Nonrelativistic Conformal Field Theory and nuclear reactions

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50 years of Quantum Chromodynamics, UCLA
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Happy birthday QCD!

Cultural impact of QCD

- QCD is the first theory where

  microscopic d.o.fs $\neq$ low energy d.o.fs

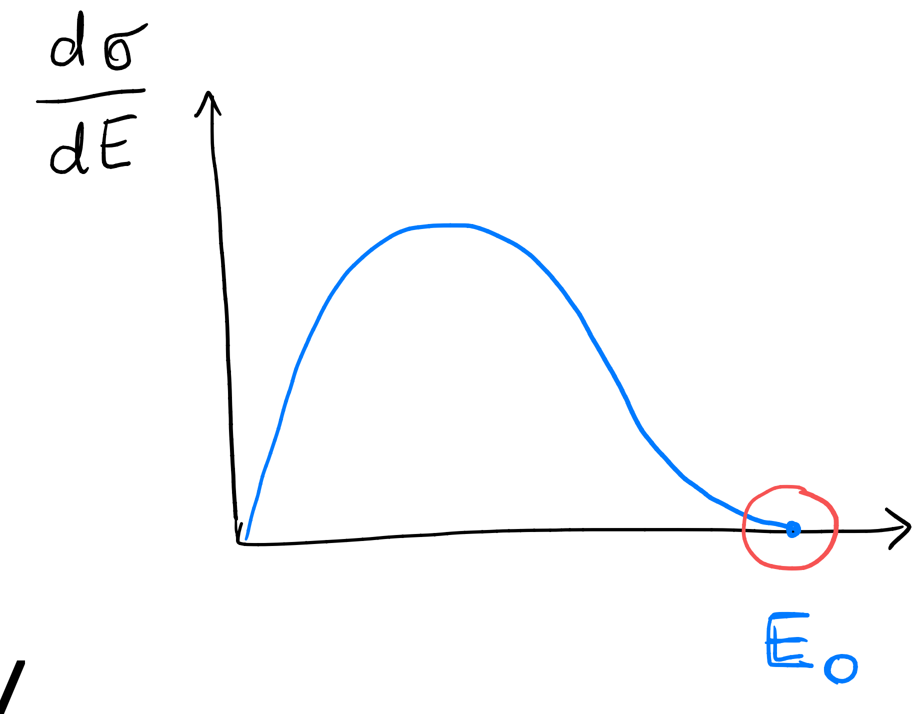
- Immense impact on condensed matter physics
  - quasiparticles in fractional quantum Hall states: fractionally charged at $\nu = 1/3$, electrically neutral fermion ("composite fermions") at $\nu = 1/2$

- "deconfinement", "partons": now standard notions in theoretical condensed matter physics

This talk

- about "chemistry of QCD" — nuclei
  - simplest chemistry: He_2 molecule (10^{-7} eV binding energy)
- processes involving very small energy scale, from 0.1 MeV to a few MeV
- Accidental fine-tuning: neutron-neutron scattering length very large $a_{nn} \approx -19$ fm
- Nontrivial effect on rates of reactions with several neutron in the final state

Example



- $\pi^- + {}^3\text{H} \to \gamma + 3n$
- $E_\gamma = E \in (0, E_0), \quad E_0 \approx 130$ MeV
- Near end point
  - $\dfrac{d\sigma}{dE} \sim (E_0 - E)^{1.77}$
  - $1.77 = \Delta - 5/2$ where $\Delta = 4.27$ is the scaling dimension of a local operator in a non-relativistic CFT

Plan

- Nonrelativistic conformal symmetry
- Nonrelativistic CFTs
- Fermions at unitarity
- Neutrons and “UnNuclear Physics”

Refs.: Y. Nishida, DTS , PRD 76, 086004 (2007)
H.-W. Hammer, DTS PNAS 118 (2021) e2108716118
S.D. Chowdhury, R. Mishra, DTS to appear

Schrödinger group

- Symmetries of the free Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi$$

- Phase rotation $M = mN$ $\quad \psi \to e^{i\alpha}\psi$
- space and time translations $\mathbf{P}, H$; rotations J_{ij}
- Galilean boosts $\mathbf{K}$ $\psi(t, \mathbf{x}) \to e^{im\mathbf{v}\cdot\mathbf{x} - \frac{i}{2}mv^2 t}\,\psi(t, \mathbf{x} - \mathbf{v}t)$
- Dilatation D $\psi(t, \mathbf{x}) \to \lambda^{3/2}\psi(\lambda^2 t, \lambda\mathbf{x})$

“Proper conformal transformation”

$$C:\ \psi(t,\mathbf{x}) \to \frac{1}{(1+\alpha t)^{3/2}} \exp\left(\frac{i}{2}\frac{m\alpha x^2}{1+\alpha t}\right)\psi\left(\frac{t}{1+\alpha t}, \frac{\mathbf{x}}{1+\alpha t}\right)$$

Schrödinger algebra

$$[X, Y]$$

$X \setminus Y$	P_j	K_j	D	C	H
P_i	0	$-i\delta_{ij}M$	$-iP_i$	$-iK_i$	0
K_i	$i\delta_{ij}M$	0	iK_i	0	iP_i
D	iP_j	$-iK_j$	0	$-2iC$	$2iH$
C	iK_j	0	$2iC$	0	iD
H	0	$-iP_j$	$-2iH$	$-iD$	0

$$[N, \text{anything}] = 0$$

Nonrelativistic CFTs

Y. Nishida, DTS, 2007

- QFTs with Schrödinger symmetry
- local operators $O(\vec{x})$ characterized by charge (mass) and dimension $[D, O(0)] = i\Delta_0 O(0), \quad [M, O(0)] = iN_O O(0)$
  example: $\psi \quad N_\psi = 1,\ \Delta_\psi = \frac{3}{2}$
  - primary operators: $[K_i, O(\vec{0})] = [C, O(\vec{0})] = 0$
- Constraints from conformal invariance:

$$\langle TO(t,\vec{x})O^\dagger(0,0)\rangle = \frac{c}{t^{\Delta_O}} \exp\left(\frac{im_O x^2}{2t}\right)$$

$[E] = 2$
$[p] = 1$

Example of NRCFTs

- Free particles
- nonrelativistic anyons (two spatial dimensions)
- Spin-1/2 fermions at unitarity
  - realized in cold atom experiments, but also approximately by neutrons

Unitarity fermions: QM

- Wave function of m spin-up and n spin-down fermions $\psi(\mathbf{x}_1, \ldots, \mathbf{x}_m; \mathbf{y}_1, \ldots, \mathbf{y}_n)$
- ψ antisymmetric under exchanging two $\mathbf{x}$'s or $\mathbf{y}$'s
- When one spin-up and one spin-down fermions approach each other:

$$\psi(\mathbf{x}, \mathbf{y}) = \frac{C}{|\mathbf{x} - \mathbf{y}|} + O(|\mathbf{x} - \mathbf{y}|) + \cdots$$

- $H = -\frac{1}{2} \sum_a \frac{\partial^2}{\partial \mathbf{x}_a^2} - \frac{1}{2} \sum_a \frac{\partial^2}{\partial \mathbf{y}_a^2}$

What are the local operators?

- First example: annihilation operator in second quantized formulation of QM
- $\langle 0 | \hat{\psi}(\vec{x}) | \Psi_{1-\mathrm{body}} \rangle = \Psi(\vec{x})$
- This is a charge-1 operator, dimension=3/2

Charge-2 local operator

- Second-quantized formulation of QM:

$$\langle 0 | \psi_\uparrow(\mathbf{x}) \psi_\downarrow(\mathbf{y}) | \Psi_{2-\mathrm{body}} \rangle = \Psi(\mathbf{x}, \mathbf{y})$$

- Limit $\mathbf{y} \to \mathbf{x}$ does not exist:

$$\langle 0 | \psi_\uparrow(\mathbf{x}) \psi_\downarrow(\mathbf{x}) | \Psi \rangle = \Psi(\mathbf{x}, \mathbf{x}) = \infty$$

- but one can define

$$O_2(\mathbf{x}) = \lim_{\mathbf{y} \to \mathbf{x}} |\mathbf{x} - \mathbf{y}| \psi_\uparrow(\mathbf{x}) \psi_\downarrow(\mathbf{y})$$

- then

$$\langle 0 | O_2(\mathbf{x}) | \Psi \rangle = \lim_{\mathbf{y} \to \mathbf{x}} |\mathbf{x} - \mathbf{y}| \Psi(\mathbf{x}, \mathbf{y}) = \text{finite}$$

Dimension of O_2

- $O_2(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{x}} |\,\mathbf{x} - \mathbf{y}\,|\, \psi_\uparrow(\mathbf{x})\psi_\downarrow(\mathbf{y})$

- $\Delta[O_2] = 2\Delta[\psi] - 1 = 2$

- cf free theory: $\Delta[\psi\psi] = 3$

Charge-3 operator

- Need to know short distance behavior of $\Psi(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y})$
- 3-body problem solved by Efimov ~ 1970

$$\Psi(\mathbf{x}_1, \mathbf{x}_2; \mathbf{y}) \sim R^{-0.2273} f(\alpha, \hat{\rho}, \hat{r})$$

$$R^2 = |\mathbf{x}_1 - \mathbf{x}_2|^2 + |\mathbf{x}_1 - \mathbf{y}|^2 + |\mathbf{x}_2 - \mathbf{y}|^2$$

$\alpha, \hat{\rho}, \hat{r}$ = 5 hyperangles

- Charge-3 operator

$$O_3(\mathbf{x}) \sim \lim_{\mathbf{x}_2 \to \mathbf{x}} \lim_{\mathbf{y} \to \mathbf{x}} R^{0.2273} \psi_\uparrow(\mathbf{x}) \psi_\uparrow(\mathbf{x}_2) \psi_\downarrow(\mathbf{y})$$

- $\Delta[O_3] = 4.2727$

cf free theory: $[\psi_\downarrow \psi_\uparrow \nabla \psi_\uparrow] = \frac{11}{2}$

Dimension of charge-3 operators

$\Delta = \frac{5}{2} + s$ where s solves an equation

$$l = 0: \quad s\cos\left(\tfrac{\pi}{2}s\right) + \frac{4}{\sqrt{3}}\sin\left(\tfrac{\pi}{6}s\right) = 0$$

$$\Delta = 4.666,\ 7.627,\ 9.614, \ldots, 2n + \frac{7}{2}$$

$$l = 1: \quad (s^2 - 1)\sin\left(\tfrac{\pi}{2}s\right) + \frac{4}{\sqrt{3}} s\cos\left(\tfrac{\pi}{6}s\right) - 4\sin\left(\tfrac{\pi}{6}s\right) = 0$$

$$\Delta = 4.273,\ 6.878,\ 8.216, \ldots.\quad 2n + \frac{5}{2}$$

Charge-4 operator

- Dimension of operator with particle number N>3 can only be obtained numerically
- Nishida and DTS 2007: equal to ground state energy of N unitary fermions in a harmonic trap
- $\Delta[O_4] = 5.0 \pm 0.1$ (cf. free theory: 8)

Two point functions

- One can compute two-point functions by inserting a complete set of states

$$\langle 0 | O(t,\mathbf{x}) O^\dagger(0) | 0 \rangle = \sum_n \langle 0 | O(0) | n \rangle e^{-iE_n t + i\mathbf{P}_n \cdot \mathbf{x}} \langle n | O^\dagger(0) | 0 \rangle$$

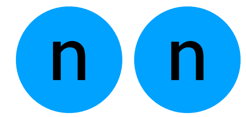
- Result

$$\langle O(t,\mathbf{x}) O^\dagger(0,\mathbf{0}) \rangle = \frac{C}{t^{\Delta_O}} \exp\left(\frac{iM_O x^2}{2t}\right)$$

- In momentum space

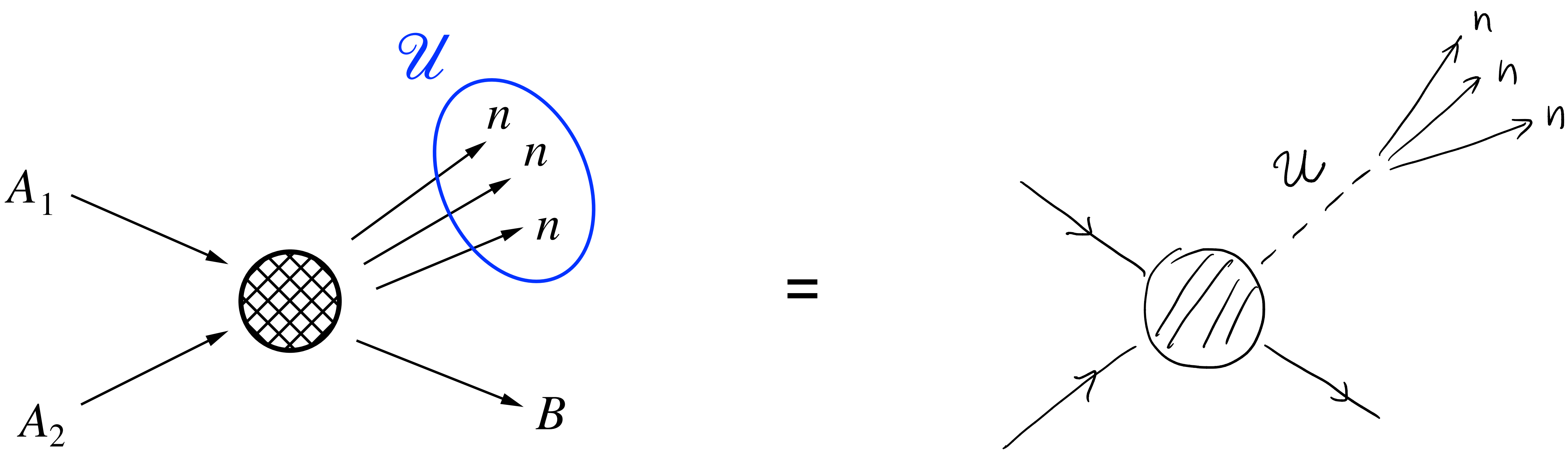
$$\langle O O^\dagger \rangle(\omega, \mathbf{p}) \sim \left(\frac{\mathbf{p}^2}{2M_O} - \omega\right)^{\Delta_O - 5/2}$$

NRCFT in real world: neutrons



- $a \approx -19$ fm, $r_0 \approx 2.8$ fm
- NRCFT in energy range between $\hbar^2/ma^2 \sim 0.1$ MeV and $\hbar^2/mr_0^2 \sim 5$ MeV
- Consequence: power-law behavior in processes with final state neutrons
- "Unnuclear Physics" Hammer, DTS 2021 nonrelativistic version of Georgi's "unparticle physics"

"UnNuclear physics"

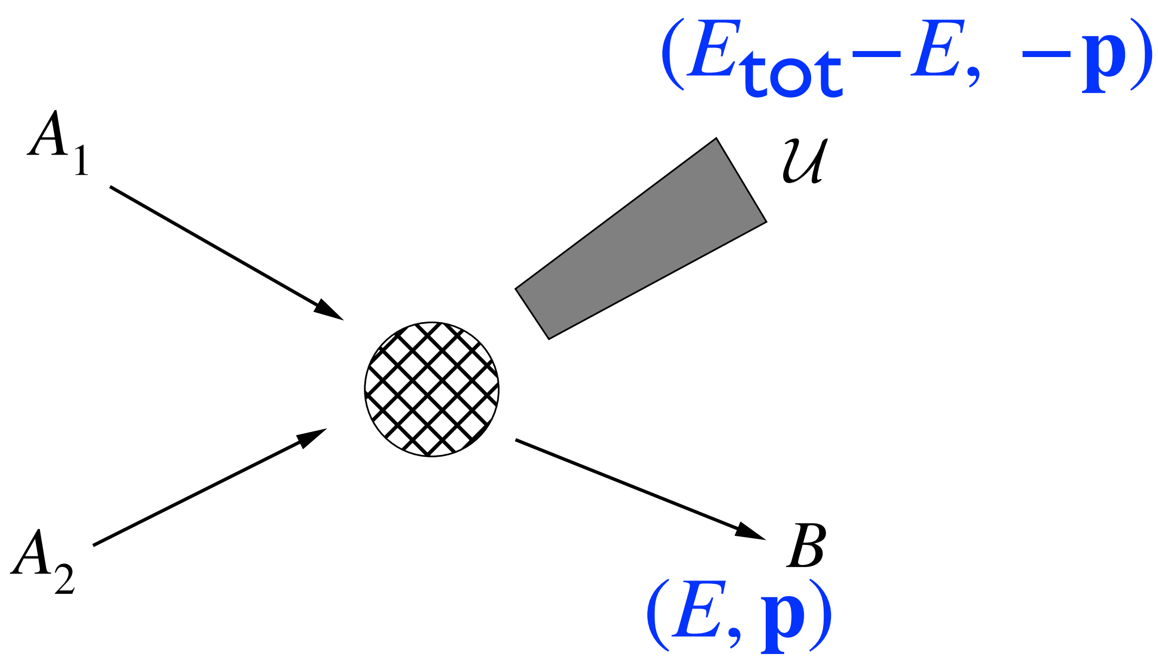


$$P(A_1 + A_2 \to B + 3n) = P(A_1 + A_2 \to B + \mathscr{U})P(\mathscr{U} \to 3n)$$

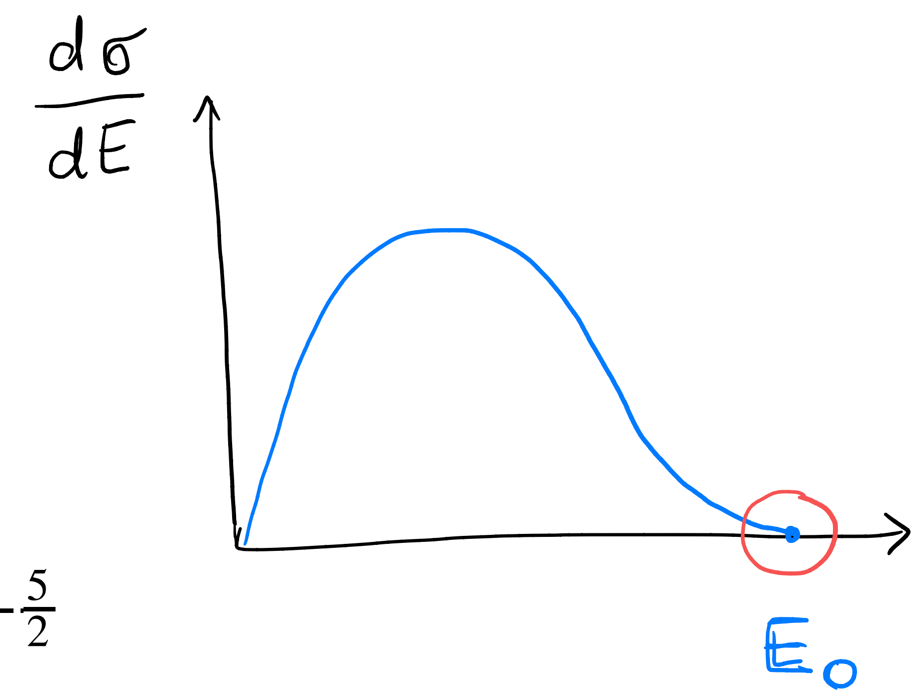
when energy scale of primary reaction is larger than $\mathscr{U} \to 3n$

$\mathscr{U}$ = "unnucleus" = field in NRCFT

Rates of unnuclear processes



$$E_{\text{tot}} = E + E_{\mathcal{U}}$$



- $$\frac{d\sigma}{dE} \sim |\mathscr{M}|^2 \sqrt{E} \times \underbrace{\operatorname{Im} G_{\mathcal{U}}(E_{\text{tot}} - E, \mathbf{p})}_{(E_0 - E)^{\Delta - \frac{5}{2}}}$$

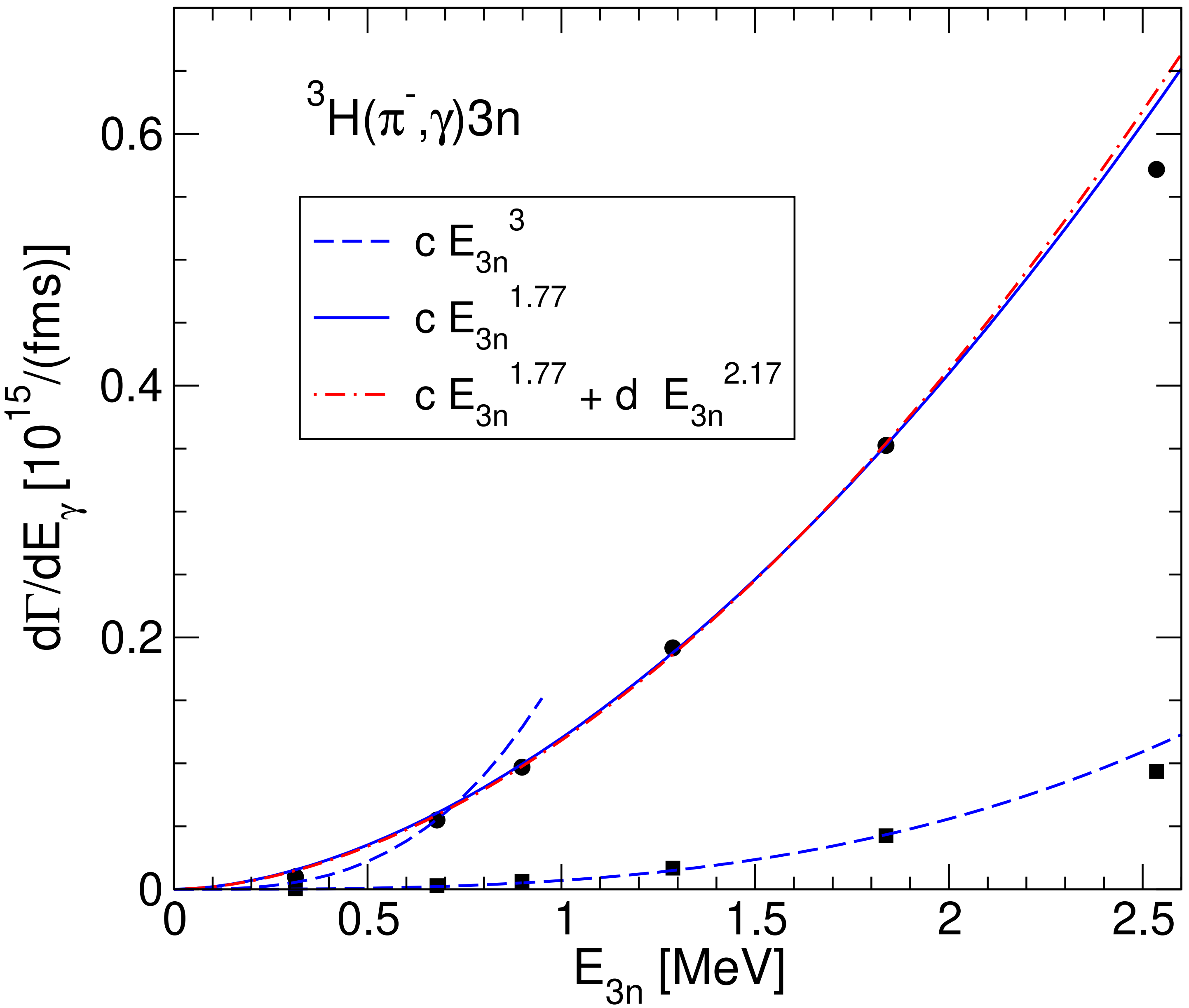
- Near end point: $\dfrac{d\sigma}{dE} \sim (E_0 - E)^{\Delta - \frac{5}{2}}$, Δ= dimension of $\mathcal{U}$

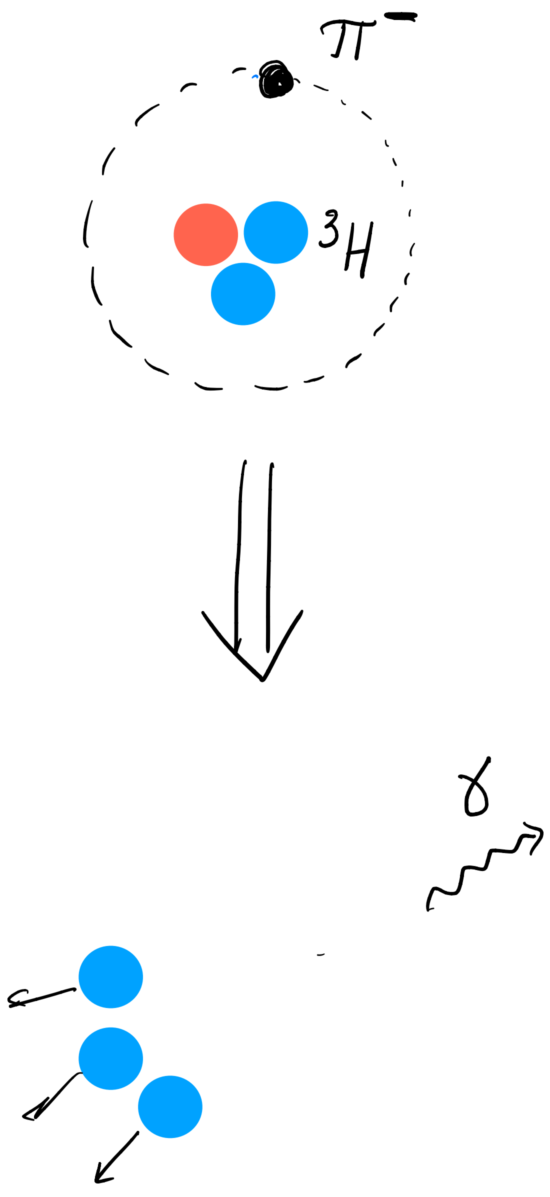
Nuclear reactions

- $$\frac{d\sigma}{dE} \sim (E_0 - E)^{\alpha} \qquad \alpha = \Delta - \frac{5}{2}$$

	Δ	α	
• $^3H + {}^3H \rightarrow {}^4He + 2n$	2	-0.5	Watson-Migdal 1950's
• $\pi^- + {}^3H \rightarrow \gamma + 3n$	4.27	1.77	
• $^4He + {}^8He \rightarrow {}^8Be + 4n$	5.0	2.5	

Comparison with "experiment"

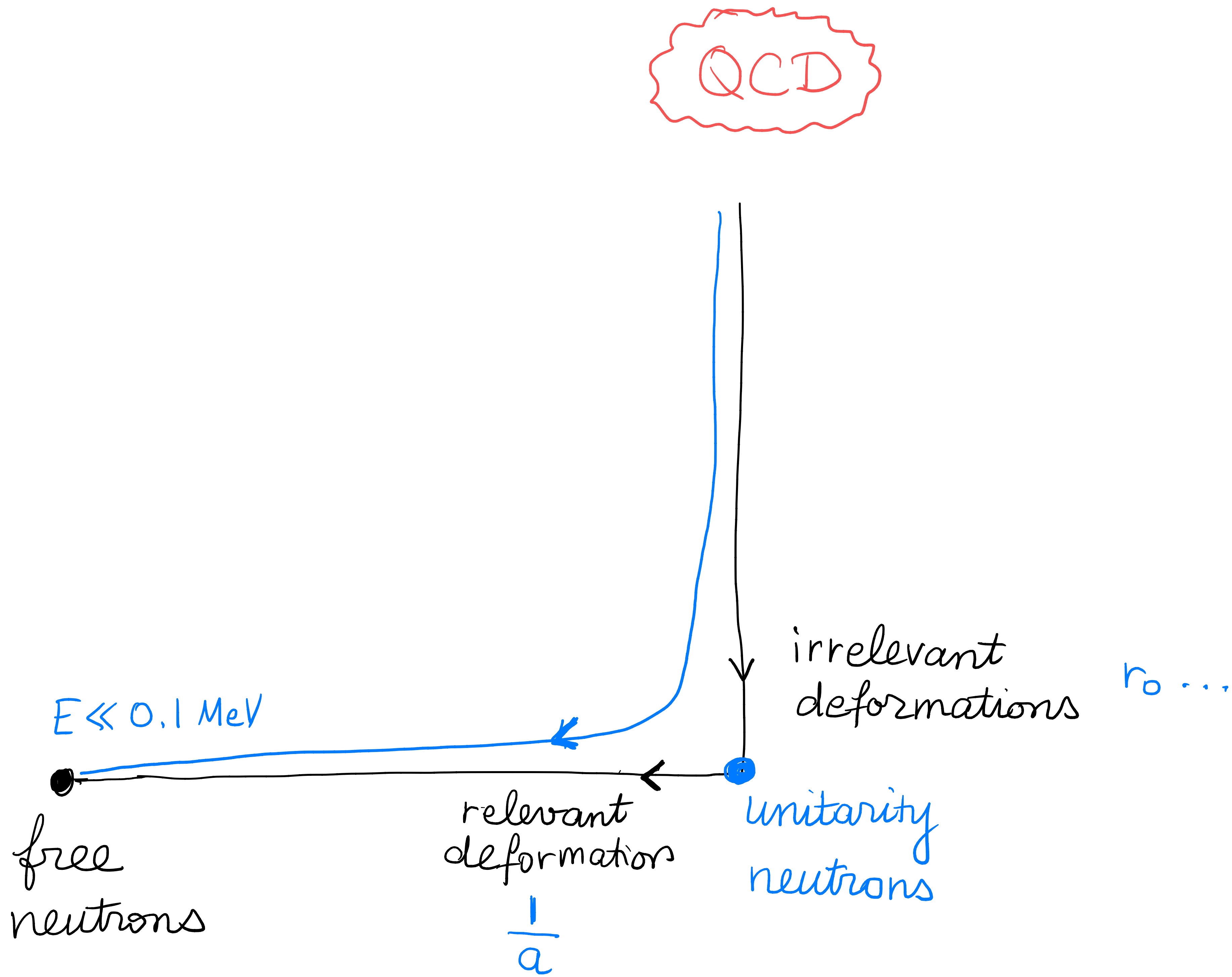




$$\pi^- + {}^3\mathrm{H} \rightarrow \gamma + 3n$$

Golak et al. PRC 98, 054001 (2018)

NRCFT as transient fixed point



Deformations of NRCFT

- Dimensional counting

$$S = \int \underbrace{dt}_{-2}\ \underbrace{d^3x}_{-3}\ \left(\mathscr{L}_{\mathrm{CFT}} + \text{deformations}\right)$$

- operators with dim < 5: relevant; dim > 5: irrelevant
- one relevant deformation: $[O_2^\dagger O_2] = 4$
- Leading Galilean-invariant irrelevant deformation:

$$O_2^\dagger\left(\partial_t + \frac{\nabla^2}{4}\right)O_2 \qquad \text{dim = 6}$$

Away from conformality

- Finite scattering length and effective range can be treated as perturbation away from NRCFT

$$L = L_{\mathrm{CFT}} + \frac{1}{a} O_2^\dagger O_2 - r_0 O_2^\dagger \left(i\partial_t + \tfrac{1}{4}\nabla^2 \right) O_2$$

- Contribution to $\langle O_3 O_3^\dagger \rangle$ can be computed using conformal perturbation theory
  S.D. Chowdhury, R. Mishra, DTS to be published

- $$\frac{d\sigma}{dE} \sim \omega^{\Delta-5/2} \left(1 + \frac{c_1}{a_0 \sqrt{m\omega}} + c_2 r_0 \sqrt{m\omega} \right)$$

Conformal perturbation theory

- $$\langle O_3(x) O_3^\dagger(0)\rangle = \frac{1}{Z}\int \mathscr{D}\psi\, O_3(x) O_3^\dagger(0) e^{iS_{\text{CFT}} + \frac{i}{a}\int_y O_2^\dagger(y) O_2(y)}$$

$$= \langle O_3(x) O_3^\dagger(0)\rangle_0 + \frac{1}{a}\int dy\, \langle O_3(x) O_2^\dagger(y) O_2(y) O_3^\dagger(0)\rangle_0$$

$$|1\rangle\langle 1| \qquad |3\rangle\langle 3|$$

$$\frac{d\sigma}{dE} \sim \omega^{\Delta - 5/2}\left(1 + \frac{c_1}{a\sqrt{m\omega}}\right)$$

$$\Delta = 4.273$$
$$c_1 = 2.642$$

Effective-range correction

- Effective range correction proportional to

$$\frac{\Gamma\left(\frac{d}{2}+s-2\right)}{\Gamma\left(3-\frac{d}{2}\right)\Gamma(d-3)}=0$$

- Vanishes at physical dimension $d=3$, but not in fractional spatial dimension
    - we do not understand this completely

Open questions

- Can one resum all $1/a$ correction and determine the correlator along the whole RG flow?
- Can be done for O_2

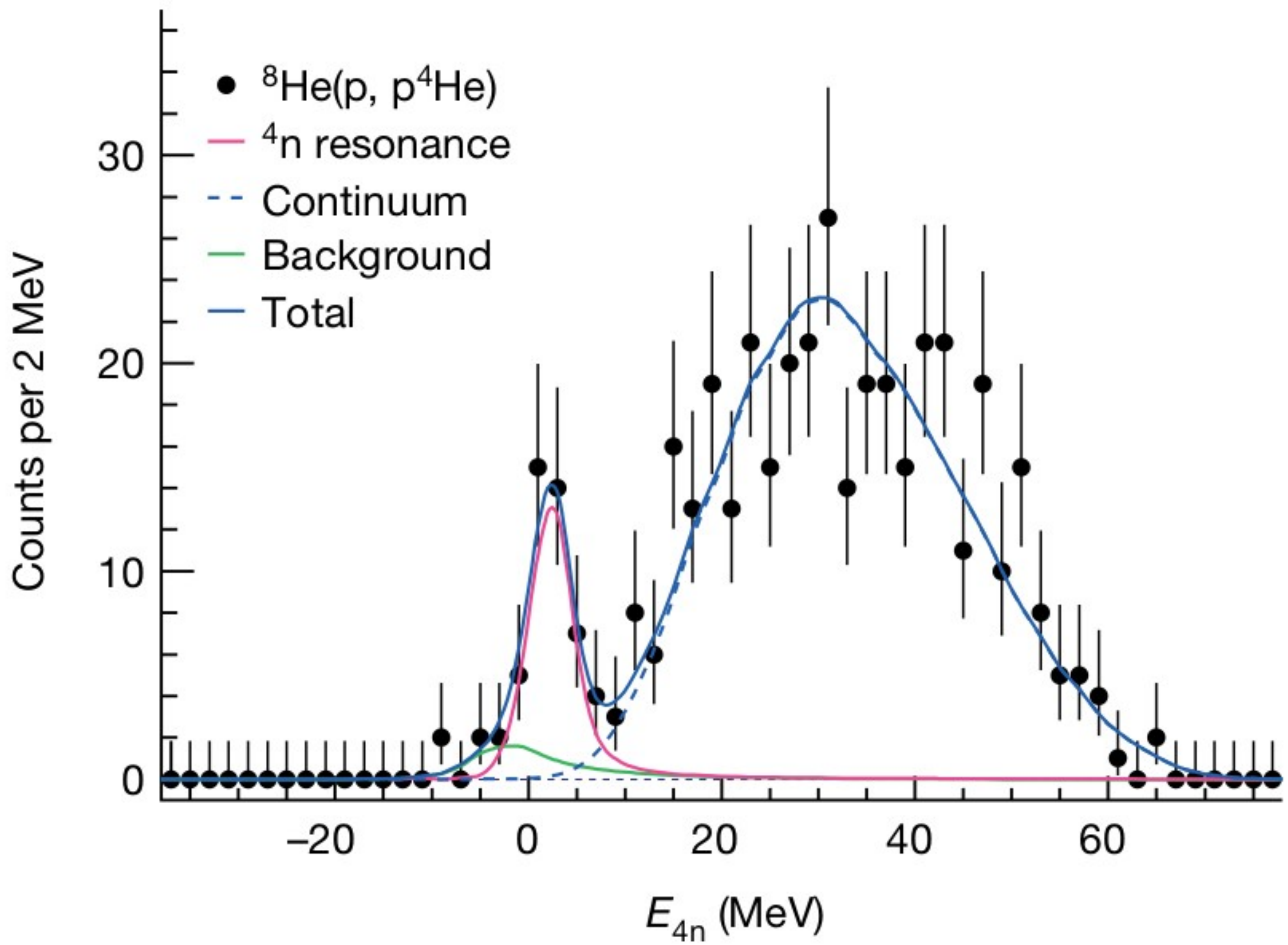
$$\langle O_2 O_2^\dagger \rangle = \left(\sqrt{\frac{p^2}{4} - p_0} - \frac{1}{a} \right)^{-1}$$

but needs to be down for charge-3 operators

- Charge-4 operators and higher?

Four-neutron production

$^8He(p,p^4He)4n$



(Duer et al Nature 2022)

Four-neutron production

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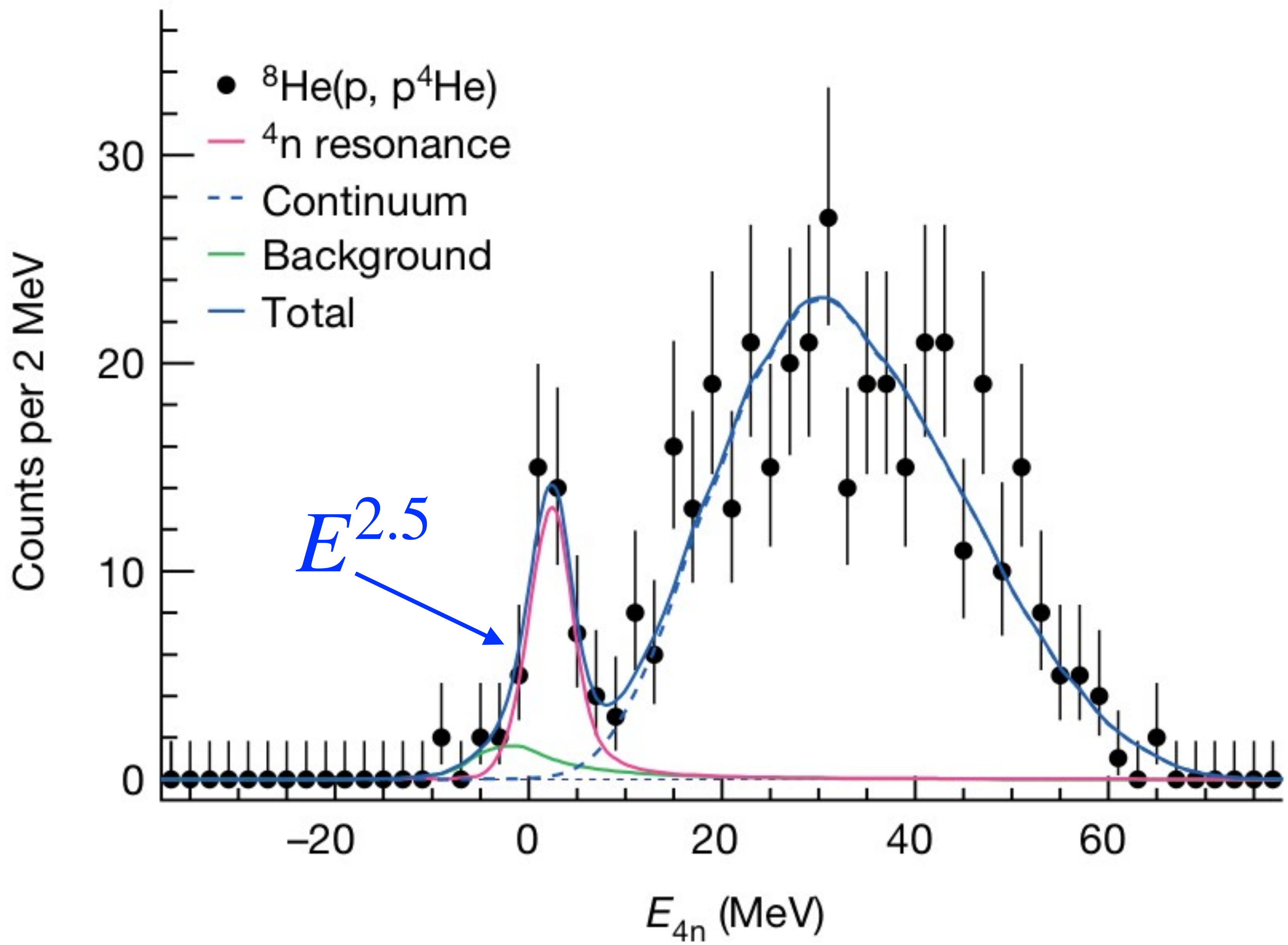

(Duer et al Nature 2022)

Conclusion

- QCD has a large cultural impact on CMP
- NR conformal field theories have Schrödinger symmetry
- Example: fermions at unitarity
- Approximately realized by neutrons in nuclear physics
- Leads to a power-law behavior of differential cross sections of certain processes near threshold
- Nonrelativistic conformal perturbation theory
- the full power of NRCFT still to be explored

Width of hypothetical multi-neutron resonances

- Suppose we have a multi-neutron resonance with $\hbar^2/ma^2 \ll \operatorname{Re} E \ll \hbar^2/mr_0^2$
- Then the width will scale as

$$\Gamma \sim E^{\Delta-5/2} = \begin{cases} E^{1.77} & N=3 \\ E^{2.5} & N=4 \end{cases}$$

DTS, M.A. Stephanov, H.-U. Yee, PRA 106(2022) L050801

New approach to two-neutron halo nuclei

- Borromean two-neutron halo nuclei (Ann),
  $A = {}^4$He, ^{9}Li, ^{20}C, …
- Small two-neutron separation energy

$$B({}^6\text{He}) = 0.975\ \text{MeV}$$
$$B({}^{11}\text{Li}) = 0.369\ \text{MeV}$$
$$B({}^{22}\text{C}) < 0.18\ \text{MeV?}$$

Hammer Ji Phillips 2017

- EFT based on smallness of B and neutron virtual energy

EFT of weakly-bound two-neutron halo nuclei

- Add two fields to the NRCFT of neutron: core ϕ, halo nucleus h
- Interaction: $h^\dagger O_2 \phi + O_2^\dagger \phi^\dagger h$
  - dimension: $\frac{3}{2} + \frac{3}{2} + 2 = 5$: marginal
  - leading-order EFT renormalizable;
  - Universal result for (charge radius)/(matter radius), E1 dipole strength function Hongo and DTS, PRL 2022

$$\frac{\mathrm{d}B(E1)}{\mathrm{d}\omega} = \frac{3}{4\pi} Z^2 e^2 \frac{12g^2}{\pi} \frac{A^{1/2}}{(A+2)^{5/2}} \frac{(\omega - B)^2}{\omega^4} \times f_{E1}\left(\frac{1}{-a\sqrt{\omega - B}}\right), \quad (29)$$

where

$$f_{E1}(x) = 1 - \frac{8}{3}x(1+x^2)^{3/2} + 4x^2\left(1 + \frac{2}{3}x^2\right). \quad (30)$$