Integrability of high-energy QCD

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Based on the work with A. Belitsky, V. Braun, S. Derkachov, J. Drummond, L. Faddeev, J. Henn, L. Lipatov, A. Manashov, D. Müller, E. Sokatchev

QCD 50, September 14, 2023

Remember the people who shaped the field

Ludwig Faddeev (1934 – 2017)

Lev Lipatov (1940 – 2017)

Very special year 2023

✔ ***50 years of discovery of asymptotic freedom in Quantum Chromodynamics***

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Ultraviolet Behavior of Non-Abelian Gauge Theories*

David J. Gross† and Frank Wilczek
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
(Received 27 April 1973)

It is shown that a wide class of non-Abelian gauge theories have, up to calculable logarithmic corrections, free-field-theory asymptotic behavior. It is suggested that Bjorken scaling may be obtained from strong-interaction dynamics based on non-Abelian gauge symmetry.

Reliable Perturbative Results for Strong Interactions?*

H. David Politzer
Jefferson Physical Laboratories, Harvard University, Cambridge, Massachusetts 02138
(Received 3 May 1973)

An explicit calculation shows perturbation theory to be arbitrarily good for the deep Euclidean Green's functions of any Yang-Mills theory and of many Yang-Mills theories with fermions. Under the hypothesis that spontaneous symmetry breakdown is of dynamical origin, these symmetric Green's functions are the asymptotic forms of the physically significant spontaneously broken solution, whose coupling could be strong.

✔ QCD has had tremendous success in describing the strong interaction at high energy

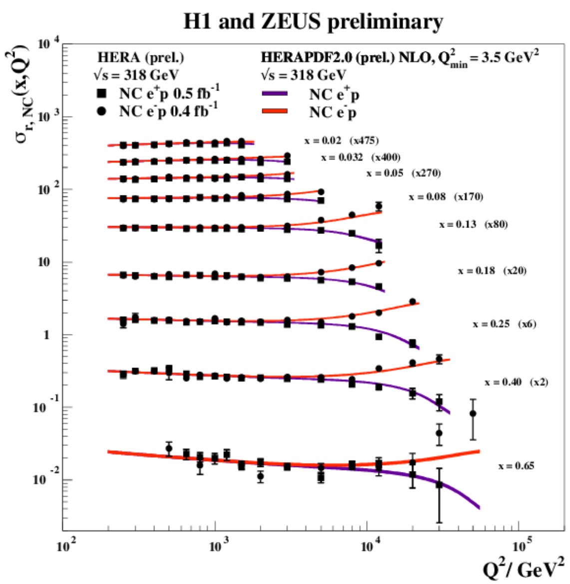
Bjorken scaling violation in DIS

$$\int_0^1 dx\, x^{n-2} N(x,q^2) \underset{q^2\to-\infty}{\sim} \text{const}(\ln q^2)^{-a_n}\left[1+O\left(\frac{1}{\ln q^2}\right)\right], \quad (5.20)$$

where

$$A_n = \frac{3C_2(R)}{22C_2(G)-8T(R)}\left[1-\frac{2}{n(n+1)}+4\sum_{k=2}^{n}\frac{1}{k}\right].$$

D. Gross, F. Wilczek, Phys.Rev.D 8 (1973) 3633

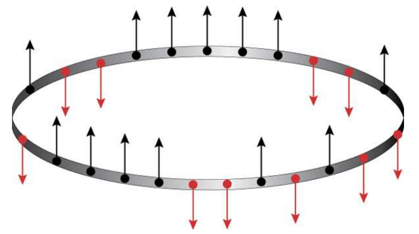


✔ Understanding quark confinement remains one of the most outstanding problem in QCD

95 years of Heisenberg spin chain model

- ✔ Heisenberg antiferromagnetic XXX spin $1/2$ chain *Heisenberg, 1928*

$$H_{\rm XXX} = -\sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1}$$

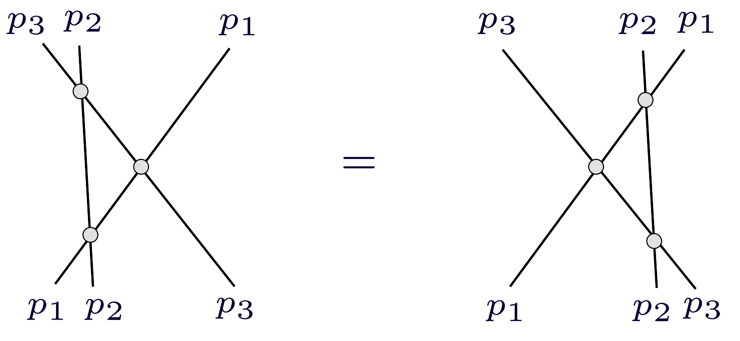


Exact solution can be found using Bethe Ansatz *Bethe, 1931*

- ✔ ***Integrable models*** – family of *solvable* quantum field-theoretical models in 2 dimensions

Infinitely many conserved charges $\to$ Elastic scattering $\to$ Factorizable S-matrices

- ✔ Two-particle S-matrix satisfies Yang-Baxter equation



Many-particle S–matrix is a product of 2-particle S–matrices

- ✔ What is the relation between QCD and two-dimensional (integrable) models?

Some hints

- ✔ Bjorken scaling violation is controlled by twist-two QCD anomalous dimensions

$$\gamma_N = \frac{\alpha_s}{2\pi}\left[4\sum_{k=1}^{N}\frac{1}{k} - \frac{2}{N(N+1)} + 1\right]$$

Depend on a special function (harmonic sum, digamma function) $\psi(x) = d\log\Gamma(x)/dx$

$$\sum_{k=1}^{N}\frac{1}{k} = \psi(N+1) - \psi(1)$$

- ✔ Meantime in the world of integrable models . . .

Generalization of Heisenberg XXX model to high spin *Faddeev,Tararov,Takhtajan'83*

$$H_n = h(\vec{S}_1\cdot\vec{S}_2) + \cdots + h(\vec{S}_n\cdot\vec{S}_1)$$

Hamiltonian depends on two particle spin $J_{12}(J_{12}+1) = (\vec{S}_1 + \vec{S}_2)^2$

$$h(\vec{S}_1\cdot\vec{S}_2) = (\log R_{12}(0))' = \psi(J_{12}+1) - \psi(1)$$

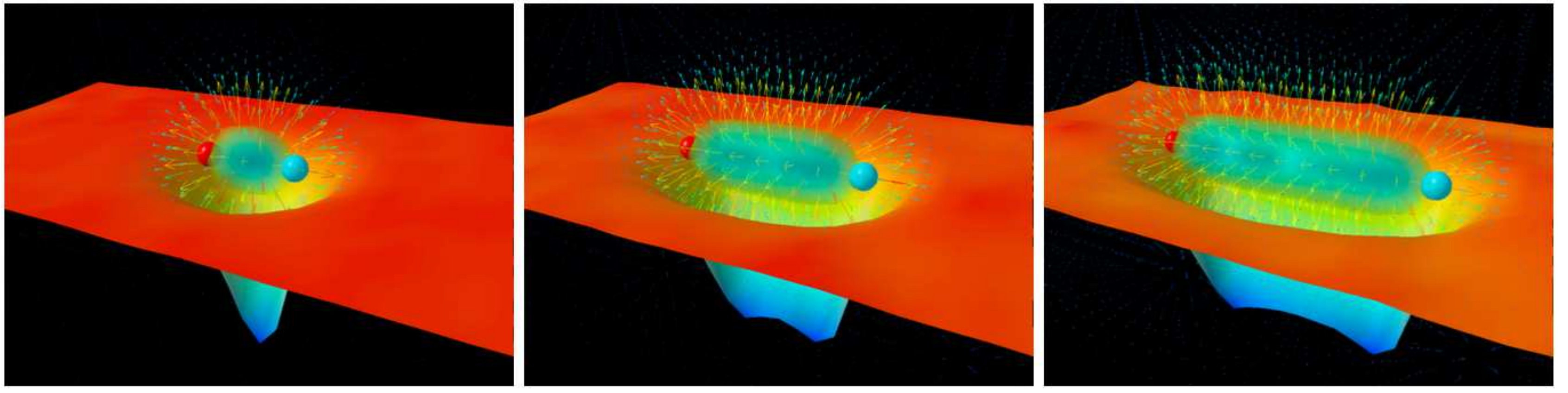
- ✔ Landau paradigm: "*A logarithm is not a function but a signal of simple underlying physics*"

$$\frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(\mu^2)} + \frac{b_0}{8\pi^2}\log(Q^2/\mu^2)$$

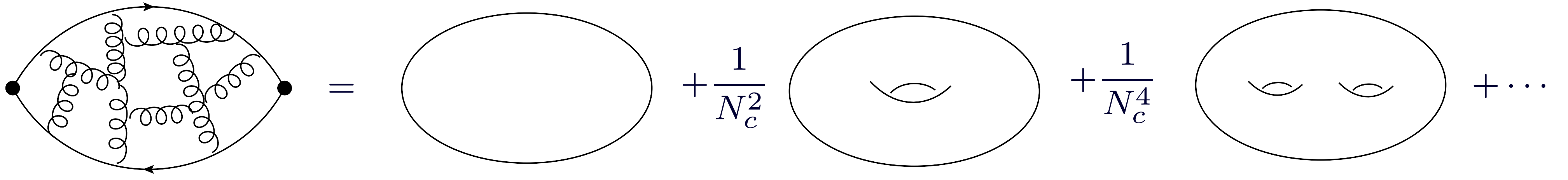
Improved version: "*A digamma is not a function but a signal of integrability*"

Strings from Quantum Chromo Dynamics

- ✔ What is an effective string theory of QCD flux tubes?



- ✔ String description naturally appears in ***large N_c limit***



$$= \quad + \frac{1}{N_c^2} \quad + \frac{1}{N_c^4} \quad + \cdots$$

Dense Feynman diagrams = Sum over 2d Riemann surfaces (string world-sheet)

- ✔ If QCD at large distances is described by a string theory, this should have some manifestation at short distances $\Longrightarrow$ look for hidden symmetries

What are the symmetries of QCD?

✔ QCD = (3+1)-dimensional Yang-Mills field theory with the $SU(N_c = 3)$ gauge group

✔ Symmetry of the *classical* theory:

- ✗ gauge symmetry,
- ✗ chiral symmetry,
- ✗ conformal symmetry, . . .

✔ Many of classical symmetries are broken on the *quantum* level

Q: Could it be that QCD possesses some hidden symmetry which

(i) does *not* exhibit itself as a symmetry of the classical Lagrangian

(ii) is only revealed on the *quantum* level

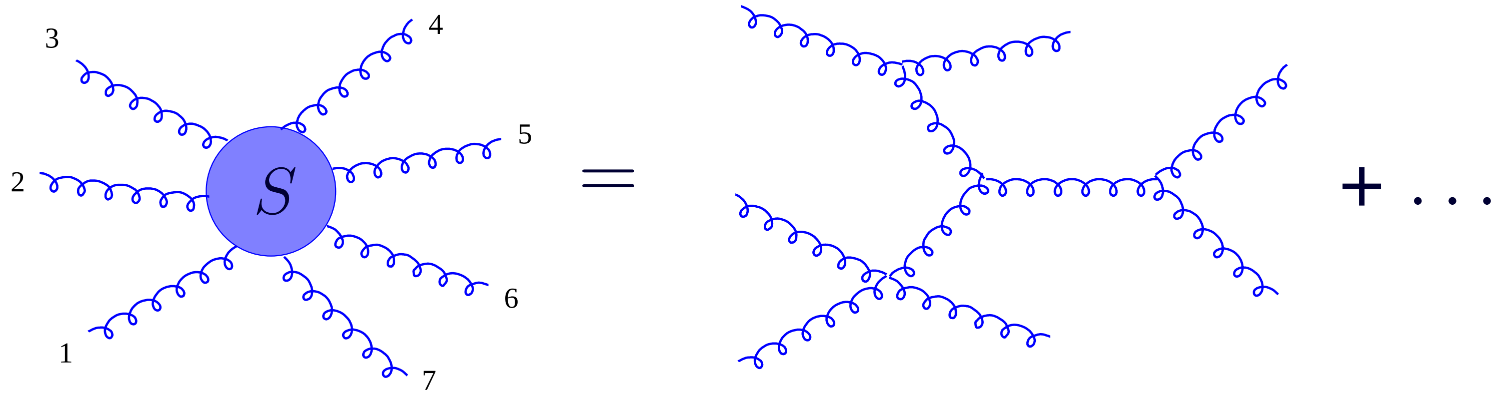
Example: Integrability in AdS/CFT correspondence

$$\mathcal{N} = 4 \text{ SYM theory} \quad \Longleftrightarrow \quad \text{type IIB string on the } AdS_5 \times S^5 \text{ background}$$

A: *Yes!* QCD at high energy is intrinsically related to *completely integrable models*

Scattering amplitudes in QCD

Tree gluon scattering amplitudes



✔ The same in QCD and in maximally supersymmetric Yang-Mills theory

Number of external gluons	4	5	6	7	8	9	10
Number of 'tree' diagrams	4	25	220	2485	34300	559405	10525900

✔ Number of diagrams grows factorially for large number of external gluons

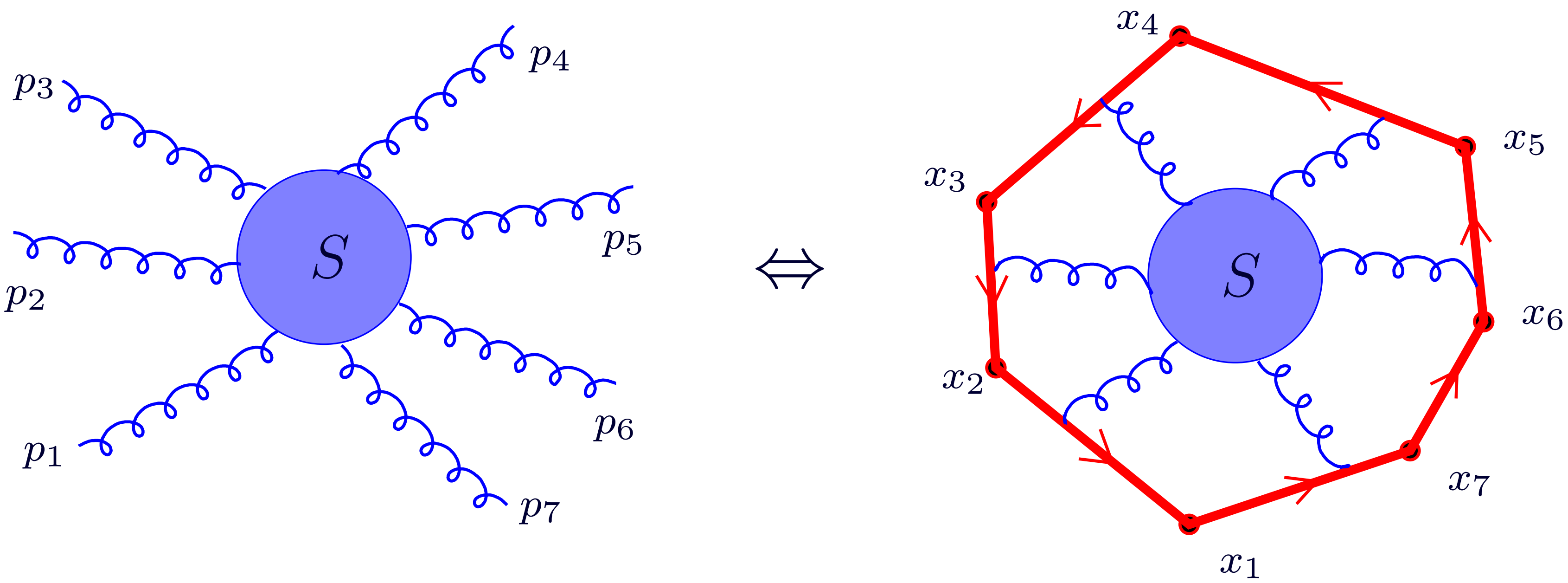
... but the final expression for tree amplitudes looks remarkably simple *Parke-Taylor'86*

$$A_n^{\text{tree}}(\underbrace{1^+2^+3^-\dots n^-}_{\text{MHV amplitude}}) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\dots\langle n1\rangle}\,, \qquad \Big[\text{spinor notations: } \langle ij\rangle = \lambda^\alpha(p_i)\lambda_\alpha(p_j)\Big]$$

✔ What is the reason for remarkable simplicity of amplitudes? ***'Dual conformal'*** symmetry

Scattering amplitudes/Wilson loops duality

- Gluon (MHV) scattering amplitude vs Light-like Wilson loop in Minkowski space-time

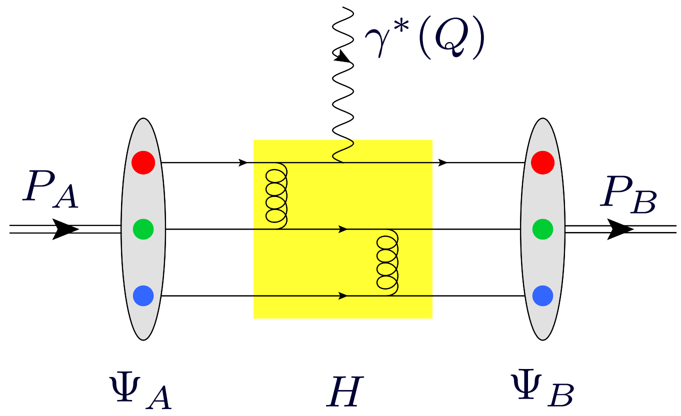


Dual coordinates $p_i = x_i - x_{i+1}$

- Dual conformal symmetry of the amplitude = conformal symmetry of the Wilson loop
- Conformal symmetry + Dual conformal symmetry = Yangian symmetry
- Exact duality relation in the planar $\mathcal{N} = 4$ SYM
- In also holds in QCD but in the (multi) Regge limit only

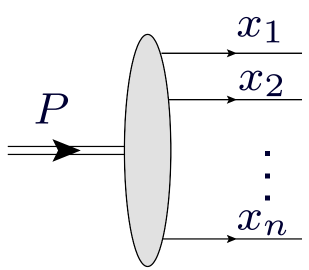
Hard processes in QCD

Electromagnetic form factor of proton at large Q^2



$$= \Psi_A \otimes H \otimes \Psi_B$$

Hadrons in the infinite momentum frame $\approx$ system of quasi-free partons with virtuality μ^2



$$= \Psi(x_1, \ldots, x_n; \mu^2) \qquad , \qquad \underbrace{0 \le x_k \le 1\,, \quad \sum_k x_k = 1}_{\text{momentum fractions}}$$

QCD factorization (scale separation)

$$F(Q^2) = \frac{1}{(Q^2)^{n-1}} \int_0^1 [dx][dy]\, \Psi_A(\{x\}; \mu^2) H(\{x, y\}, Q^2/\mu^2, \alpha_s(\mu^2)) \Psi_B(\{y\}; \mu^2)$$

Distribution amplitudes are nonperturbative, hard function is perturbative

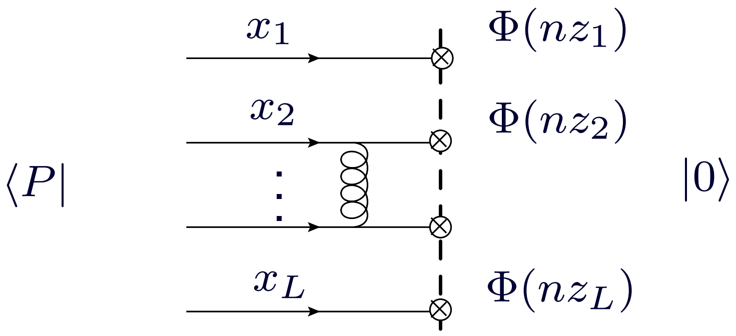
Perturbative QCD can be used to predict Q^2-dependence (= scaling violation)

Hadron distribution amplitudes

- ✔ Nonperturbative definition *[Brodsky, Lepage'79],[Efremov,Radyushkin'79]*

$$\langle P|\Phi(nz_1)\dots\Phi(nz_L)|0\rangle_{\mu^2} \overset{n^2=0}{=} \int_0^1 [dx]\, e^{-i(Pn)\sum_i z_i x_i}\,\Psi(x_1,\dots,x_L;\mu^2)$$

Correlation functions of parton fields on the light front = Sum of plane waves



Parton fields $\Phi = \{$quark, gluon$\}$ connected by gauge links

- ✔ Moments of distribution amplitudes $\Longleftrightarrow$ local operators:

$$\widetilde{\Psi}_{k_1\dots k_L} = \int [dx]\, x_1^{k_1}\dots x_L^{k_L}\,\Psi(x_1,\dots,x_L;\mu^2) = \langle P|(D_+^{k_1}\Phi)\dots(D_+^{k_L}\Phi)|0\rangle_{\mu^2}$$

- ✔ Scale dependence of the distribution amplitudes

$$\mu\frac{d}{d\mu}\widetilde{\Psi}_{k_1\dots k_L} = \sum_{m_j}\underbrace{V(k_i|m_j)}_{\text{mixing matrix}}\widetilde{\Psi}_{m_1\dots m_L}$$

Conventional QCD approach

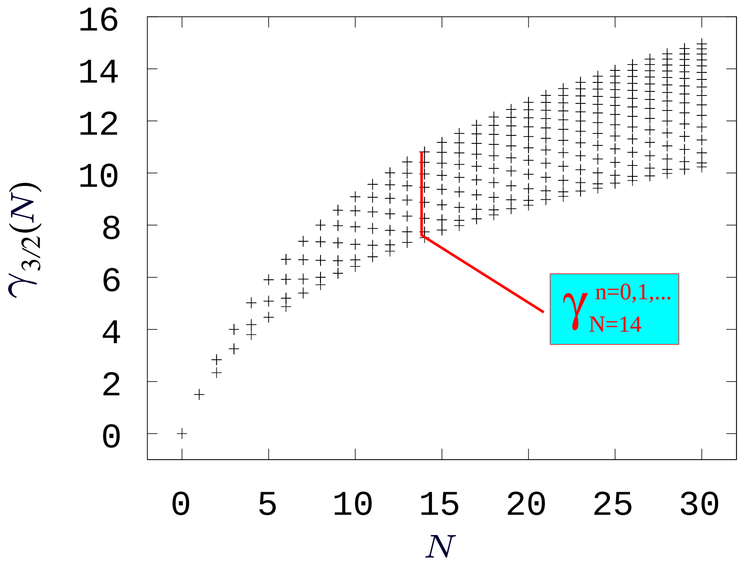
- ✔ Diagonalize the mixing matrix and find the spectrum of anomalous dimensions
- ✔ Example: helicity$-3/2$ baryon distribution amplitude $[q = q^{\uparrow}(x) + q^{\downarrow}(x),\, q^{\uparrow(\downarrow)} = \frac{1\pm\gamma_5}{2} q]$

$$q^{\uparrow}(z_1 n) q^{\uparrow}(z_2 n) q^{\uparrow}(z_3 n) \longrightarrow (D_+^{k_1} q^{\uparrow})\,(D_+^{k_2} q^{\uparrow})\, q^{\uparrow}(0) + [\text{total derivatives}]$$

  - ✗ Mixing matrix:

$$\sum_{n_1+n_2=N} V(k_1,k_2|n_1,n_2)\,\Psi^{(\ell)}_{n_1,n_2} = \gamma^{(\ell)}_{3/2}(N)\,\Psi^{(\ell)}_{k_1,k_2}\,, \qquad (\ell = 0,\dots,N)$$

  - ✗ Rich spectrum of anomalous dimensions:



- (Almost) all levels are double degenerate
- Where does this structure come from?

  Conformal symmetry + Integrability!

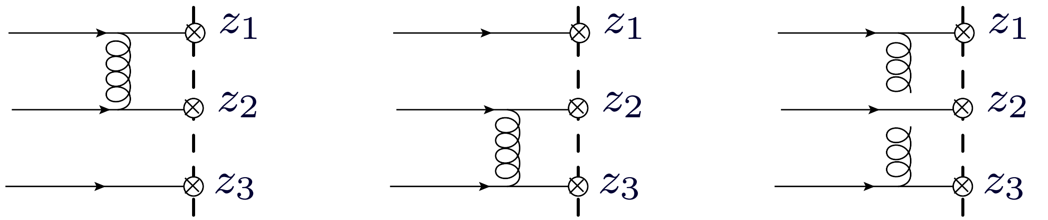
Integrability on the light-cone

- ✔ Callan-Symanzik equation (helicity$-\frac{3}{2}$ baryon operator $B \equiv q^{\uparrow}(z_1 n) q^{\uparrow}(z_2 n) q^{\uparrow}(z_3 n)$)

$$\mu \frac{d}{d\mu} B(z_1, z_2, z_3) = [\mathbb{H} \cdot B](z_1, z_2, z_3),$$

- ✔ One-loop dilatation operator:

$\mathbb{H} =$



$+ O(\alpha_s^2)$

- ✔ Two-particle structure:

$$\mathbb{H} = \frac{\alpha_s N_c}{2\pi} \left[\mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{13}\right] + O(\alpha_s^2)$$

Displaces quark fields along the light-cone

$$\mathcal{H}_{12} B(z_1, z_2, z_3) = \int_0^1 \frac{d\alpha\,\alpha}{1-\alpha} \left[B(z_1 - \alpha z_{12}, z_2, z_3) + B(z_1, z_2 + \alpha z_{12}, z_3) - 2B(z_1, z_2, z_3)\right]$$

- ✔ QCD evolution = Quantum mechanics on the light cone

$\mathbb{H} =$ 3 particle Hamiltonian with nearest neighbour interaction

Conformal symmetry on the light-cone

- ✔ Conformal symmetry is broken in QCD but the conformal anomaly affects the anomalous dimensions starting from two loops only

- ✔ One-loop dilatation operator in QCD inherits conformal symmetry of the classical Lagrangian!

- ✔ Full conformal symmetry reduces on the light-cone $x_\mu = z n_\mu$ ($n^2 = 0$) to its $SL(2)$ subgroup:

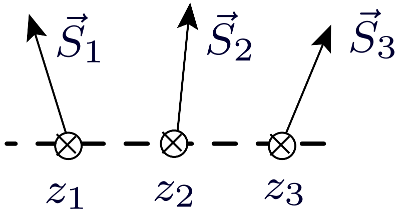
$$S_- = -\frac{d}{dz}\,, \qquad S_+ = z^2\frac{d}{dz} + 2z\,, \qquad S_0 = z\frac{d}{dz} + 1$$

  Can be interpreted as spin operators

$$[S_\alpha, S_\beta] = i\epsilon_{\alpha\beta\gamma} S_\gamma$$

- ✔ Dilatation operator is the spin-chain Hamiltonian

$$\mathbb{H} = h(\vec{S}_1 \cdot \vec{S}_2) + h(\vec{S}_2 \cdot \vec{S}_3) + h(\vec{S}_3 \cdot \vec{S}_1)$$



  - ✗ Number of sites = number of quark operators
  - ✗ Spin operators = Generators of the ‘collinear’ conformal group

Integrability on the light-cone (II)

- ✔ QCD anomalous dimensions are eigenvalues of the dilatation operator

$$\mathbb{H}\,\Psi_N(z_1,z_2,z_3) = \gamma_N\,\Psi_N(z_1,z_2,z_3)$$

- ✔ $SL(2)$ invariant form of the dilatation operator

$$\mathbb{H} = \frac{\alpha_s N_c}{\pi}\left[\mathcal{H}_{12} + \mathcal{H}_{23} + \mathcal{H}_{13}\right]$$

Two-particle Hamiltonian

$$\mathcal{H}_{12} = \psi(J_{12}) - \psi(1)\,, \qquad J_{12}(J_{12}-1) \equiv (\vec{S}_1 + \vec{S}_2)^2$$

Coincides with the Heisenberg $SL(2;\mathbb{R})$ spin chain Hamiltonian!

- ✔ One-loop dilatation operator $\equiv$ Hamiltonian of the $SL(2,\mathbb{R})$ Heisenberg spin chain

- ✔ The spectrum of anomalous dimensions can be found exactly using the Bethe Ansatz

$$\gamma_N = \frac{\alpha_s N_c}{\pi}\sum_{k=1}^{N}\frac{1}{\lambda_k^2+1}\,, \qquad \left(\frac{\lambda_k+i}{\lambda_k-i}\right)^3 = \prod_{j\neq k}^{N}\frac{\lambda_k-\lambda_j-i}{\lambda_k-\lambda_j+i}$$

$\{\lambda_1,\ldots,\lambda_N\}$ = Bethe roots

Integrable "zoo" in multi-color QCD

- ✔ Interaction between partons with the *aligned* helicities (quarks $q^{\uparrow}$, gluons $G^{\uparrow}$) is integrable

  One-loop dilatation operator in QCD = Hamiltonian of $SL(2,\mathbb{R})$ Heisenberg magnet:

  - ✗ Three-quark states:

  $$[q^{\uparrow}(z_1)q^{\uparrow}(z_2)q^{\uparrow}(z_3)] \Longrightarrow \textit{closed} \text{ spin } j_q = 1 \text{ chain}$$

  - ✗ Multi-gluon states:

  $$[G^{\uparrow}(z_1)G^{\uparrow}(z_2)...G^{\uparrow}(z_L)] \Longrightarrow \textit{closed} \text{ spin } j_g = 3/2 \text{ chain}$$

  - ✗ Antiquark-Glue-Quark states:

  $$[\bar{q}(z_1)\, G^{\uparrow}(z_2)...G^{\uparrow}(z_{L-1})q(z_L)] \Longrightarrow \textit{open inhomogeneous} \text{ spin chain}$$

- ✔ Integrability is broken in the 'mixed' helicity sectors (ex: helicity$-1/2$ states $[q^{\uparrow}q^{\downarrow}q^{\uparrow}]$)
  - ✗ Symmetry breaking terms generate a mass gap in the spectrum of γ's [scalar diquarks]
  - ✗ ... but they do not affect large spin asymptotics

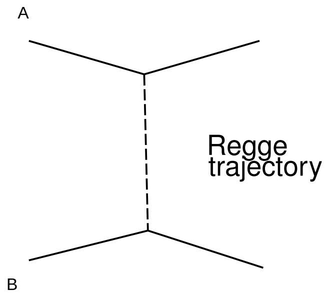
  $$\gamma(S) = 2\ \underbrace{\Gamma_{\rm cusp}(\alpha_s)}_{\text{cusp anom.dim.}}\ \ln S + S^0 \times (\text{nonintegrable terms})$$

- ✔ Important applications to QCD phenomenology

Scattering in QCD at high-energy (Regge asymptotics)

✔ Regge phenomena in strong interactions (since 60's):

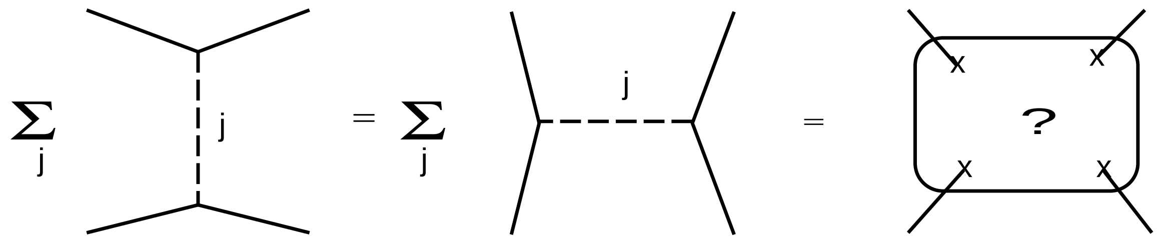
$$\sigma_{\mathrm{AB}}(s) =$$



$$= \sum_j \beta_A^j(t)\beta_B^j(t)\, s^{\alpha_j(t)-1}$$

Scattering amplitudes grow at high energy s as a power $\sim s^{\alpha_j(t)}$

✔ Dual model:



Regge trajectories + duality condition = Hadronic string (?)

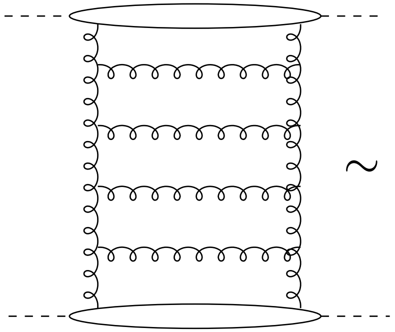
✔ High-energy asymptotics in QCD: interaction induces large corrections which need to be resummed to all order of perturbation theory *Balitsky-Fadin-Kuraev-Lipatov '78*

$$\sigma_{\mathrm{AB}}(s) = \sum_{n=0,1,\dots} w_n \left(g_s^2 \ln s\right)^n \sim s^{\alpha_{\mathbb{P}}-1}$$

BFKL Pomeron + Unitarity

- Leading contribution: BFKL Pomeron ($\lambda = g_s^2 N_c/(4\pi^2)$)

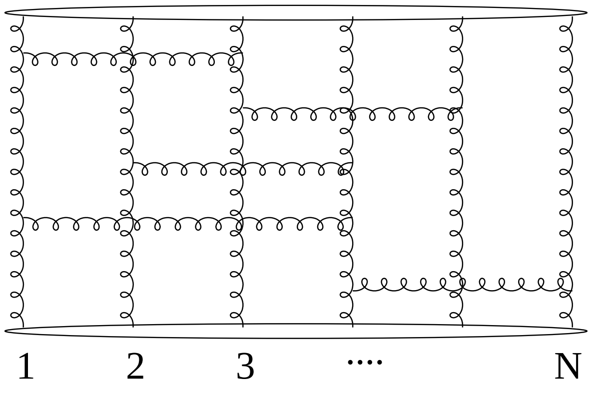
$$\sigma_{\rm LO} = \sum_{\rm rungs}$$



$$\sim \lambda^2 \frac{\exp(4\ln 2 \cdot \lambda \ln s)}{\sqrt{\lambda \ln s}} \sim \underbrace{s^{4\ln 2 \cdot \lambda}}_{\text{violates unitarity}}$$

- BFKL Pomeron + Unitarity $\Longrightarrow$ generalized ladder diagrams

$$\sum_{N=2,3,..}$$



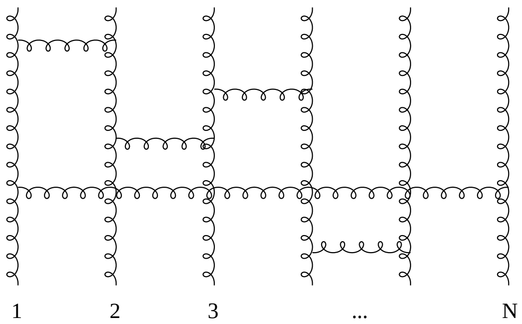
Elastic pair-wise interaction of $N = 2, 3, ...$ particles

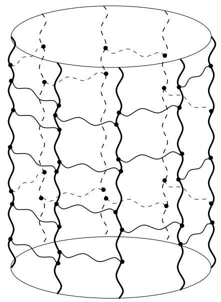
Rapidities of gluons $y = \log(k_+/k_-)$ are strongly ordered

Nontrivial QCD dynamics occurs on the two-dimensional transverse plane $k_\perp = (k_1, k_2)$

Color-singlet compound gluonic states

✌ The effective QCD Hamiltonian $\mathcal{H}_N$ has remarkable properties in the planar limit:

$\mathcal{A}(s,t) =$ 

$=$  $\sim \exp\left(y\mathcal{H}_N\right) = s^{\mathcal{H}_N}$

$\hookrightarrow$ Elastic scattering of N reggeized gluons

✔ The Bartels-Kwiecinski-Praszalowicz equation $\equiv$ 2-dim Schrödinger equation

$$\mathcal{H}_N \Psi\underbrace{(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)}_{2-\text{dim coordinates}} = E_N \Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)$$

✔ $\Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N) =$ colour-singlet compound states built from N reggeized gluons

✔ High-energy asymptotics of the scattering amplitudes is governed by these states

$$\mathcal{A}(s,t) \sim -is \sum_{\substack{N-\text{gluon}\\ \text{states}}} (i\lambda)^N \underbrace{s^{\lambda E_N}}_{\text{Regge behaviour}} \beta_N(t)$$

✔ Intercept = maximal energy E_N = *Eigenvalues of the Heisenberg* $SL(2;\mathbb{C})$ *spin chain!*

Integrability of high-energy QCD

- ✔ Integrability emerges as a hidden symmetry of the *effective* QCD dynamics in *different* limits:
  - ✗ Scale dependence of multi-particle distribution amplitudes (=dilatation operator)
  - ✗ High-energy (Regge) behaviour of scattering amplitudes
- ✔ Does the $SL(2,\mathbb{R})$ integrability hold beyond one-loop (beware of the *broken* conformal symmetry)? Yes, it does!
- ✔ Integrability is *not* tied to the conformal symmetry but it requires the *planar limit*
- ✔ Integrability is a general feature of (super) Yang-Mills theories in four dimensions
- ✔ In maximally supersymmetric Yang-Mills theory it can be understood using the AdS/CFT correspondence

What is the origin of integrability in QCD?