Confinement and Dimensional Transmutation

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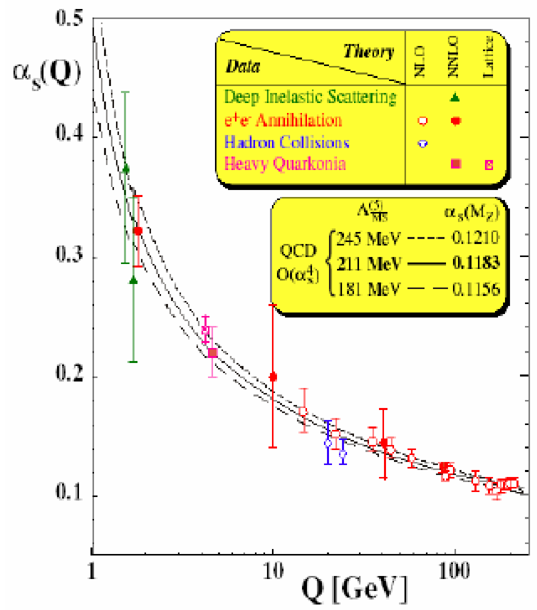
QCD: a perfect quantum field theory

$$\mathcal{L} = \sum_q \bar{\psi}_{q,a}(i\gamma^\mu \partial_\mu \delta_{ab} - g_s \gamma^\mu t^C_{ab} \mathcal{A}^C_\mu - m_q \delta_{ab})\psi_{q,b} - \frac{1}{4} F^A_{\mu\nu} F^{A\,\mu\nu}$$

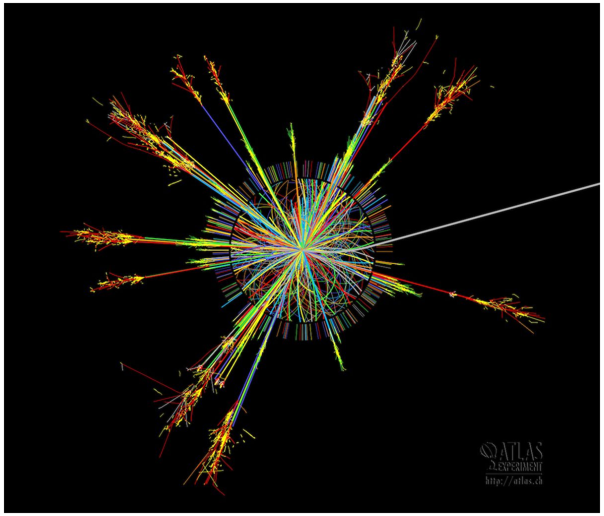
- Becomes weakly coupled at short distances: Asymptotic Freedom
  Gross & Wilczek; Politzer

- It is the part of the Standard Model that can stand alone: it need not be embedded in a bigger theory.
- Admits a lattice definition that matches onto the Asymptotic Freedom while allowing the theory to be defined non-perturbatively at long distances.



The Confinement Problem

- Why are colored quarks and gluons not observed as asymptotic states?
- Confined into colorless hadrons.
- On very short time scales, the propagation of colored objects has been observed in a myriad of high-energy collider events.
- Eventually, they hadronize.
- Only colorless bound states are recorded by the detectors.



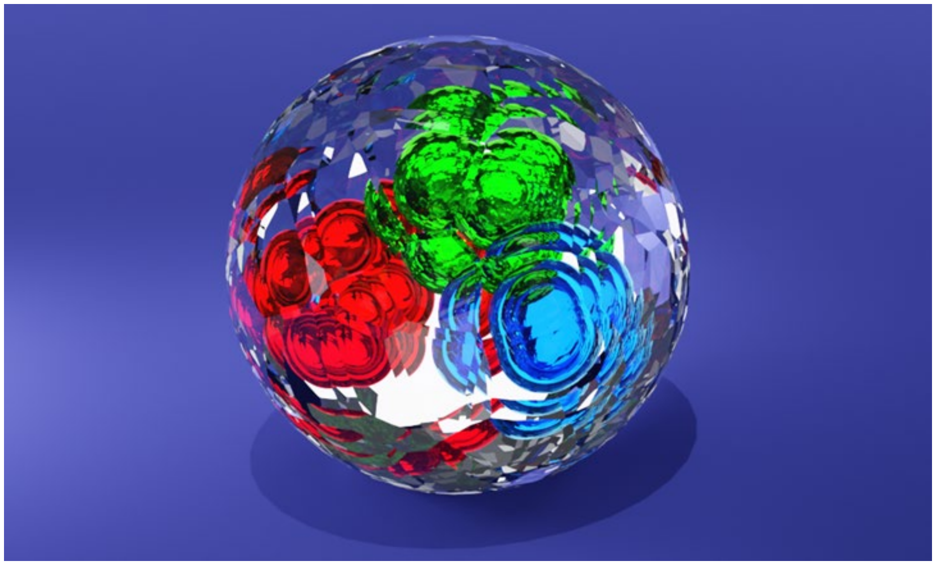
Clay Millenium Problems

Yang-Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Status: Unsolved

Simons Collaboration on Confinement and QCD Strings

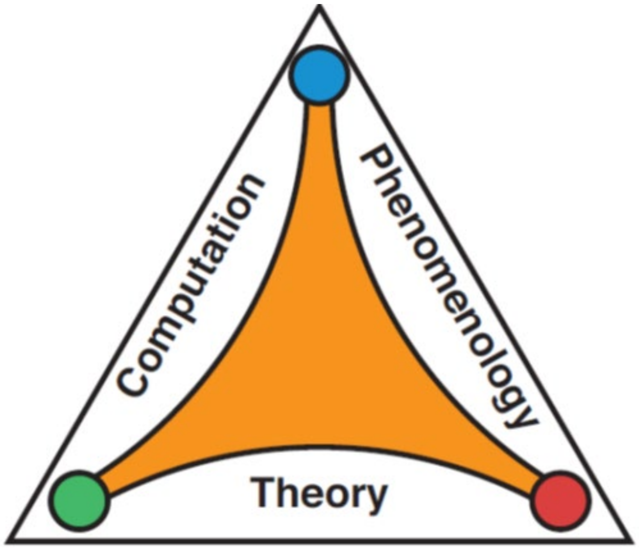


Motivational quote:

"It's about unleashing the strong force..."

From the "Oppenheimer" movie

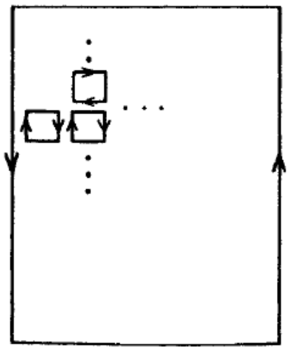
- A always
- B be
- C collaborating



Euclidean Lattice SU(N) Theory

- The gauge field kinetic term is encoded in the plaquette terms.

$$S = -(1/2g^2) \sum_{n,\,\mu\nu} \mathrm{tr}\, U_\mu(n)\, U_\nu(n+\mu)\, U_{-\mu}(n+\mu+\nu)\, U_{-\nu}(n+\nu) + \mathrm{h.c.}$$



- In the strong coupling expansion where these terms are treated as perturbations, the Area Law of the Wilson loop is obvious. $\left\langle \prod_C \exp[iB_\mu(n)] \right\rangle \approx \exp[-F(g^{-2})A]$
- To obtain the continuum limit, one needs to interpolate to the weak coupling limit on lattice scale $\frac{1}{g^2(a)} = \frac{1}{g_0^2} + \frac{b_0}{8\pi^2}\log(a_0/a)$

Dimensional Transmutation

- The beta function of QCD is

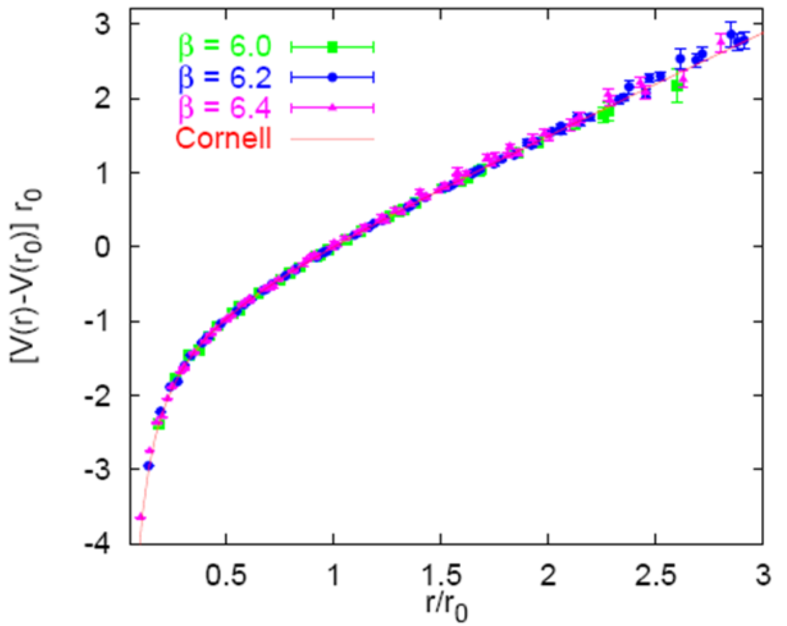
$$\mu \frac{dg}{d\mu} = -b_0 \frac{g^3}{(4\pi)^2} + O(g^5) \ , \qquad b_0 = \frac{11}{3} N - \frac{2}{3} N_f$$

- We need to send the lattice spacing a_0 -> 0 (and the number of lattice points to infinity) keeping the torus size fixed. And simultaneously send g_0 in the lattice action to zero.
- The QCD scale is exponentially small compared to inverse lattice spacing

$$\Lambda_{QCD} \sim a_0^{-1} e^{-(4\pi)^2/(2 b_0 g_0^2)}$$

- The bound state masses and square root of string tension should be pure numbers times this quantity.

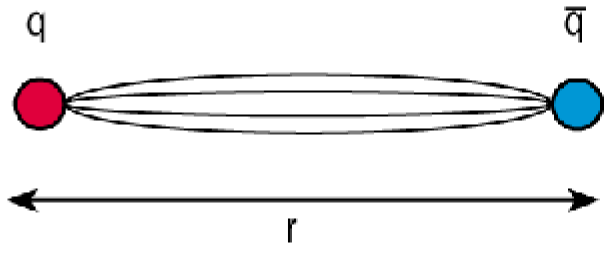
- This is strongly suggested by Monte Carlo simulations starting with those of Wilson; Creutz; … in the mid-70s.
- For example, the quark-antiquark potential looks asymptotically linear.
- But numerically it is hard to take the strict continuum limit.

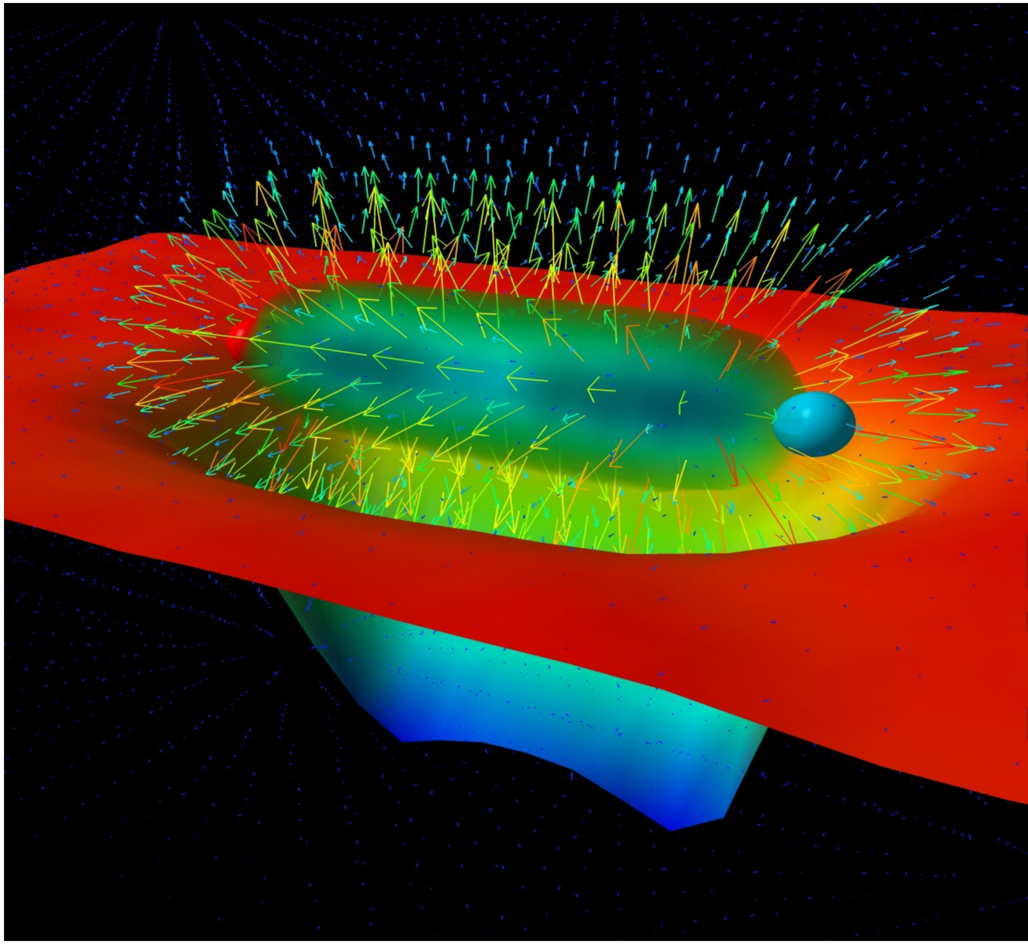


- Can the asymptotic linearity be proven?

Confining Flux Tube

- At distances much smaller than 1 fm, the quark-antiquark potential is nearly Coulombic.
- At larger distances the potential should be linear (Wilson) due to formation of confining flux tubes. Their dynamics is described by the Nambu-Goto area action with corrections.





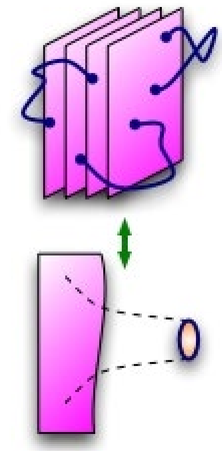
Large N Yang-Mills Theories

- Connection of gauge theory with string theory is strengthened in `t Hooft's generalization from 3 colors (SU(3) gauge group) to N colors (SU(N) gauge group).
- Make N large, while keeping the `t Hooft coupling fixed:

$$\lambda = g^2_{\mathrm{YM}} N$$

- The probability of snapping a flux tube by quark-antiquark creation (meson decay) is 1/N. The string coupling is 1/N.

D-Branes vs. Geometry



- Dirichlet branes led string theory back to gauge theory in the mid-90's. Polchinski
- A stack of N Dirichlet 3-branes realizes $\mathcal{N}$=4 supersymmetric SU(N) gauge theory in 4 dimensions. It also creates a curved background of 10-d theory of closed superstrings

$$ds^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \left(-(dx^0)^2 + (dx^i)^2\right) + \left(1 + \frac{L^4}{r^4}\right)^{1/2} \left(dr^2 + r^2 d\Omega_5^2\right)$$

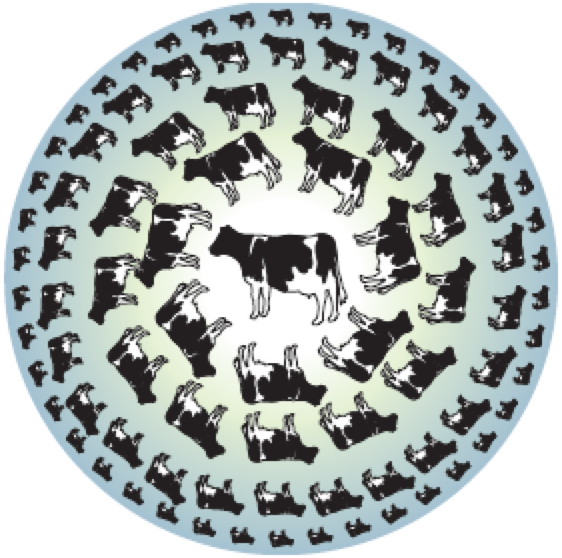
which for small r approaches $AdS_5 \times \mathbf{S}^5$

whose radius is related to the coupling by $L^4 = g_{\rm YM}^2 N \alpha'^2$

The AdS/CFT Duality

Maldacena; Gubser, IRK, Polyakov; Witten

- Relates conformal gauge theory in 4 dimensions to string theory on 5-d Anti-de Sitter space times a 5-d compact space. For the $\mathcal{N}$=4 SYM theory this compact space is a 5-d sphere.
- The geometrical symmetry of the AdS_5 space realizes the conformal symmetry of the gauge theory.
- The AdS space-time is a generalized hyperboloid. It has negative curvature.
- Over 25 years of vigorous research.



- When a gauge theory is strongly coupled, the radius of curvature of the dual AdS_5 and of the 5-d compact space becomes large: $\frac{L^2}{\alpha'} \sim \sqrt{g_{\rm YM}^2 N}$

- String theory in such a weakly curved background can be studied in the effective (super)-gravity approximation, which allows for a host of explicit calculations. Corrections to it proceed in powers of

$$\frac{\alpha'}{L^2} \sim \lambda^{-1/2}$$

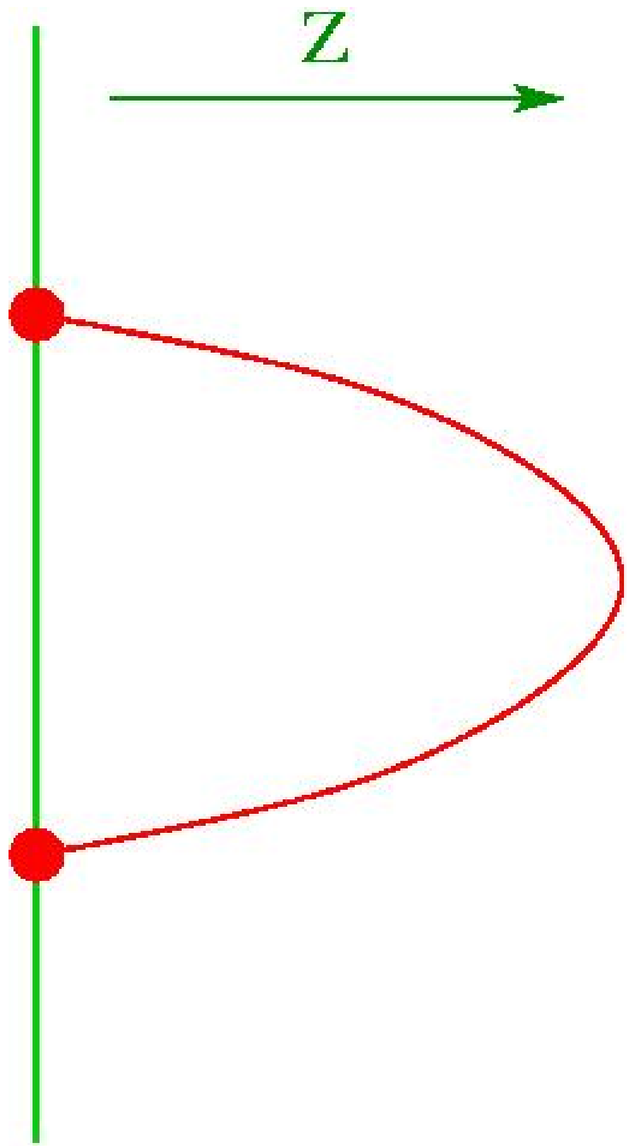
- Feynman graphs instead develop a weak coupling expansion in powers of λ. At weak coupling the dual string theory becomes difficult.

The quark anti-quark potential

- The z-direction of AdS is dual to the energy scale of the gauge theory: small z is the UV; large z is the IR.

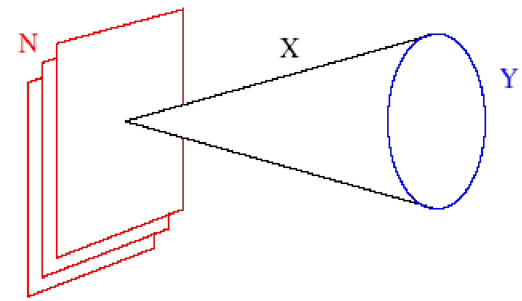
- The quark and anti-quark are placed at the boundary of Anti-de Sitter space (z=0), but the string connecting them bends into the interior (z>0). Due to the scaling symmetry of the AdS space, this gives Coulomb potential Maldacena; Rey, Yee

$$V(r) = -\frac{4\pi^2\sqrt{\lambda}}{\Gamma\left(\frac{1}{4}\right)^4 r}$$



Conebrane Dualities

- To reduce the number of supersymmetries in AdS/CFT, we may place the stack of N D3-branes at the tip of a 6-d Ricci-flat cone X whose base is a 5-d Einstein space Y:



$$ds_X^2 = dr^2 + r^2 ds_Y^2$$

- Taking the near-horizon limit of the background created by the N D3-branes, we find the space AdS_5 x Y, with N units of RR 5-form flux, whose radius is given by

$$L^4 = \frac{\sqrt{\pi}\kappa N}{2\,\mathrm{Vol}(Y)} = 4\pi g_s N \alpha'^2 \frac{\pi^3}{\mathrm{Vol}(Y)}$$

- This type IIB background is conjectured to be dual to the IR limit of the gauge theory on N D3-branes at the tip of the cone X.

D3-branes on the Conifold

- The conifold is a Calabi-Yau 3-fold cone X described by the constraint $\sum_{a=1}^{4} z_a^2 = 0$ on 4 complex variables.
- Its base Y is a coset $T^{1,1}$ which has symmetry $SU(2)_A$x$SU(2)_B$ that rotates the z's, and also $U(1)_R$: $z_a \rightarrow e^{i\theta} z_a$
- The Einstein metric on $T^{1,1}$ is

$$\mathrm{d}s^2_{T^{1,1}} = \frac{1}{9}\Big(\mathrm{d}\psi + \sum_{i=1}^{2} \cos\theta_i \mathrm{d}\phi_i\Big)^2 + \frac{1}{6}\sum_{i=1}^{2}\Big(\mathrm{d}\theta_i^2 + \sin^2\theta_i \mathrm{d}\phi_i^2\Big)$$

where $\theta_i \in [0,\pi], \phi_i \in [0,2\pi], \psi \in [0,4\pi]$

- The topology of $T^{1,1}$ is S^2 x S^3.

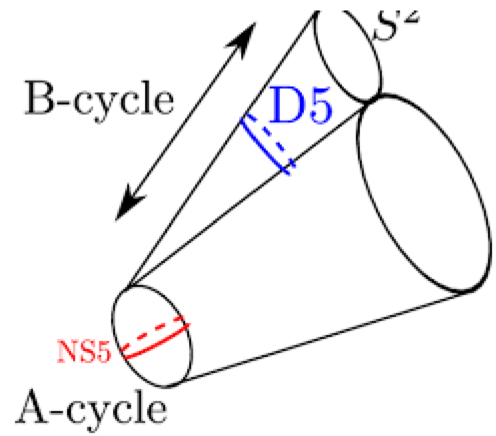
- The $\mathcal{N}$=1 SCFT on N D3-branes at the apex of the conifold has gauge group SU(N)xSU(N) coupled to bifundamental chiral superfields A_1, A_2, in $(\mathbf{N}, \overline{\mathbf{N}})$, and B_1, B_2 in $(\overline{\mathbf{N}}, \mathbf{N})$. IRK, Witten
- The R-charge of each fields is ½. This insures $U(1)_R$ anomaly cancellation.
- The unique $SU(2)_A$x$SU(2)_B$ invariant, exactly marginal quartic superpotential is added:

$$W = \epsilon^{ij}\epsilon^{kl}\,\mathrm{tr}\,A_iB_kA_jB_l$$

- This gauge theory also has a baryonic U(1) symmetry under which $A_k \to e^{ia} A_k$; $B_l \to e^{-ia} B_l$.
- It is dual to type IIB string theory on $AdS_5 \times T^{1,1}$

Confinement and Warped Throat

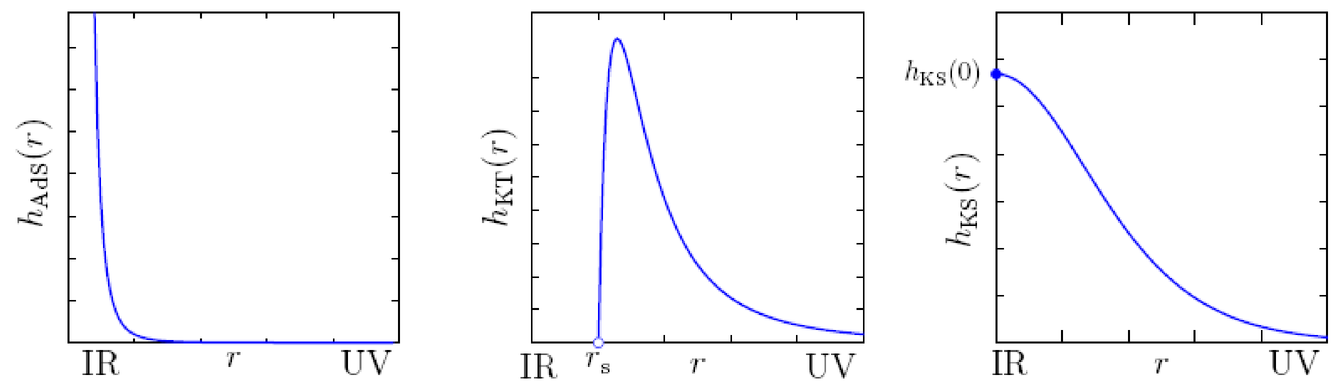
- To break conformal invariance, change the gauge theory: add to the N D3-branes M D5-branes wrapped over the sphere at the tip of the conifold. Gives SU(N+M) x SU(N).
- The 10-d geometry dual to the gauge theory on these branes is the warped deformed conifold (IRK, Strassler)



$$ds_{10}^2 = h^{-1/2}(y)\big(-(dx^0)^2 + (dx^i)^2\big) + h^{1/2}(y) ds_6^2$$

- ds_6^2 is the metric of the deformed conifold, a Calabi-Yau space defined by the following constraint on 4 complex variables:

$$\sum_{i=1}^{4} z_i^2 = \varepsilon^2$$



- Comparison of warp factors in the AdS, warped conifold, and warped deformed conifold cases. The warped conifold solution, which has a naked singularity (IRK, Tseytlin) should be interpreted as asymptotic (UV) approximation to the correct solution.

IR Behavior of the Conifold Cascade

- Here the dynamical deformation of the conifold renders the solution smooth, and explains the IR dynamics of the gauge theory.
- Dimensional transmutation in the IR from the logarithms in the UV. . The dynamically generated confinement scale is $\sim \varepsilon^{2/3}$
- The pattern of R-symmetry breaking is the same as in the SU(M) SYM theory: Z_{2M} -> Z_2
- Yet, the IR gauge theory is somewhat more complicated.

- In the IR the gauge theory is SU(2M) x SU(M). The SU(2M) gauge group effectively has $N_f = N_c$.
- The baryon and anti-baryon operators

$$\mathcal{A} = \epsilon^{i_1 \dots i_{N_c}} A^{a_1}_{\alpha_1 i_1} \dots A^{a_{N_c}}_{\alpha_{N_c} i_{N_c}}$$

$$\mathcal{B} = \epsilon_{i_1 \dots i_{N_c}} B^{i_1}_{\dot{\alpha}_1 a_1} \dots B^{i_{N_c}}_{\dot{\alpha}_{N_c} a_{N_c}}$$

  acquire expectation values and break the U(1) symmetry under which $A_k \to e^{ia} A_k$; $B_l \to e^{-ia} B_l$. Hence, we observe confinement without a mass gap: due to $U(1)_{baryon}$ symmetry breaking there is a Goldstone boson and its massless scalar superpartner.
- Fundamental string at the bottom of conifold is dual to the chromoelectric flux tube.
- Confinement without a mass gap!

- The quark anti-quark potential is qualitatively similar to that found in numerical simulations of QCD (graph shows lattice QCD results by G. Bali et al with $r_0 \sim 0.5$ fm).
- The dual gravity provides a `hyperbolic cow' approximation, i.e. a toy model, for QCD.

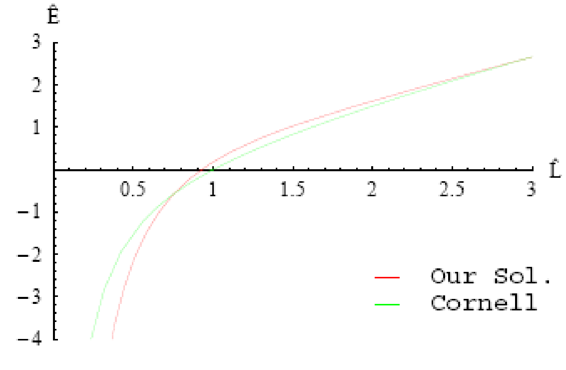
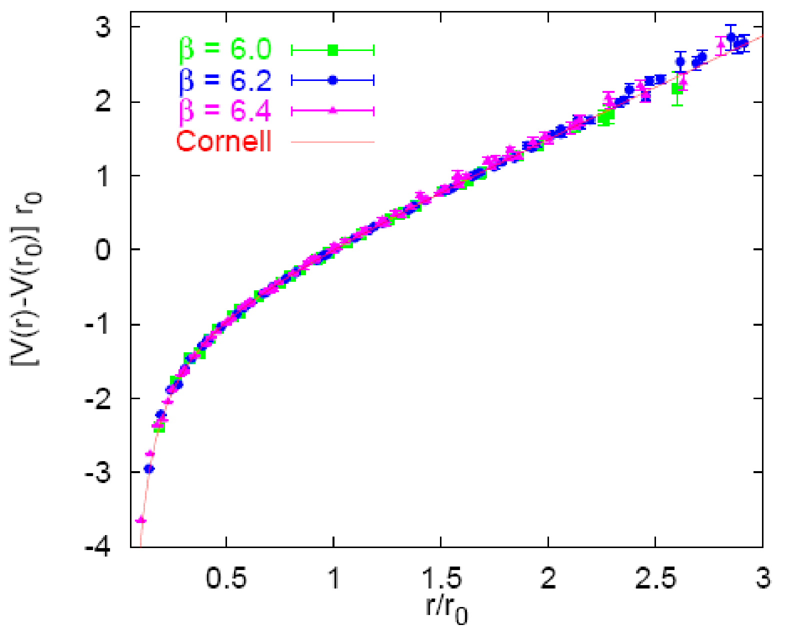


Figure 11: Comparison to the Cornell model



Schwinger Model (1962)

- Quantum Electrodynamics in 1+1 dimensions coupled to a charged fermion of mass m. Admits a theta-angle Coleman, Jackiw, Susskind (1975)

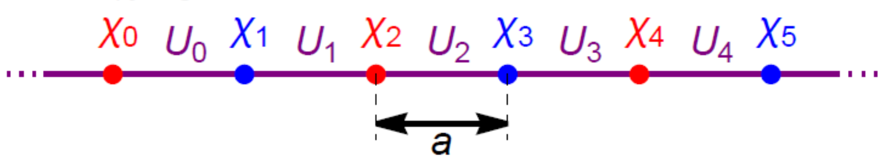
$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\theta}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ - A\!\!\!/ - m)\Psi$$

- Exactly solvable for m=0 where it reduces to the free Schwinger boson of mass $e/\sqrt{\pi} \approx 0.56419e$
- It is the tightly bound state of electron and positron in the linear electrostatic potential.

Lattice Hamiltonian Approach

- Using the staggered fermions Kogut, Susskind, Banks (1975)

$$H = \frac{e^2 a}{2} \sum_{n=0}^{N-1} \left(L_n + \frac{\theta}{2\pi} \right)^2 + m_{\text{lat}} \sum_{n=0}^{N-1} (-1)^n \chi_n^\dagger \chi_n - \frac{i}{2a} \sum_{n=0}^{N-1} \left[\chi_n^\dagger U_n \chi_{n+1} - \chi_{n+1}^\dagger U_n^\dagger \chi_n \right]$$



- The Gauss Law Constraints Hamer, Zheng, Oitmaa (1997)

$$L_n - L_{n-1} = Q_n \,, \qquad Q_n \equiv \chi_n^\dagger \chi_n - \frac{1-(-1)^n}{2}$$

$$[L_n, U_m] = \delta_{nm} U_n \qquad \{\chi_n, \chi_m^\dagger\} = \delta_{nm}$$

- The lattice approach revisited with the surprising result Dempsey, IRK, Pufu, Zan, arXiv: 2206.05308

$$m_{\text{lat}} = m - \frac{1}{8}e^2 a$$

- At m=0 the improved Hamiltonian is preserved by a "discrete chiral symmetry:" shift by one lattice unit accompanied by $\theta \to \theta + \pi$
- The mass shift greatly improves the extrapolation of strong coupling expansions

$$\omega_1 - \omega_0 = 1 + 2\mu + \frac{2x^2}{1+2\mu} - \frac{2(5+2\mu)x^4}{(1+2\mu)^3} + \frac{4(59+68\mu+24\mu^2+4\mu^3)x^6}{(1+2\mu)^5(3+2\mu)}$$

$$\mu = 2m_{\text{lat}}/(e^2 a) \qquad x^2 = y = 1/(ea)^4$$

- In earlier work the massless Schwinger model was assumed to be described by μ=0, and extrapolation to large x did not seem to give good results.
- We instead set $\mu = -1/4$ to obtain

$$\delta\omega = \frac{1}{2} + 4y - 72y^2 + 2224y^3 + O(y^4)$$

- Pade extrapolating to weak coupling we find

$$E_1 - E_0 \approx \left(\frac{19}{188}\right)^{1/4} e \approx 0.56383e$$

- This reproduces the mass of Schwinger boson with error < 0.1 %.

Two-Flavor Schwinger Model

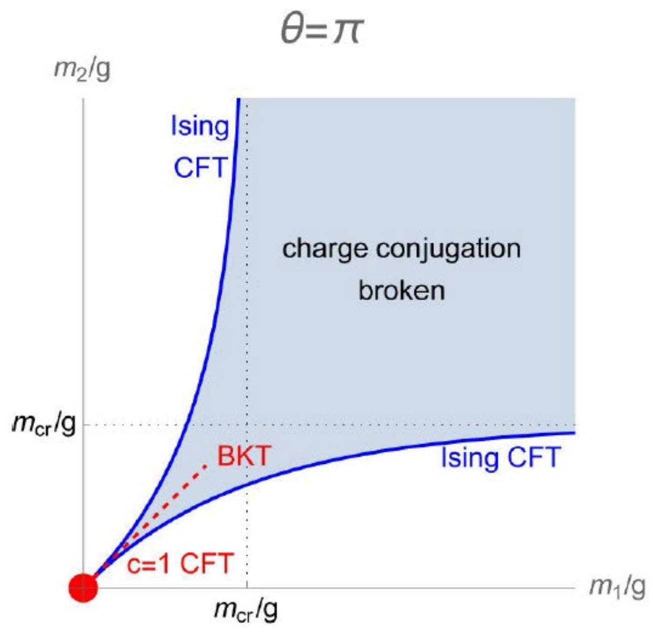
- For m=0 it is a conformal field theory coupled to a massive field. The CFT is a massless compact boson at the self-dual radius where it has SU(2) x SU(2) symmetry.
- Charge conjugation symmetry

$$C : \qquad A_\mu \to -A_\mu \, , \qquad \Psi_\alpha \to \gamma^5 \Psi_\alpha^*$$

- Is preserved by the lagrangian for theta=0 or pi. At $\theta = \pi$ it can be broken spontaneously. Reminiscent of spontaneous CP breaking in QCD. Dashen; Creutz; Gaiotto, Kapustin, Komargodski, Seiberg

Phase Diagram at Zero Temperature

- Our proposal Dempsey, IRK, Pufu, Soegaard, Zan, arXiv: 2305.04437



- In particular, C is broken along the SU(2) invariant line $m_1 = m_2 = m$ where dimensional transmutation takes place for small m/g.

Analogy in 4D

- An analogous phase diagram of 4D QCD as a function of up and down quark masses appeared in Creutz's talk yesterday. See his papers and the original work by Dashen.
- There the shaded region is where the CP violation occurs.

Bosonized 2-flavor model

- Form two combinations of scalar fields

$$\phi_+ = 2^{-1/2}(\phi_1 + \phi_2 + \tfrac{1}{2}\pi^{-1/2}\theta)$$

$$\phi_- = 2^{-1/2}(\phi_1 - \phi_2)$$

$$\mathcal{L}_{\rm bos} = -\frac{1}{4g^2}F_{\mu\nu}^2 - \frac{\phi_+}{\sqrt{\pi}}\epsilon^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu\phi_+)^2 + \frac{1}{2}(\partial_\mu\phi_-)^2$$
$$+\frac{e^\gamma}{\pi}m\mathcal{M}N_{\mathcal{M}}\cos\left[\sqrt{2\pi}\phi_+ - \frac{\theta}{2}\right]N_{\mathcal{M}}\cos\left[\sqrt{2\pi}\phi_-\right]$$

- Integrating out the gauge field, makes the plus field massive. There is also a massless minus field.

- For generic theta, the effective Lagrangian is

$$\mathcal{L}_{\rm eff} = \frac{1}{2}(\partial_\mu \phi_-)^2 + m\mu \frac{e^\gamma}{\pi} \cos\frac{\theta}{2} N_\mu \cos\left[\sqrt{2\pi}\phi_-\right]$$

- The mass term is a relevant operator of dimension ½ which induces RG flow to a theory with mass gap $\sim |m \cos(\theta/2)|^{2/3} g^{1/3}$
- This vanishes for $\theta = \pi$. Could this theory be gapless for small m?! Georgi
- We don’t think so, but the energy gap is non-perturbatively small.

- When $\theta = \pi$

$$\mathcal{L}_{\text{bos}} = \frac{1}{2}(\partial_\mu \phi_-)^2 + \frac{e^{3\gamma} I_s m^2}{8\pi^2 \mu^2} \Big(\mu^2 N_\mu \cos(\sqrt{8\pi}\phi_-) + 2\pi e^{-2\gamma} (\partial_\mu \phi_-)^2 \Big) + O(m^4)\,,$$

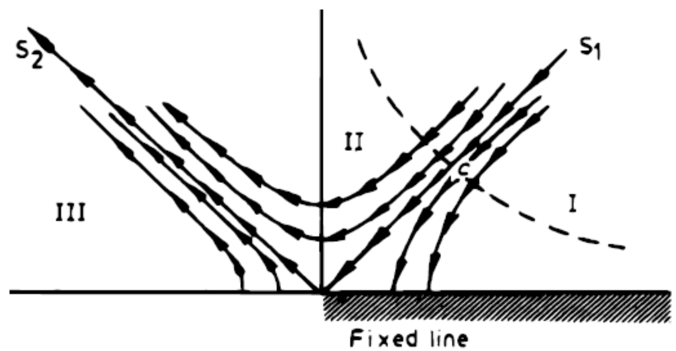
- This was derived by Coleman in 1976, but he did not study the logarithmic RG flow of the two nearly marginal operators.
- This is the Berezinskii-Kosterlitz-Thouless flow in the sine-Gordon model

$$\mathcal{L} = \frac{1-\delta}{2}(\partial_\mu \phi)^2 + \frac{\alpha e^{2\gamma}}{32\pi} \mathcal{M}^2 N_{\mathcal{M}} \cos(\sqrt{8\pi}\phi)$$

- The beta functions are

$$\beta_{\overline{\alpha}} = 2\overline{\alpha}\overline{\delta}\,, \qquad \beta_{\overline{\delta}} = \frac{1}{32}\overline{\alpha}^2$$

- The SU(2) invariant flow is along $\overline{\alpha} = -8\overline{\delta}$

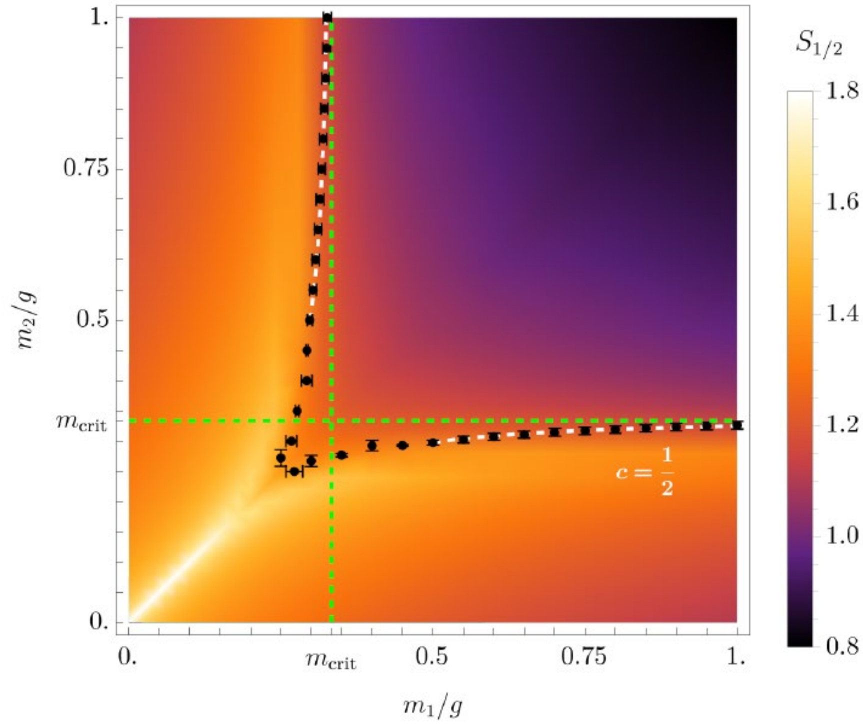


- Starting with the bare values $\alpha = \frac{8e^{\gamma} I_s}{4} \frac{m^2}{g^2} = -8\delta$

$$\beta_{\overline{m}} = M\frac{d\overline{m}}{dM} = -\frac{e^{\gamma} I_s}{4g^2}\overline{m}^3$$

$$E_{\text{gap}} \sim e^{-A\frac{g^2}{m^2}}\,, \qquad A = \frac{2e^{-\gamma}}{I_s} \approx 0.111$$

- This exponentially small scale is analogous to the appearance of Λ_{QCD}
- Numerical evidence in support of the exponentially small gap and our proposed phase diagram is provided by the Entanglement Entropy



Lattice Hamiltonian calculation

- Analogously to the 1-flavor case, we adopt

$$H = \frac{g^2 a}{2} \sum_{n=0}^{N-1} \left(L_n + \frac{\theta}{2\pi} \right)^2 + \sum_{\alpha=1}^{N_f} m_{\mathrm{lat},\alpha} \sum_{n=0}^{N-1} (-1)^n c_{n,\alpha}^\dagger c_{n,\alpha}$$
$$- \frac{i}{2a} \sum_{n=0}^{N-1} \sum_{\alpha=1}^{N_f} \left(c_{n,\alpha}^\dagger U_n c_{n+1,\alpha} - c_{n+1,\alpha}^\dagger U_n^\dagger c_{n,\alpha} \right) .$$

- The mass shift is $m_{\mathrm{lat},\alpha} = m_\alpha - \frac{N_f g^2 a}{8}$
- Very important in the 2-flavor case. For m=0 a discrete chiral symmetry is preserved. It is the lattice translation by one site.

$$\theta = 0$$

- For m<<g, the low-energy dynamics is governed by the sine-Gordon with $\beta = \sqrt{2\pi}$
- Four light particles with masses $\sim m^{2/3} g^{1/3}$
- An SU(2) triplet of pseudoscalar pions followed by a singlet sigma meson.
- The ratio of masses is $\sqrt{3}$ Coleman
- A recent Hamiltonian calculation including the mass shift reproduces this ratio well. Itou, Matsumoto, Tanizaki; arXiv:2307.16655

Discussion

- For $\theta = \pi$ and m<<g, the low-lying bound states have exponentially small masses compared to the UV scale g.
- Calculating them numerically is an interesting challenge.
- The lattice Hamiltonian approach with Tensor Network methods gives precision results in 1+1 dimensions.
- Worth revisiting also in 2+1 and 3+1.
- Connection with quantum simulation and computation.