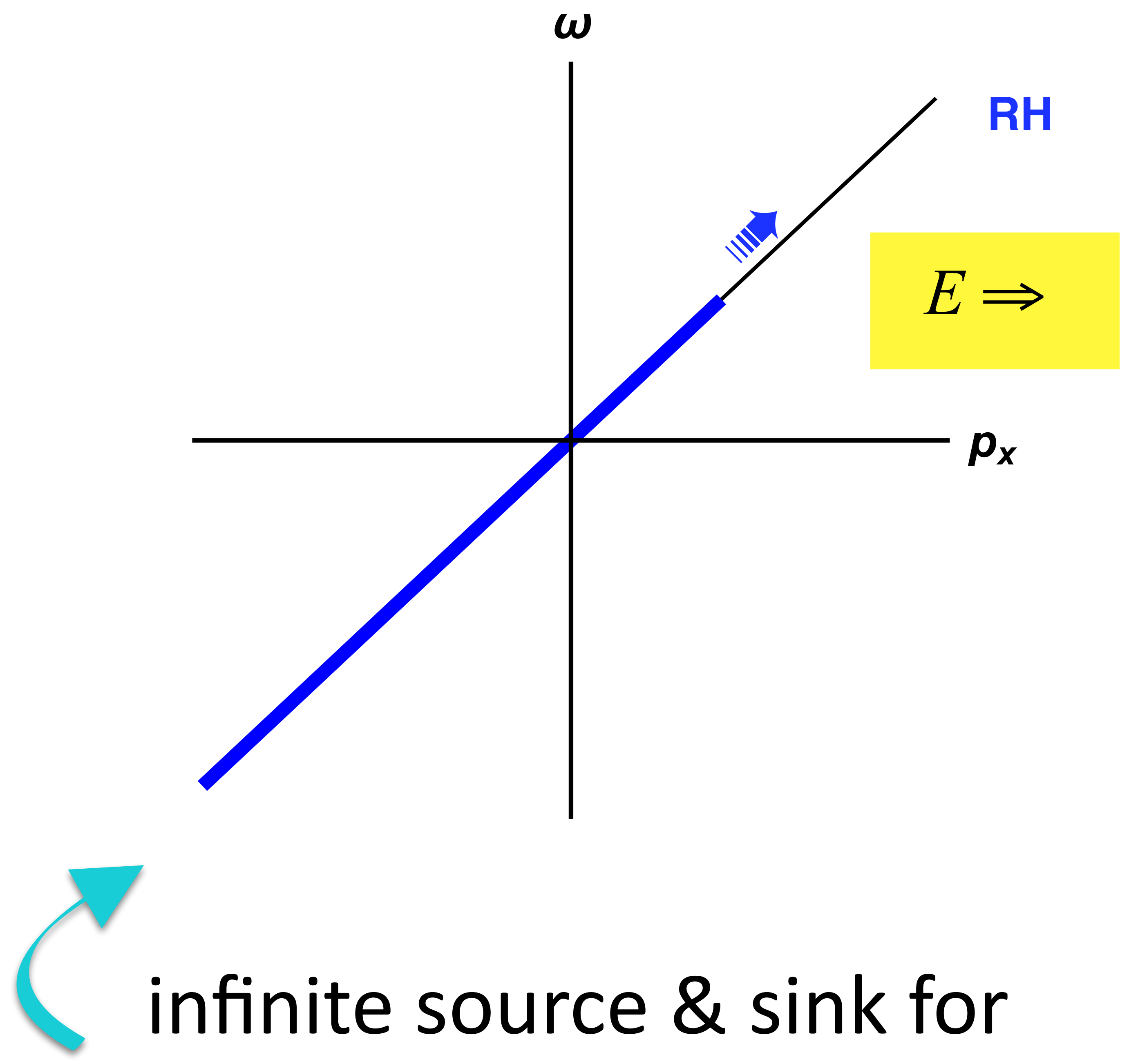
No chirality fermion in 2+1 dimension bulk

Chirality ⇔ ε tensor ⇔ topology

…but where is the topology?

But what happens when you only have a RH fermion (eg, domain wall fermion)?

Massless chiral fermion in an electric field E, 1+1 dim



infinite source & sink for fermions

$$dp = qEdt$$

$$\frac{dn}{dt} = \frac{dp}{2\pi} = \frac{qE}{2\pi}$$

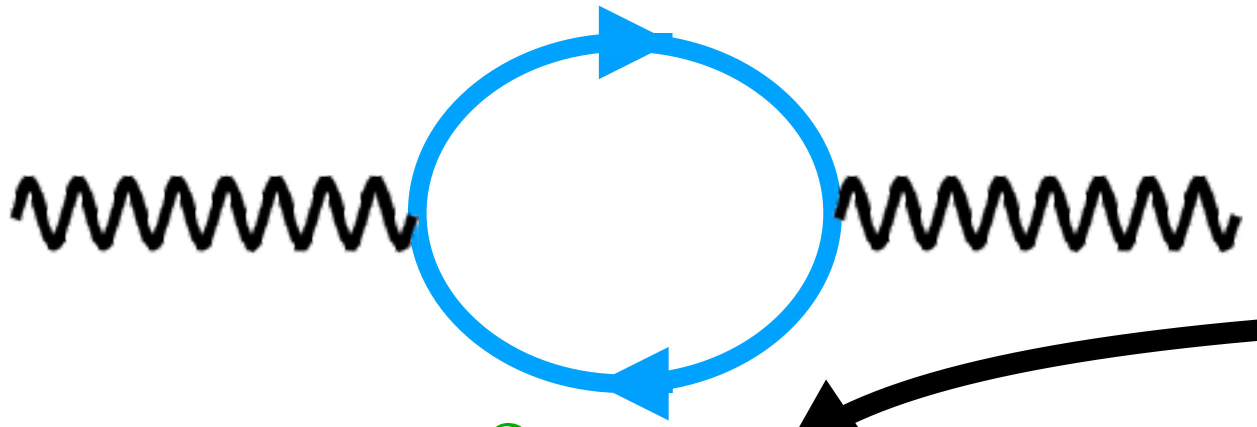
- Electric charge is violated!
- Sick theory!
- Hilbert's hotel: there's always room for more (or less) in the Dirac sea...

This phenomenon occurs in 1+1 and 3+1 dimensions, but not 2+1 where electric charge is always conserved...

.... so how can this state appear at the edge of a 2+1 dimensional theory of massive fermions?

Callan and Harvey's resolution:

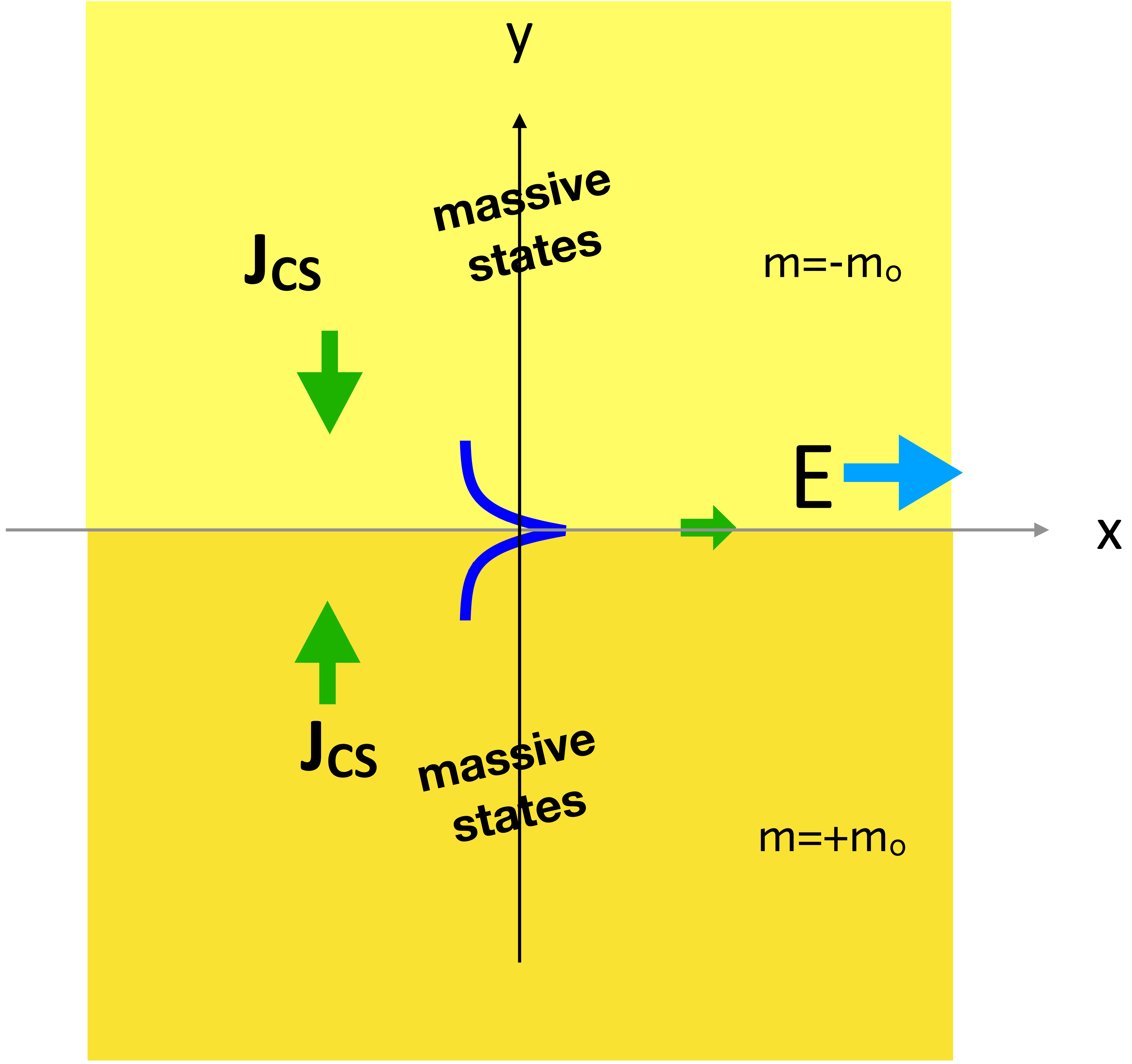
Integrating out the heavy fermions off the domain wall gives rise to a Chern Simons term



$$\mathcal{L}_{\rm CS} = \frac{q^2}{16\pi}\frac{m}{|m|}\epsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma$$

Heavy bulk states do not decouple!

ε(y) due to domain wall



$$J^\mu_{\rm CS} = \frac{\delta\mathcal{L}_{\rm CS}}{\delta A_\mu} \quad\Longrightarrow\quad \partial_\mu J^\mu_{\rm CS} = \delta(x_2)\frac{q^2}{2\pi}E$$

The charge building up on domain wall flows in **from the bulk**, accounting for the anomalous divergence of the domain wall fermion

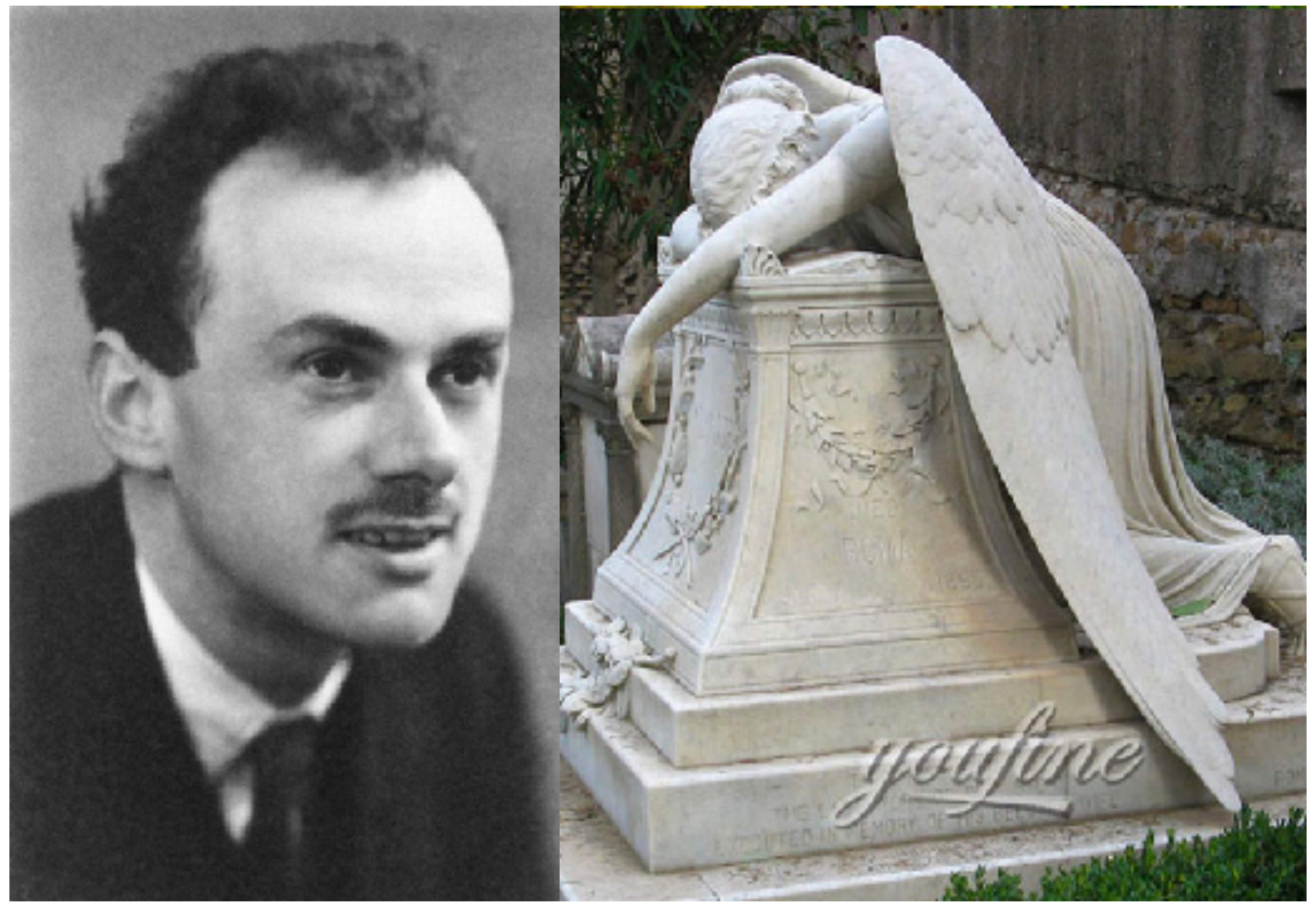
In the late 1980s I started thinking about whether this effect could be used to maintain chiral symmetry in lattice QCD.

"Forget it!" said lattice practitioners, "d=4 is hard enough"

In creating lattice QCD Wilson revolutionized how one thinks about field theory & its infinities

Greatest Generation I

Dispairing

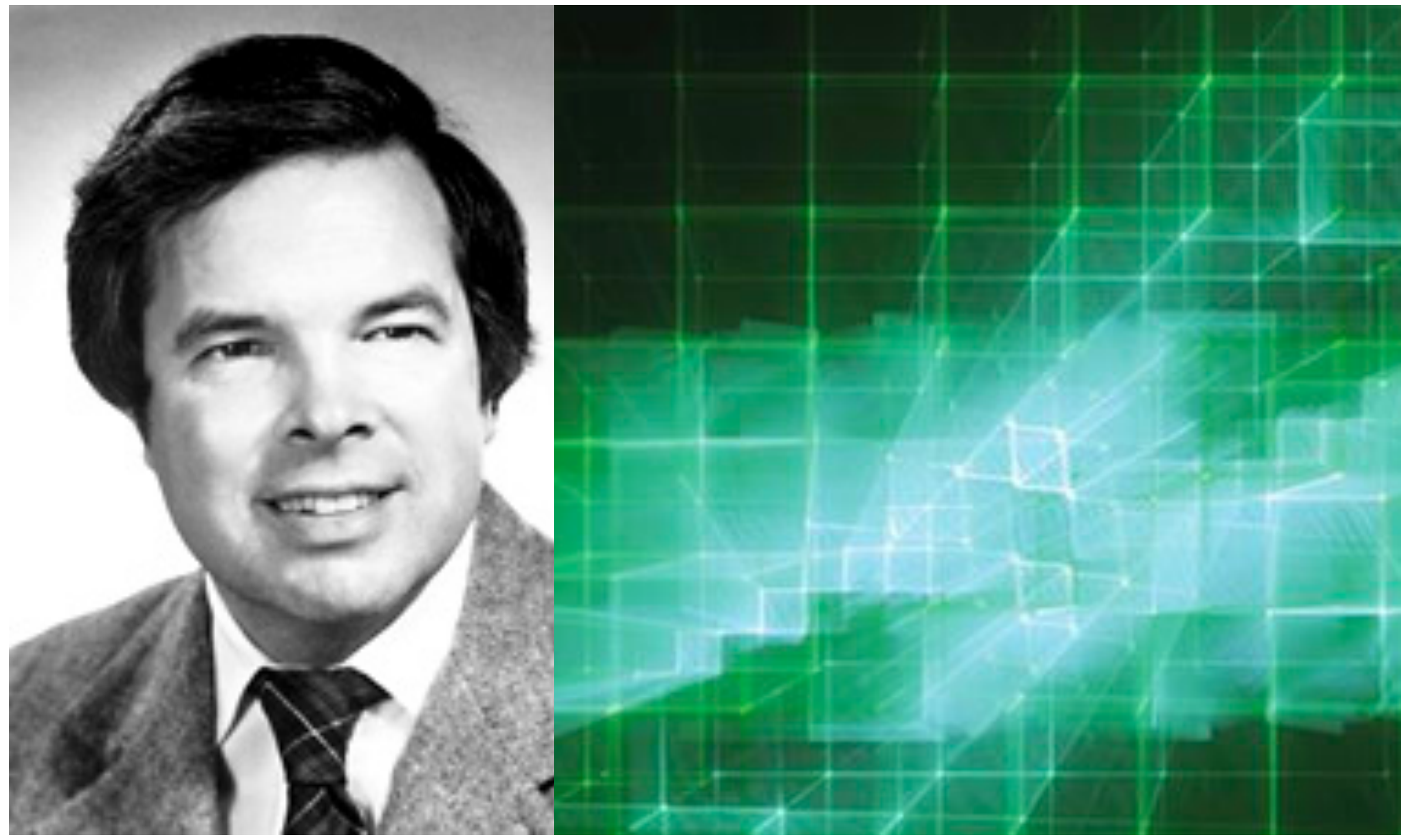


Greatest Generation II

Smug

Silent Generation

Pragmatic



One issue when putting QCD on the computer: problems with chirality

Nielsen-Ninomiya

$$S = \int_{-\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^4}\, \overline{\Psi}_{-\mathbf{p}} \tilde{D}(\mathbf{p})\Psi(\mathbf{p})$$

wanted:

1. $\tilde{D}(\mathbf{p})$ is a periodic, analytic function of p_μ;
2. $D(\mathbf{p}) \propto \gamma_\mu p_\mu$ for $a|p_\mu| \ll 1$;
3. $\tilde{D}(\mathbf{p})$ invertible everywhere except $p_\mu = 0$;
4. ~~$\{\Gamma, \tilde{D}(\mathbf{p})\} = 0$.~~

one can only have **three** out of four desired properties

e.g., Wilson fermions:

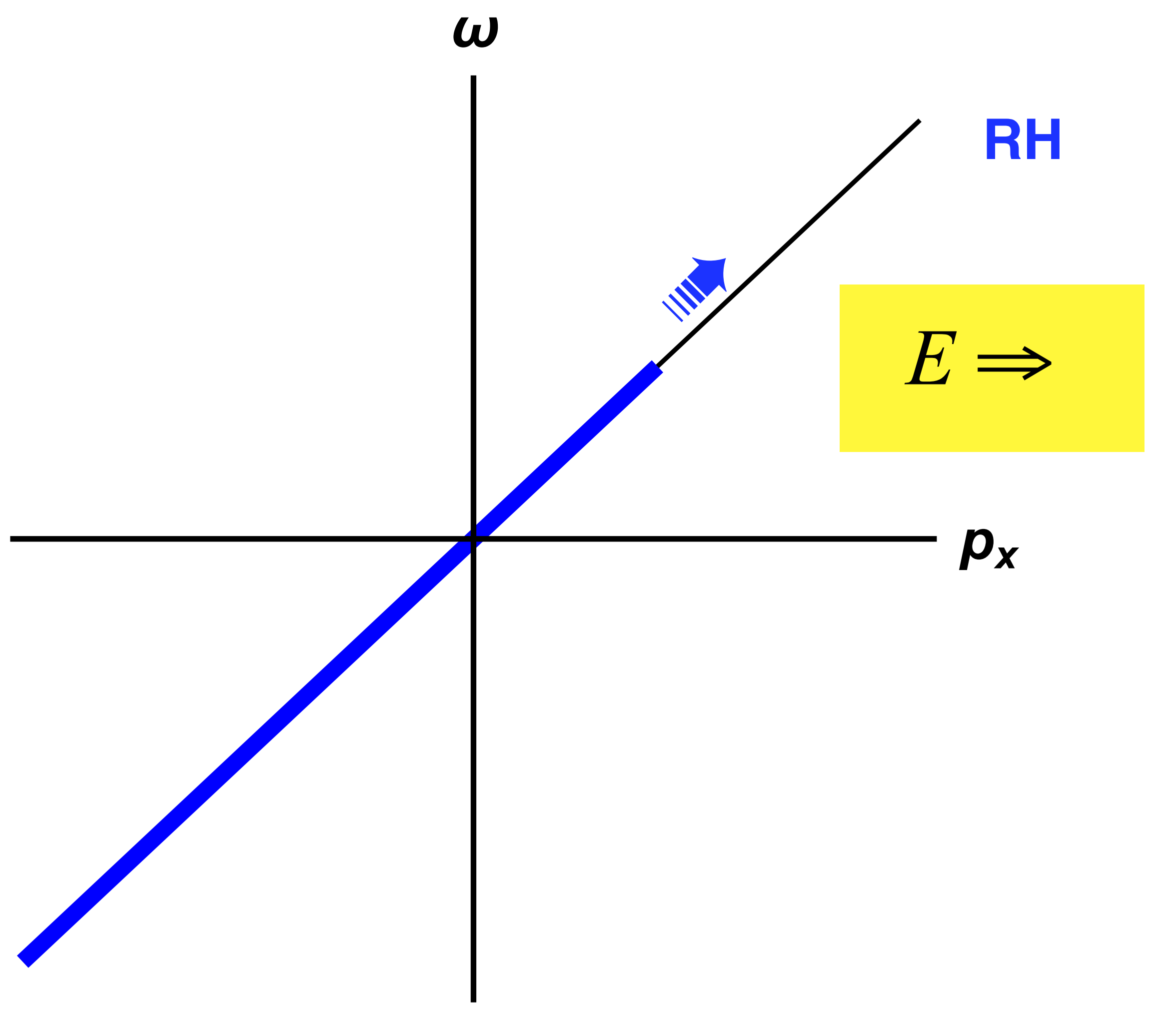
Lattice covariant derivatives

$$\mathcal{L} = \bar{\psi}\left(\not{D} + m + aD^2\right)\psi$$

violates chiral symmetry

global chirality symmetry becomes unnatural; gauged chirality symmetry is impossible

Nielsen-Ninomiya can be thought of as a consequence of anomalies

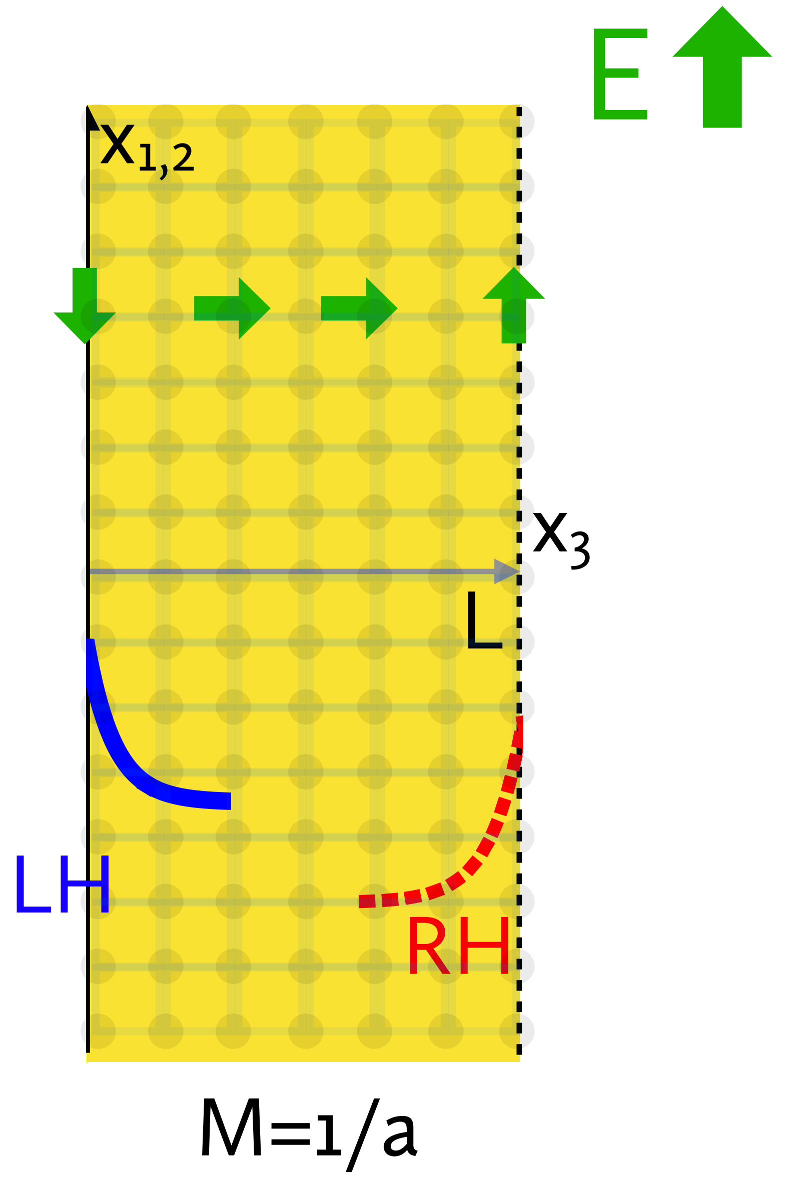


infinite source & sink for fermions?

The Hilbert Hotel:

There's always room for more in the vacuum... but not on the lattice

Can lattice QCD account for a symmetry good up to anomalies? ...add extra dimension!



Wilson fermions in **4+1** dimensions can have chiral fermions on the 3+1 dimensional edge-surfaces without fine tuning.

DBK, PLB 288 (1992) 342

Lattice covariant derivatives

$$\to \bar{\psi}\left[\not{D} + M + a\frac{R}{2}D^2\right]\psi$$

mass

Wilson coupling

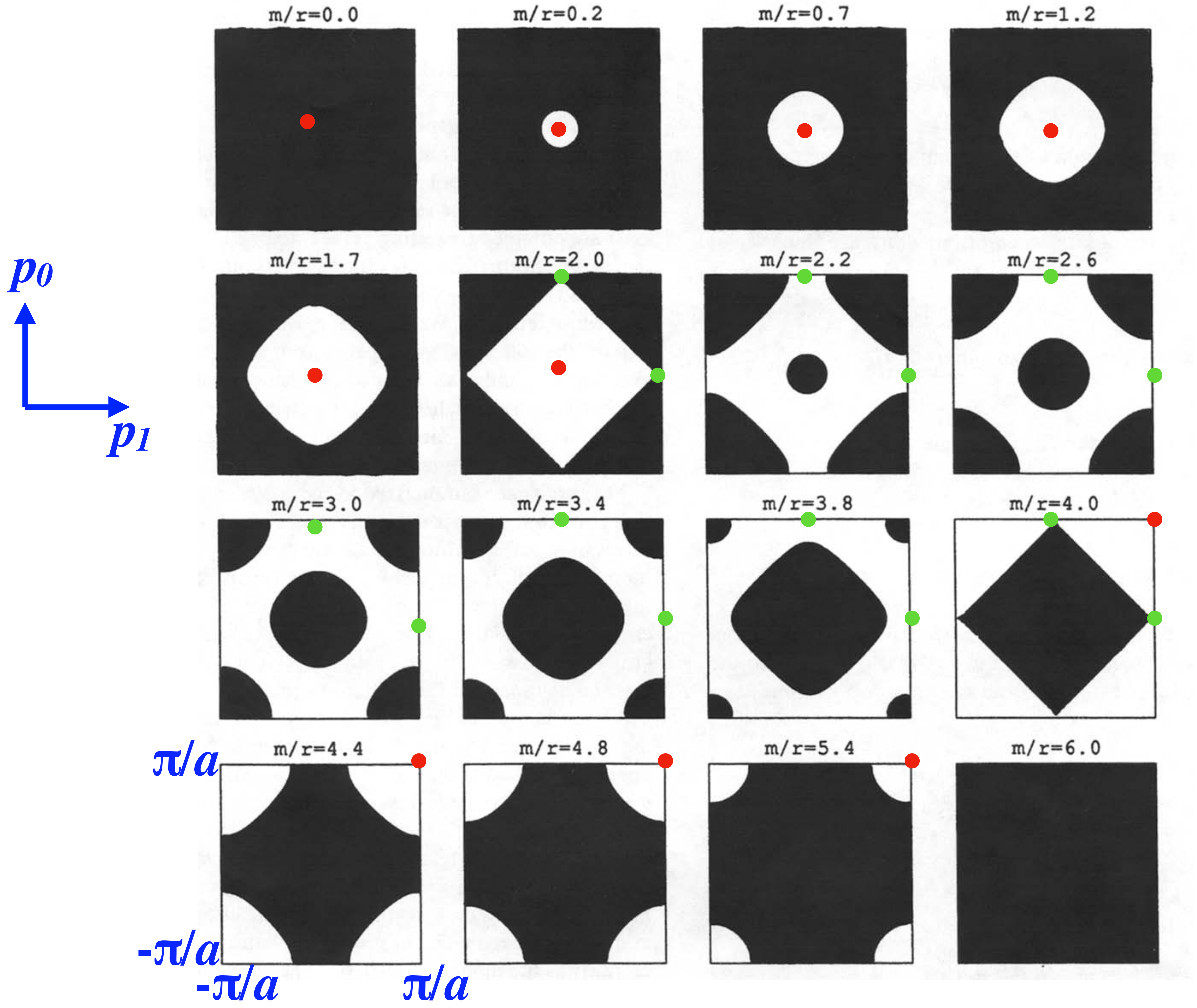
One finds: massless edge states & Chern-Simons currents in the bulk w/o doublers.

Result: a tool for studying QCD with light quarks without additive mass renormalization, but with the chiral anomaly

But… also discovered something peculiar and unexpected:

# of edge states changes abruptly at critical values of M/R = 0,2,4,6.

K. Jansen, M. Schmaltz Physics Letters B 296 (1992) 374–378



The Brillouin zone for edge state on 1+1 d surface, different values of m/r.

m = mass   r = Wilson operator coefficient

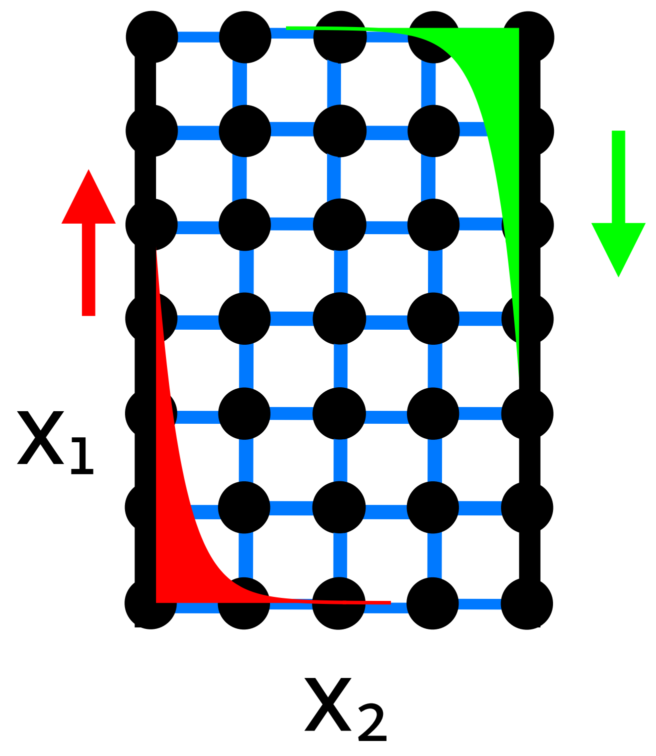
Chiral edge fermion can have momenta in the **WHITE** region

No chiral fermions with momenta in the **BLACK** region

Poles in propagator:

- None for m/r <0
- 1 RH for 0<m/r<2
- 2 LH for 2<m/r<4
- 1 RH for 4<m/r<6
- None for m/r>6

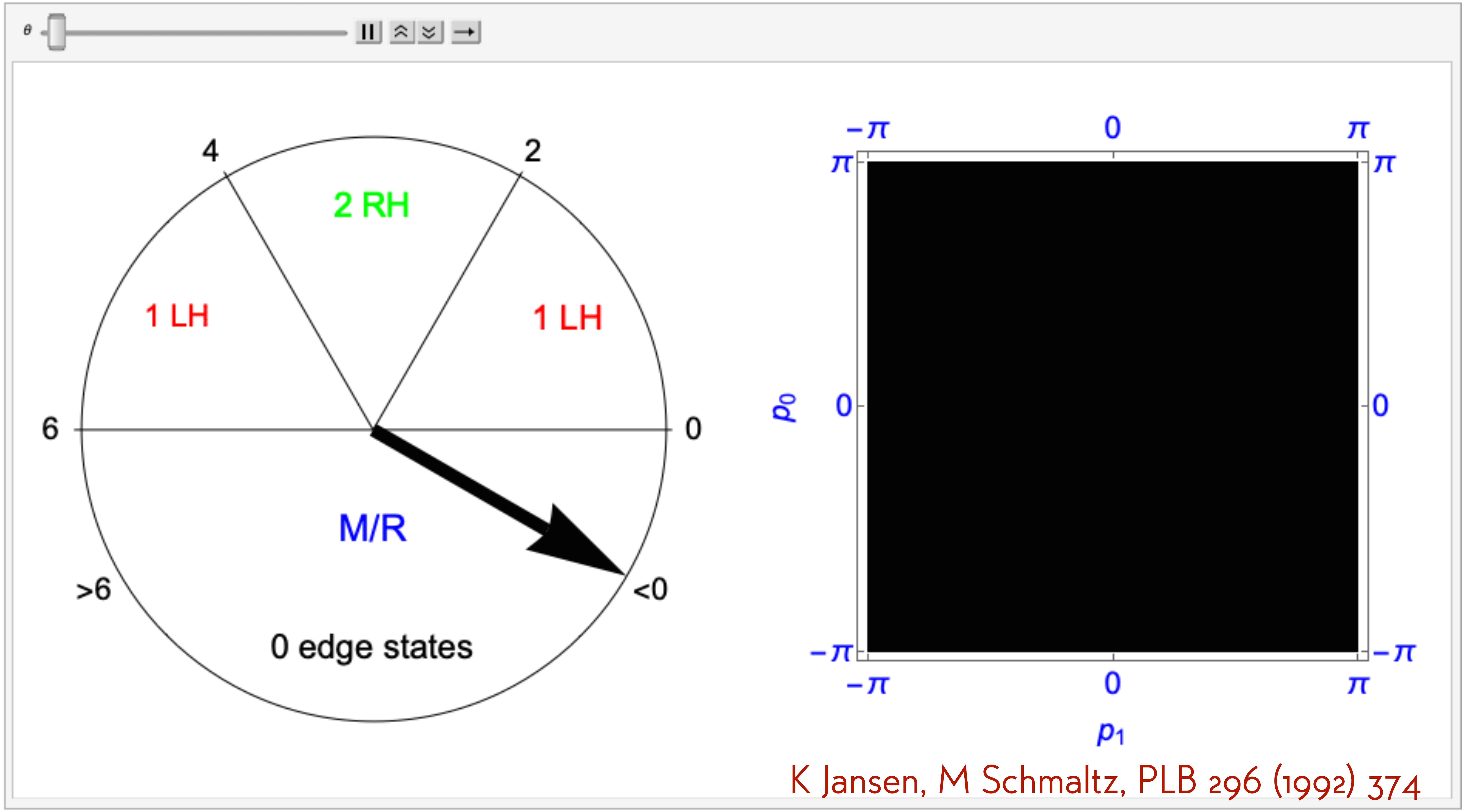
Surprise! dialing mass/Wilson coefficient ratio M/R:



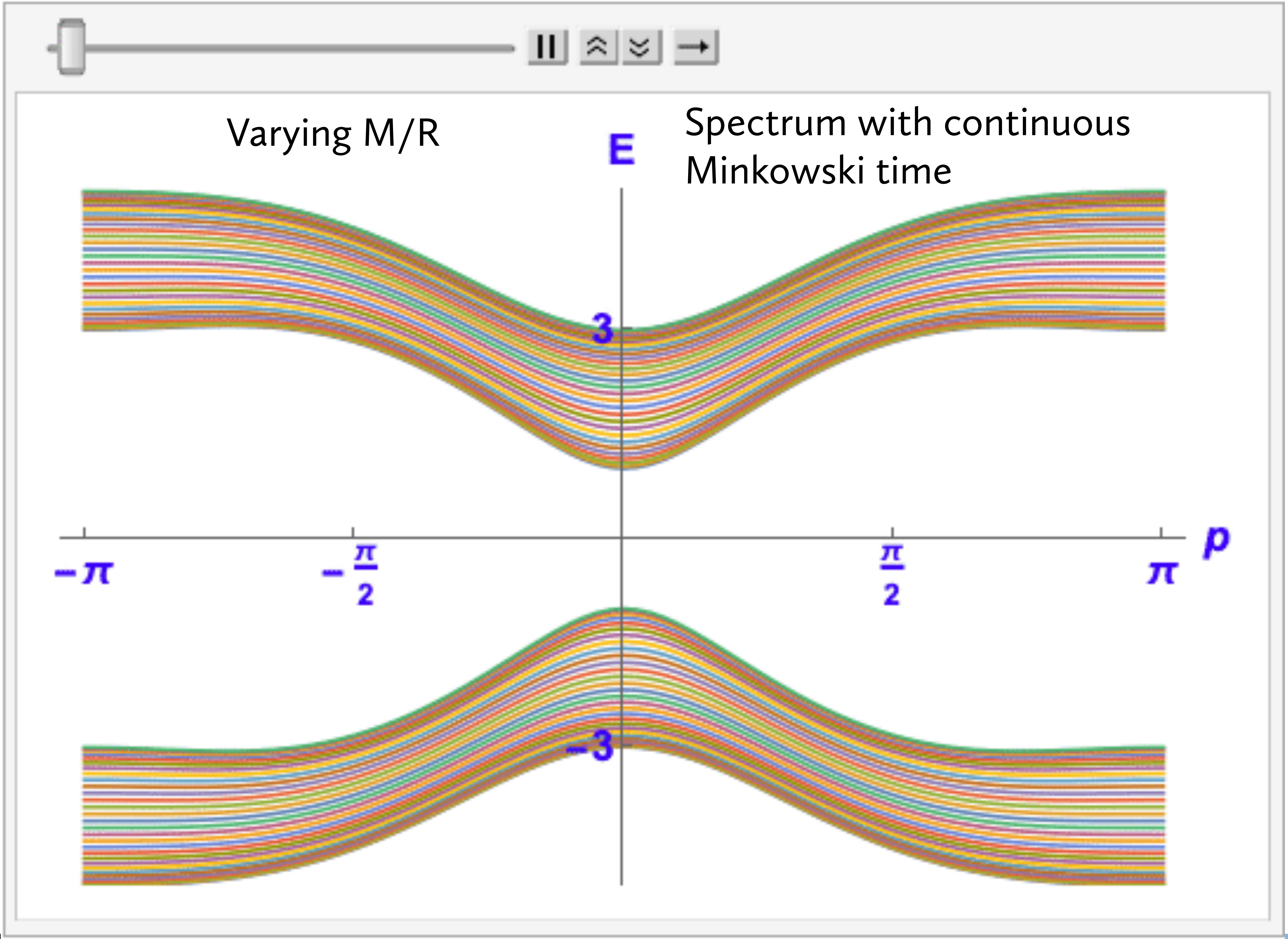
Euclidian $\{p_0,p_1\}$ Brillouin zone

Black= no chiral edge modes ✖

White= chiral edge modes ✓

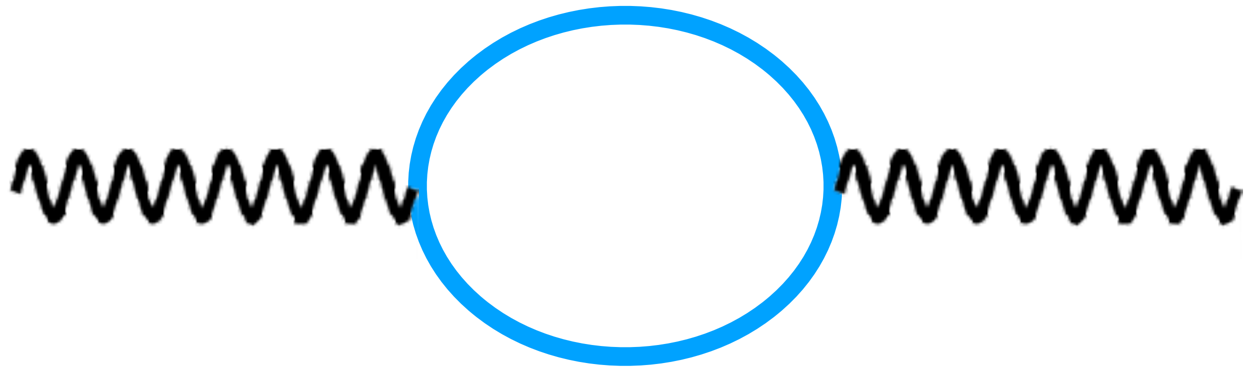


K Jansen, M Schmaltz, PLB 296 (1992) 374


Varying M/R
Spectrum with continuous
Minkowski time
E
3
-3
p
-π
-π/2
π/2
π

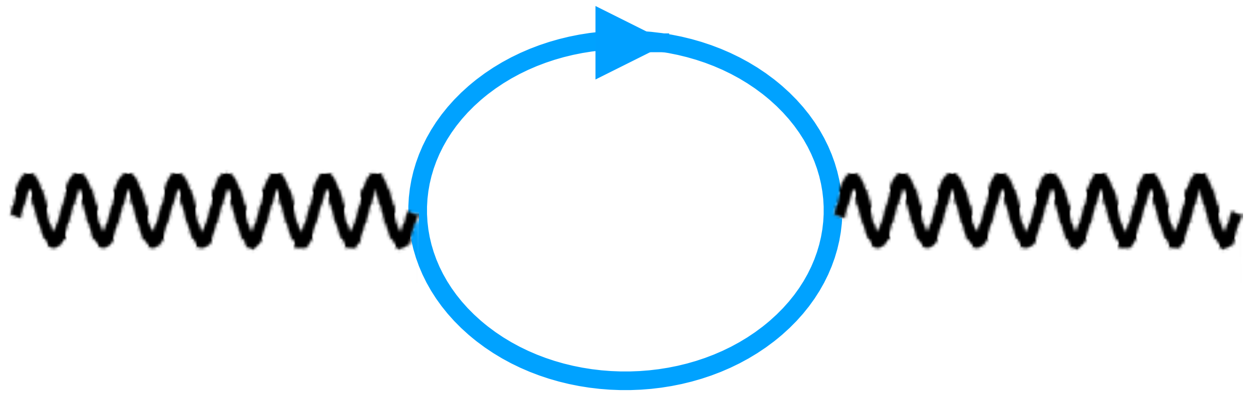
This means that the Chern-Simons current must change abruptly, by integers.

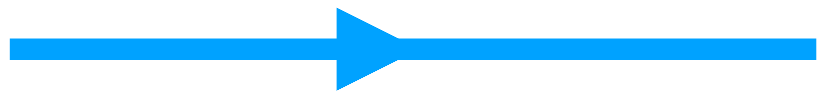
How can a 1-loop Feynman diagram behave like that??



Topology! (Golterman, Jansen, DBK 1992)

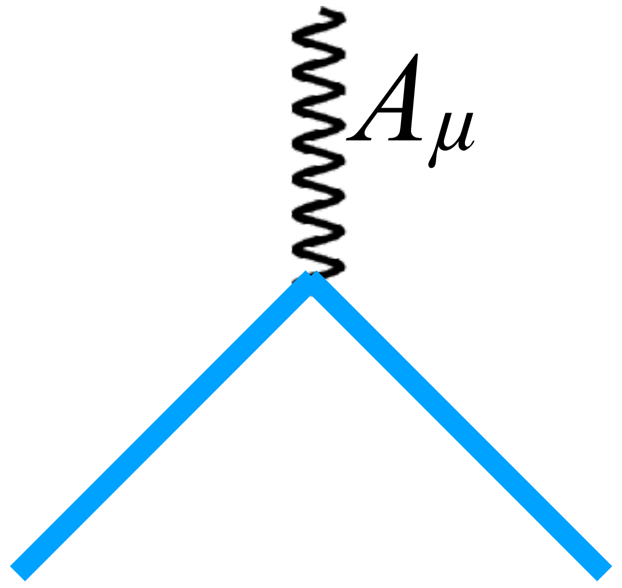
Topology on a lattice?? Topology in a 1-loop Feynman diagram??





p $S(p)$

$$S^{-1}(p) = a(p) + b_\mu(p)\gamma_\mu$$



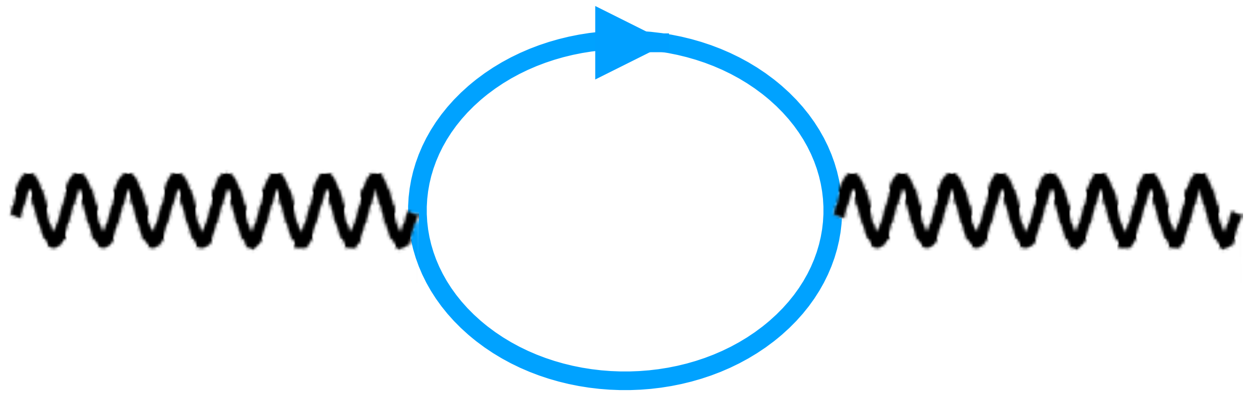
$$\Gamma^\mu = i\frac{\partial S^{-1}(p)}{\partial p_\mu}$$

Gauge invariance relates photon coupling and the fermion propagator

CS coefficient $\propto \epsilon_{\alpha\beta\gamma}\int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}(S\partial_\alpha S^{-1})(S\partial_\beta S^{-1})(S\partial_\beta S^{-1})$

only depends on the (d+1)-dimensional

unit vector: $\hat{n}(p) = \frac{\{a, \mathbf{b}\}}{\sqrt{a^2 + \mathbf{b}\cdot\mathbf{b}}}$



$$\propto \ \epsilon_{\alpha\beta\gamma} \int \frac{d^3p}{(2\pi)^3} \mathrm{Tr}(S\partial_\alpha S^{-1})(S\partial_\beta S^{-1})(S\partial_\beta S^{-1})$$

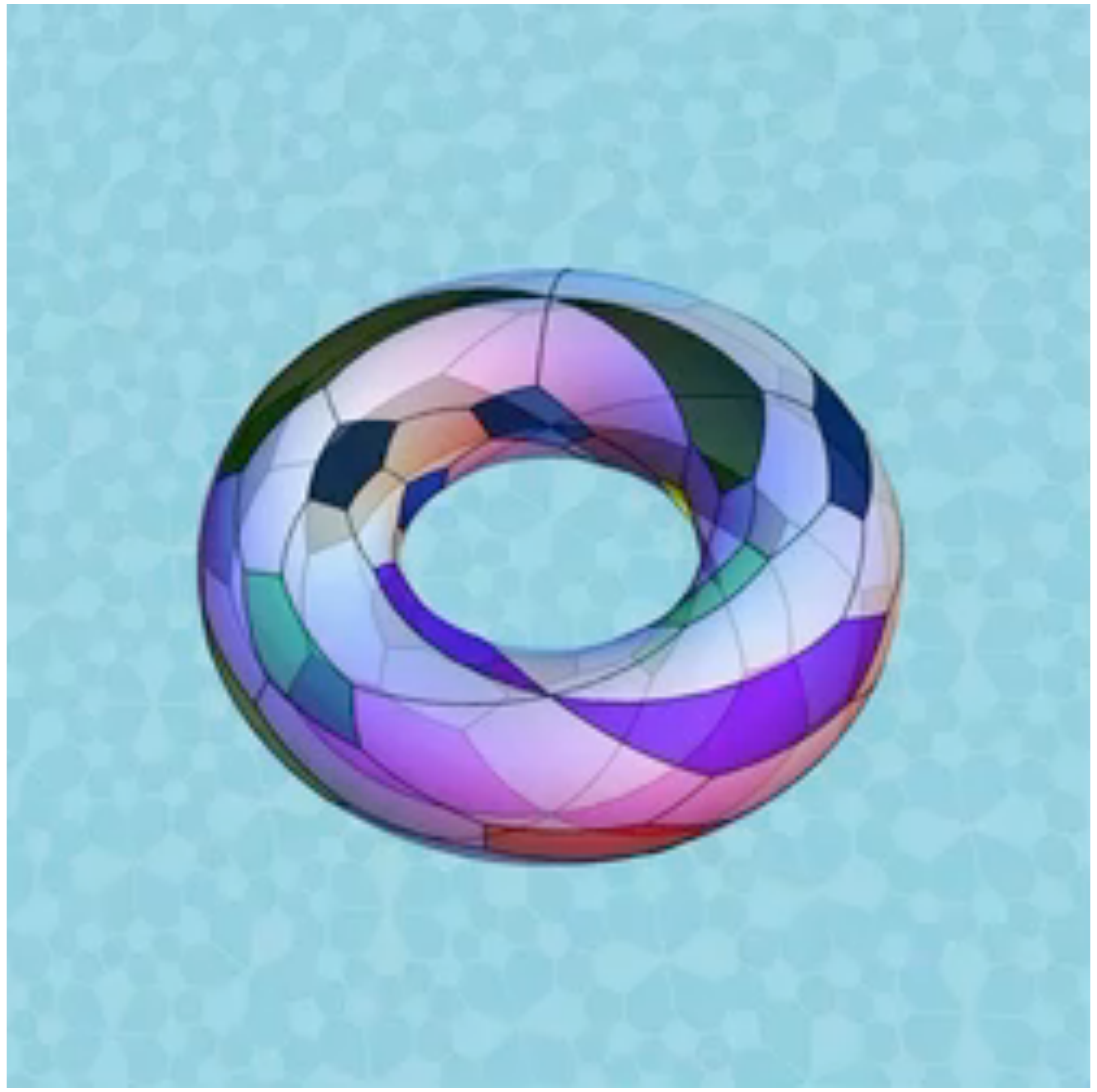
only depends on the 4-component

unit vector: $\hat{n}(p) = \frac{\{a, \mathbf{b}\}}{\sqrt{a^2 + \mathbf{b}\cdot\mathbf{b}}}$

- n(p) = map from momentum space (3-torus!) to “spin” (3-sphere!) for 1+1 dim chiral fermion
- n(p) = map from momentum space (5-torus!) to “spin” (5-sphere!) for 3+1 dim chiral fermion
- **The Feynman diagram computes the winding number of that map**
- The integer winding number jumps when topology is destroyed: the bulk goes gapless, edge state delocalizes
- For Wilson fermions, this occurs at m/r = 0,2,4,6

This is the integer quantum hall effect, QFT version of TKNN - Thouless, Kohomoto Nightingale, den Nijs, 1982)

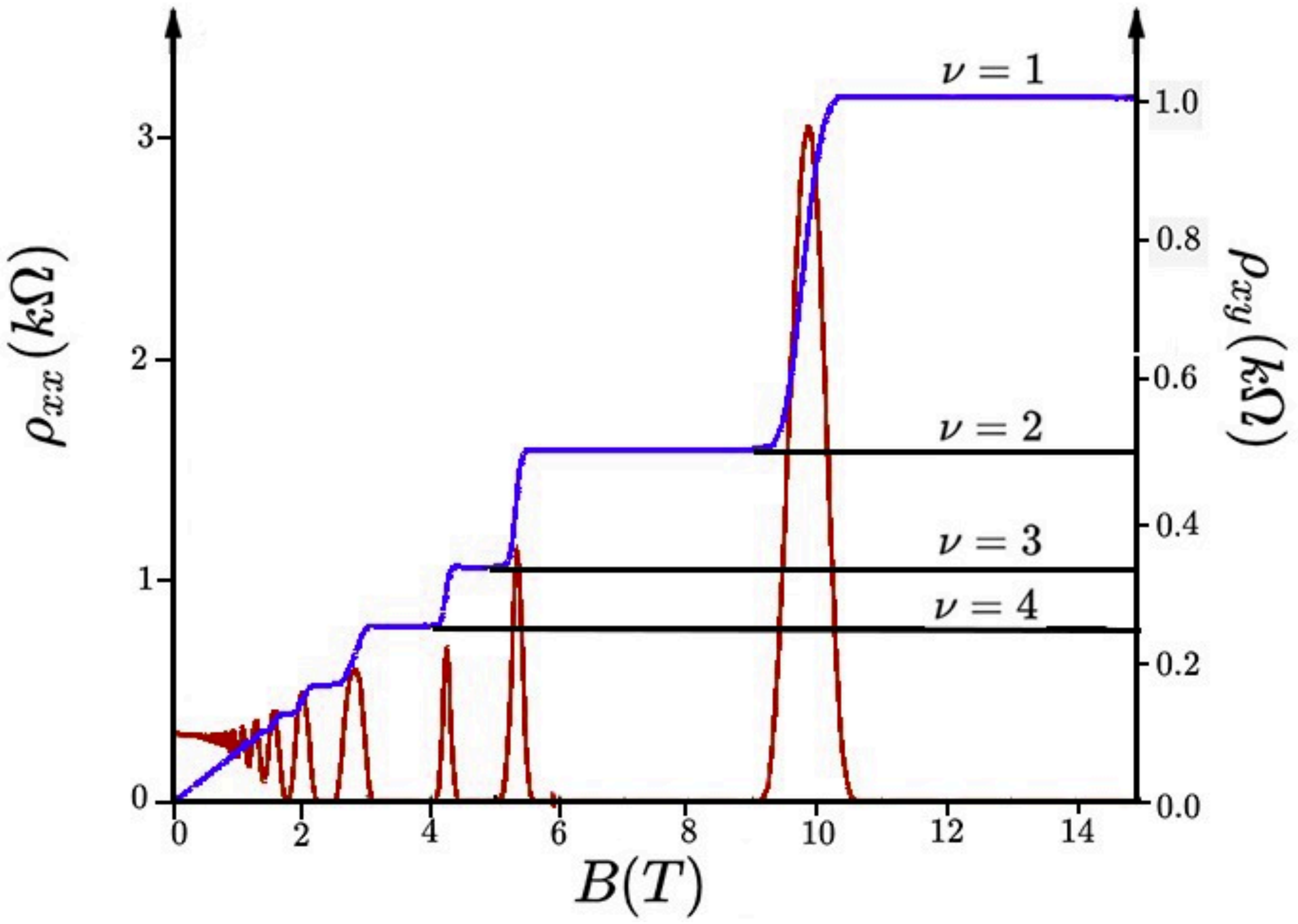
Visualizing a map from 2-torus to 2-sphere with winding number 2:

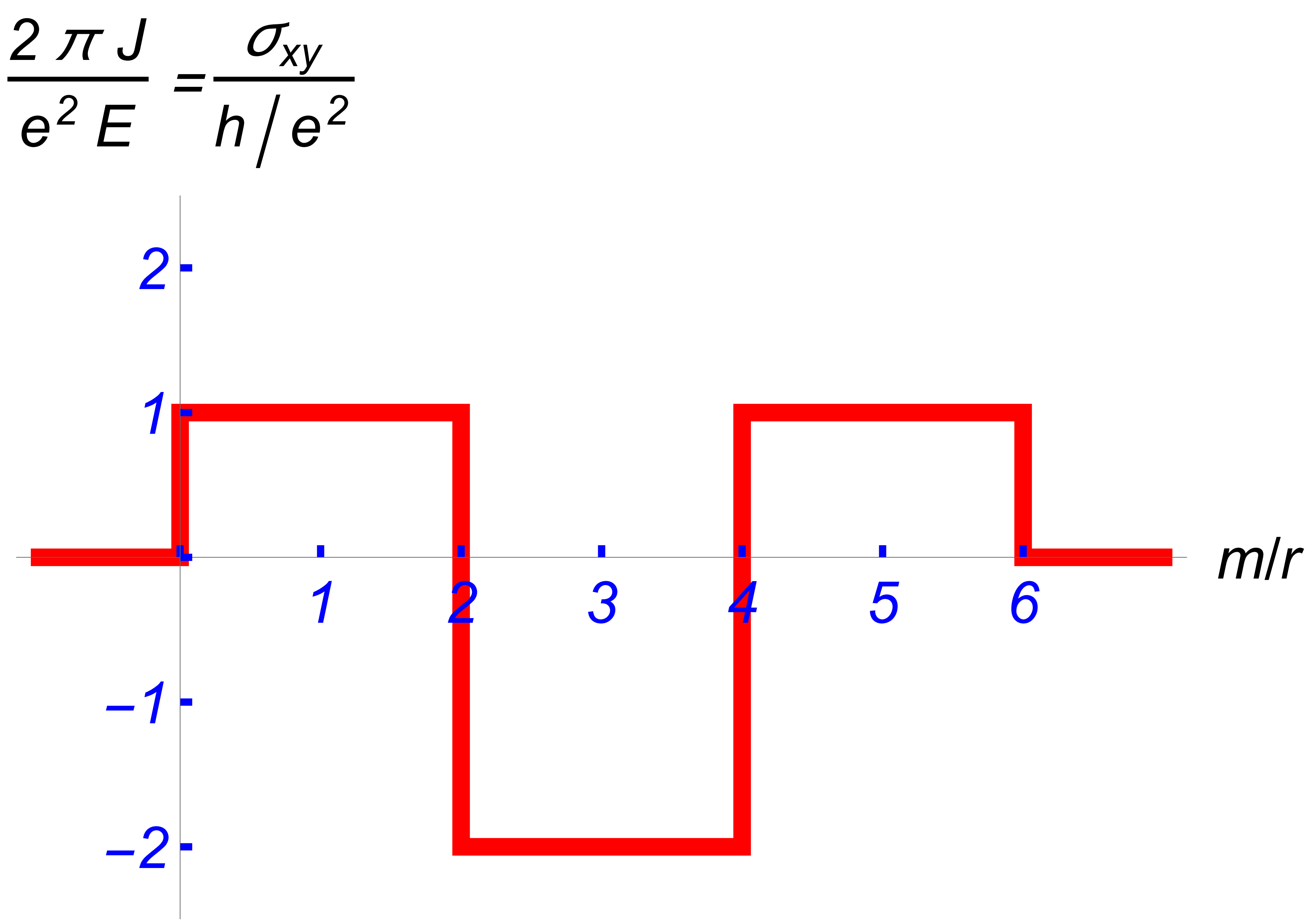


credit: http://www.mathematicaguidebooks.org/soccer/

Resistivity for IQHE

Conductivity for Wilson fermions



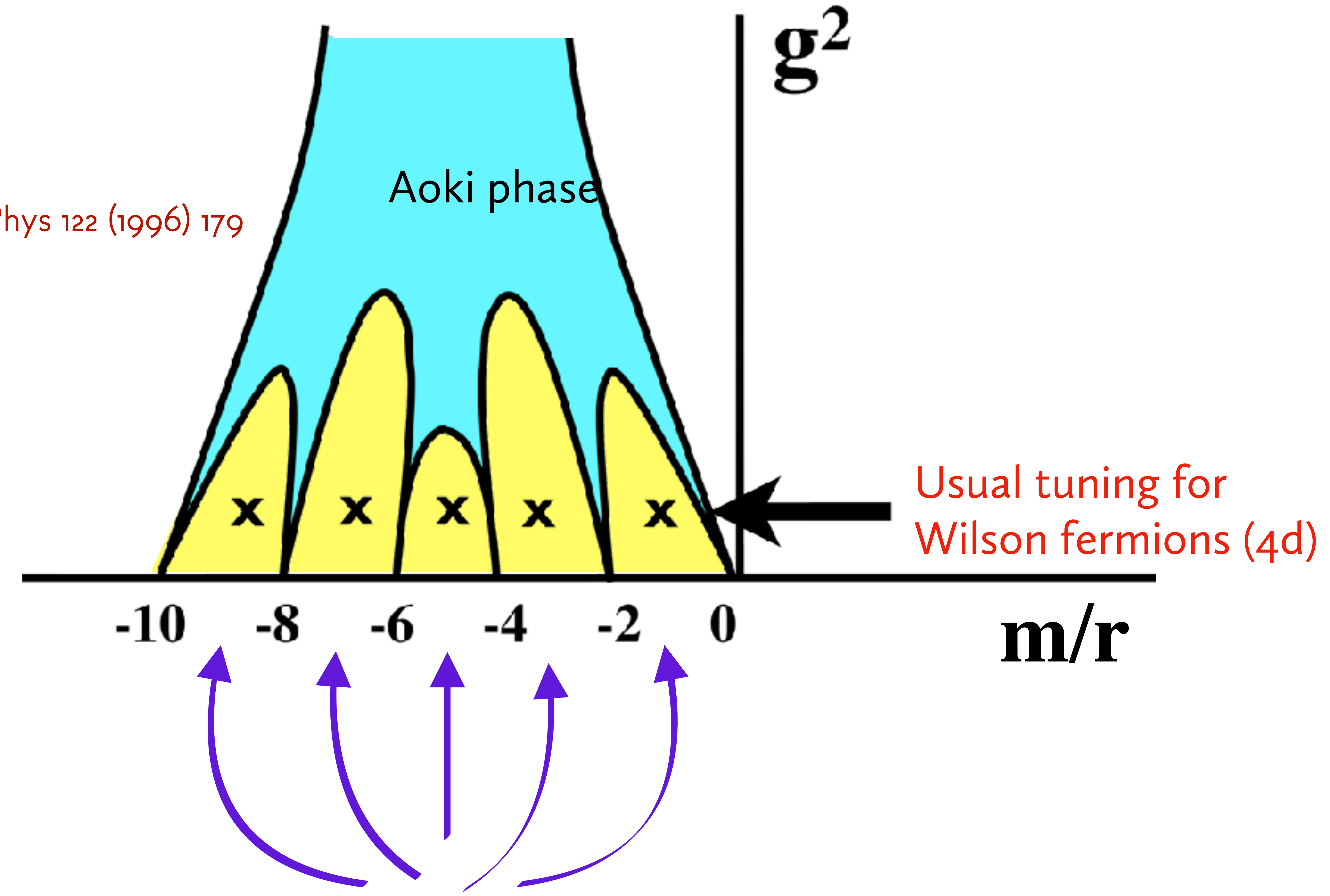


Magnetic field = source of T-reversal symmetry violation

Fermion mass = source of T-reversal symmetry violation

Phase diagram for QCD with Wilson fermions in 5d Euclidian spacetime

S Aoki, Prog Th Phys 122 (1996) 179



Topological phases, corresponding to 1,4,6,4,1 flavors

ANOMALIES AND FERMION ZERO MODES ON STRINGS AND DOMAIN WALLS

C.G. CALLAN, Jr.[1] and J. A. HARVEY[2]
Jadwin Hall, Princeton University, Princeton, New Jersey 08544, USA

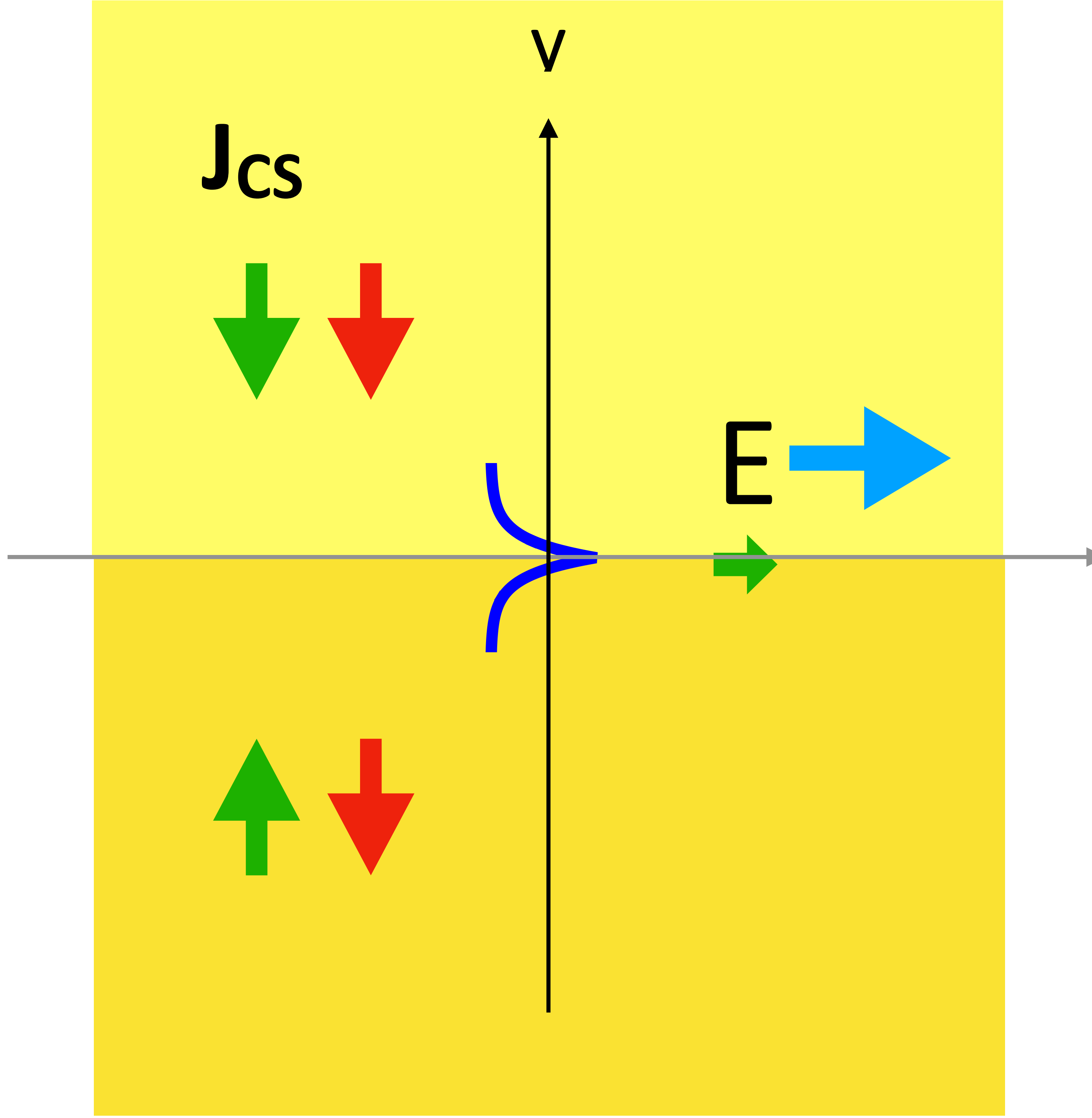
Received 10 September 1984

The current is conserved off the string and its divergence on the string exactly matches the anomaly. It is amusing to note that the geometry of this current flow is like that of the Hall effect: current flows in a direction perpendicular to the applied electric field. ^ Integer Quantum

This omission is a BIG DEAL… Haldane won the 2016 Nobel Prize for his **1988** paper showing that the IQHE only requires time reversal violation, not magnetic fields… all shown in the Callan-Harvey paper. How did they miss this?

1. CM and PT theorists too often see each other as aliens

2. C&H were only interested in the divergence of the current: no regulator = noncompact momentum space = no topology. Regulators do not decouple from CS term! Current off by half integer.



3. If C&H had computed "resistance" =E/(2 J) they would have found 1/2π … = h/e^2 (Von Klitzing quantum of resistance)

4. Their model didn't have any parameter that could render the bulk gapless at critical values, so they couldn't see the resistivity jump (unlike in lattice theories)

Where has the IQHE ⇔ QCD connection gone since the early 1990s?

In the 2000s CMT realized that the IQHE was an example of a large number of examples of topological materials in different dimensions.

Kitaev 2010: 10 topological classes that can can be classified in a periodic table.
Each of these theories has an analog in relativistic theories of free massive fermions with different T, C properties, with Dirac or Majorana masses.

Two interesting avenues to develop the connections?

1. The Ginsparg-Wilson relation and overlap fermions: bulk-boundary correspondence without the bulk

The Ginsparg & Wilson (1983) asked: what does a low energy fermion operator with "perfect" chiral symmetry look like? (eg, only broken by anomalies).
Answer: D satisfies

$$\{\gamma_5, D\} = aD\gamma_5 D$$

a = lattice spacing

$$\{\gamma_5, D^{-1}\}_{xy} = a\gamma_5\delta(x-y)$$

This result was ignored for 14 years because they had no solution to the equation

Neuberger (1997) found 4d lattice solution for QCD
the overlap operator, developed by Narayanan & Neuberger, derived by integrating out bulk modes in 5d domain wall fermion theory

Luscher (1998) showed that a Ginsparg-Wilson operator not only gets the anomaly correctly, but also forbids additive renormalization of fermion mass, mixing of operators with different chiral transformation properties, etc.

Ongoing work shows that analogs of the Ginsparg-Wilson equation and its solutions exist for other classes of topological insulators and superconductors. Can see exotic effects such as discrete time reversal anomaly for lattice Majorana fermions, etc.
(Clancy, DBK, Singh, to appear)

Perhaps this work can be of use for understanding the dynamics of edge states in various real topological materials?

2. New approaches to nonperturbatively regulate chiral gauge theories (eg, the Standard Model)

a) One approach (Eichten-Preskill (1986), Wen (2013), Cenke, Wang, You…)

- Start with vector-like SM as lattice domain wall theory with fermions on one wall, mirror fermions on the other
- use interactions to gap the mirror fermions without breaking gauge symmetry
- Motivated by Fidkowski-Kitaev model (2010) where edge states mod 8 flavors can be gapped, with Z_8 ’t Hooft anomaly for time reversal symmetry

b) Another possible approach (as in Grabowska, DBK 2016):

- generic 4d chiral gauge theory written as 5d theory
- can look 4d if gauge anomalies cancel

New life to idea due to recent advances in CM context: eg, Witten, Yonekura (2020)

Partition function for edge/bulk theory in background gauge field may be written as

$$\sqrt{|\det \not{D}|}\, e^{i\pi\eta(A_\mu)}$$

edge state Dirac fermion

Eta-invariant of bulk Dirac operator with self-adjoint boundary condition (CS term +…)

Possible to compute η on the lattice?

Collapses to local 4d theory when gauge anomalies cancel?

Conclusions

Hidden momentum space topology exists in QCD which connects it to modern condensed matter theories of topological materials

This connection is currently used routinely to simulate QCD when chiral symmetry is important

Technology discovered in lattice QCD might be interesting in CM context?

Technology discovered in topological materials might provide the key to nonperturbatively regulating chiral gauge theories, such as the Standard Model?