Applications of Topology in Yang-Mills Theory and QCD

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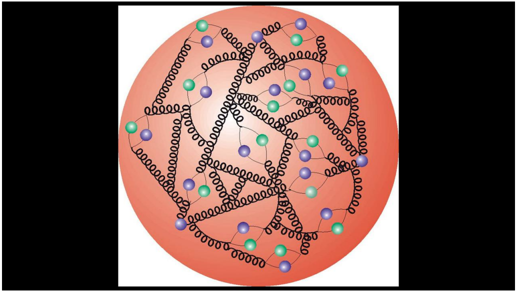


HAPPY
50th
BIRTHDAY!

This presentations is based on
[D.Gaiotto, A.Kapustin, ZK, N.Seiberg],
[ZK, T.Sulejmanpasic, M.Unsal]
[D.Gaiotto, ZK, N.Seiberg].
[ZK, K. Ohmori, K. Roumpedakis, S. Seifnashri]

Gauge theories with gauge group $SU(N)$ and adjoint matter admit twisted boundary conditions, as emphasized by 't Hooft. (I will review it quickly.)

A Property of Electric and Magnetic Flux in Nonabelian Gauge Theories

Gerard 't Hooft (Utrecht U.)
Jan, 1979

20 pages
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In recent years these were re-analyzed in detail in a new language [Gaiotto, Kapustin, Seiberg, Willett]. In combination with a better understanding of topology and anomalies this allowed to derive many new testable preditions.

Twisted Boundary Conditions: A Lightening Reminder

It is helpful to think about $U(1)$ gauge theory with particles of charge $q_i \in \mathbb{Z}$. A standard fact that goes back to Dirac is that

$$\int_{\mathbb{T}^2} F = 2\pi n \ , \quad n \in \mathbb{Z} \ .$$

When we study the theory on, say, $\mathbb{T}^4$ or $S^2 \times \mathbb{T}^2$ we have to sum over these magnetic fluxes.

Now assume $q_i \in N\mathbb{Z}$, i.e. all the dynamical charges are multiples of N. Our prescription is still to sum only over configurations with

$$\int_{\mathbb{T}^2} F = 2\pi n \ , \quad n \in \mathbb{Z}$$

since the allowed bundles are fixed by the gauge group and not by the representations for the matter fields.

't Hooft said that instead we can define a family of partition functions,

$$Z_{m_1,m_2} \ ,$$

with $m_{1,2} \in \mathbb{Z}_N$ where

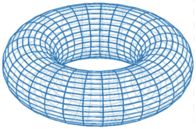
$$\int_{\mathbb{T}^2_i} F = 2\pi \frac{m_i}{N} + 2\pi n \ , \quad n \in \mathbb{Z} \ ,$$

Since $SU(N)$ Yang-Mills theory (with or without additional adjoint matter) essentially only has charges which are multiples of N, we again have parameters $m_i \in \mathbb{Z}_N$ which are called "center vortices," or "" t Hooft fluxes," associated with any two-cycle in the four-dimensional space.

A Modern Point of View

Center vortices are some sort of twisted boundary conditions, and twisted boundary conditions are associated to chemical potentials, and chemical potentials are associated to symmetries.

This is the $\mathbb{Z}_N$ one-from (center) symmetry which is associated with a discrete conserved unitary defined on a two-dimensional space.



If we insert this surface in the 1-2 directions, then we have a 't Hooft vortex in the 3-4 directions. The gauge field that is associated to this symmetry is a two-form $\mathbb{Z}_N$ gauge field,

$$B_{\mu\nu} \ .$$

We can therefore re-interpret 't Hooft's modified partition functions Z_{m_1,m_2} in a more conventional way, as corresponding to turning on chemical potentials m_i for the gauge field $B_{\mu\nu}$.

Here we see an important issue. As always in QFT, partition functions are defined up to local counter-terms. One can make local counter-terms from the gauge field B:

$$2\pi i \frac{k}{2N} \int_{\mathcal{M}_4} B \cup B \ , \quad k \in \mathbb{Z}_{2N} \ .$$

Their mere influence on the partition functions Z_{m_1,m_2} is

$$Z_{m_1,m_2} \to e^{\frac{2\pi ik}{N} m_1 m_2} Z_{m_1,m_2}$$

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Note: even though the counter-term is labeled by $k \in \mathbb{Z}_{2N}$, the $\mathbb{T}^4$ partition function is only sensitive to $k \mod N$. This is a general feature on spin manifolds and from now on we will assume $k \in \mathbb{Z}_N$.

The only dimensionless coupling in Yang-Mills theory is the theta angle

$$\frac{i\theta}{8\pi^2}\int d^4x Tr(F \wedge F)$$

New insights have been obtained due to the following fact. It turns out that

$$\theta \to \theta + 2\pi$$

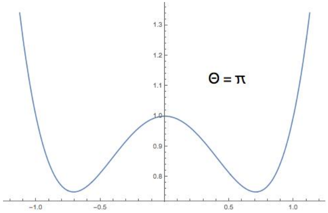
leaves *ALMOST* everything invariant. But the counter-term k jumps as

$$k \to k + 1$$

Therefore in the presence of center vortices the partition function of Yang-Mills theory does not have $\theta \to \theta + 2\pi$ periodicity. Instead it jumps by a phase.

This leads to interesting conclusions. If for every θ the vacuum was trivially gapped and confined then changing θ could not lead to a discontinuous jump in the phase. Further, it can be shown that something special has to happen at $\theta = \pi$.

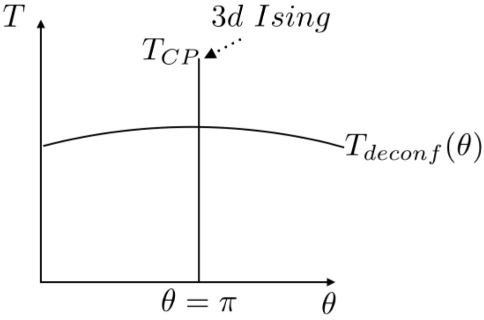
This leads to a proposal that Yang Mills theory at $\theta = \pi$ has two degenerate confined vacua (related by CP). This is certainly true for large enough N and it may or may not be true already for $N = 2$.



The order parameter is the condensate $\langle Tr(F \wedge F) \rangle$.

At high temperatures the CP symmetry is restored but if it were to be restored in the confining phase we would again have a contradiction with $k \to k+1$.
Therefore, at sufficiently large N (probably $N \geq 3$) we expect the following phase diagram for Yang-Mills theory:

$$T_{CP} \geq T_{deconf}(\theta = \pi)$$

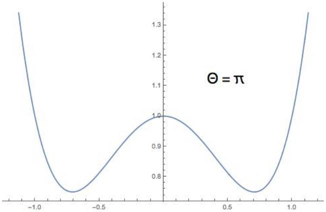


- In holographic models, $T_{CP} = T_{deconf}(\theta = \pi)$ and the story checks out. I do not know why the two transitions coincide and if we should expect this to be true in pure Yang-Mills theory, too.
- We can repeat the same story for the theory with one adjoint Weyl fermion λ_α, i.e. $\mathcal{N} = 1$ SYM theory. Then we would conclude that

$$T_{axial} \geq T_{deconf}$$

with T_{axial} stands for the temperature of the $\lambda \to e^{\pi i k/N}\lambda$ chiral symmetry restoration. It seems that lattice simulations support it [F. Karsch and M. Lutgemeier].

The two vacua at $\theta = \pi$ are confined and gapped. They are pretty boring.
What happens if we try to interpolate between these two vacua?



We get a domain wall, where on one side we have the first vacuum and on the other side we have the second vacuum. The domain wall cannot be trivial.

While $2\pi i \frac{k}{2N} \int_{\mathcal{M}_4} B \cup B$ is a sensible term on closed four-dimensional spaces, when we have a boundary, like the Chern-Simons term, it is not gauge invariant. For the Chern-Simons term, this famously leads to the edge modes in the IQHE.

Here, the jump from $k = 0$ to $k = 1$ implies that the domain wall cannot support confined external quarks. Unlike the external quarks in the bulk, the external quarks are deconfined(!) on the wall and furthermore the exeternal quarks become anyonic(!) even though their original spin is 0 or $1/2$. For example, the spin of the fundamental external quark is now $\frac{1}{2N} \mod \frac{1}{2}$.

A minimal conjecture consistent with the jump $k \to k+1$ is that the domain wall carries an Abelian fractional quantum Hall state

$$U(1)_N \ .$$

The Wilson lines are all deconfined.

Many additional applications of such ideas have been discussed in the literature. Consider QCD with one fundamental quark, $N_f = 1$.

$$\mathcal{L} = \mathcal{L}_{YM} + \bar{\Psi}\gamma^\mu D_\mu \Psi + \overline{\tilde{\Psi}}\gamma^\mu D_\mu \tilde{\Psi} + M\tilde{\Psi}\Psi + c.c.$$

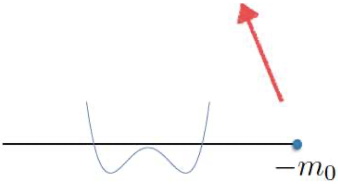
We will take $M \in \mathbb{C}$ so that the theta angle can be ignored.

This theory clearly has no center vortices and no m_i. For M real the theory has CP symmetry. For $M \to +\infty$ it is reduced to YM with $\theta = 0$ and for $M \to -\infty$ it is reduced to YM with $\theta = \pi$.

Since $M = me^{i\theta}$ is a complex parameter the 1st order line has to terminate at a massless point. We conjecture that it has one massless pseudo scalar there

$me^{i\theta}$

$$\frac{1}{2} f_{\eta'}^2 \left[(\partial \eta')^2 - (m - m_0)\Lambda \eta'^2 \right] + \frac{\chi \Lambda^4}{24} \eta'^4$$



We see that one flavor QCD has a chiral Lagrangian of a sort. There is a point with an exactly massless pseudo-scalar, η'. This does not happen for larger N_f.

There is nothing special at $M = 0$ and tuning to that point is quite meaningless.

For $M < -m_0$ we have a domain wall between the two CP breaking vacua. For large negative M it has to be the TFT $U(1)_N$ since toplogical field theories cannot be destabilized easily.

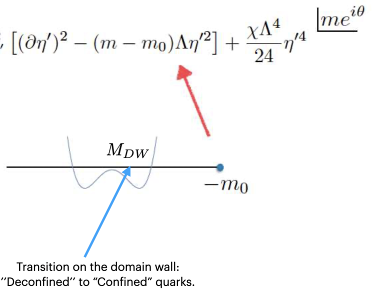
For $M + m_0 \ll \Lambda$ the domain wall is that of a η'^4 theory and hence trivial.

Therefore there is yet another special point in the phase diagram, $M_{DW} < -m_0$ such that at M_{DW} the domain wall theory changes from $U(1)_N$ to a trivial theory. A model in 2+1 dimensions on the domain wall that achieves that is

$$\frac{N}{4\pi} ada + |D\phi|^2 - (M - M_{DW})|\phi|^2 + |\phi|^4 \ .$$

Domain wall theories in QCD therefore exhibit interesting 3d dynamics with non-Landau-Ginzburg transitions.

$$\frac{1}{2} f_{\eta'}^2 \left[(\partial\eta')^2 - (m - m_0)\Lambda\eta'^2\right] + \frac{\chi\Lambda^4}{24}\eta'^4$$



Finally, let us consider adjoint QCD in 1+1 dimensions: a single Majorana fermion in the adjoint representation of $SU(N)$

$$\mathcal{L} = \mathrm{Tr}\left[\frac{1}{4g^2} F \wedge \star F + i\Psi^T \gamma^\mu D_\mu \Psi + M \Psi^T \Psi \ .\right]$$

Symmetries:

- At $M = 0$ we have an axial $\mathbb{Z}_2$ symmetry. It acts by

$$\mathbb{Z}_2 : \Psi_+ \to -\Psi_+$$

$$\mathbb{Z}_2 : \Psi_- \to \Psi_-$$

- For all M we have a center (one-form) $\mathbb{Z}_N$ symmetry. Inserting the charge operator (which is a point operator) creates a center vortex.

This model was discussed in the 90s [Gross, Klebanov, Matytsin, Smilga, Hashimoto...]

We can deform the two dimensional adjoint QCD model by four-fermion interactions. There are two quartic interactions that do not break the chiral symmetry and do not break charge conjugation symmetry:

$$O_1 = Tr(\Psi_+\Psi_+\Psi_-\Psi_-) \ , \quad O_2 = Tr(\Psi_+\Psi_-)Tr(\Psi_+\Psi_-) \ .$$

These are marginal.

As before we have a counter-term for the two-form $\mathbb{Z}_N$ gauge field

$$2\pi i \frac{k}{N} \int B$$

It turns out that under the chiral $\mathbb{Z}_2$ we have $k \to k + N/2$.
This happens only for even N. Therefore $\mathbb{Z}_2$ is not a symmetry in the presence of center vortices!
There are many explicit proofs of that, a particularly nice one is in [Cherman-Tanizaki, Unsal] who showed it through some mod 2 index.

This has strong implications.

- The line $W_F^{N/2}$ is deconfined. Namely, probe quarks in representations with $N/2$ boxes are deconfined.
- The $\mathbb{Z}_2$ symmetry is spontaneously broken.

For $M \neq 0$ confinement of $W_F^{N/2}$ is restored and the axial $\mathbb{Z}_2$ is explicitly broken.

For the special case that the operators $O_{1,2}$ are not included the model turns out to have an exponentially large number $\sim 4^N$ of non-invertible symmetries at $M = 0$. These constrain the dynamics quite dramatically. We forced to have complete deconfinement of all lines and $\sim 2^N$ degenerate ground states. At infinite N we have a Hagedorn transition at zero temperature.

Also the non invertible symmetries allow to compute the k string tension:

$$M << g : \epsilon_k \sim gM \sin(\pi k/N) \ .$$

[Armoni, Dorigoni, Veneziano] obtained the same formula at large N from arguments about Eguchi-Kawai reduction! [Herzog, Klebanov] obtained a similar formula in a holographic setting.

Thanks and happy birthday!