The Basics of Loop Quantum Gravity

Discussion 2: Gravity as a Gauge Theory

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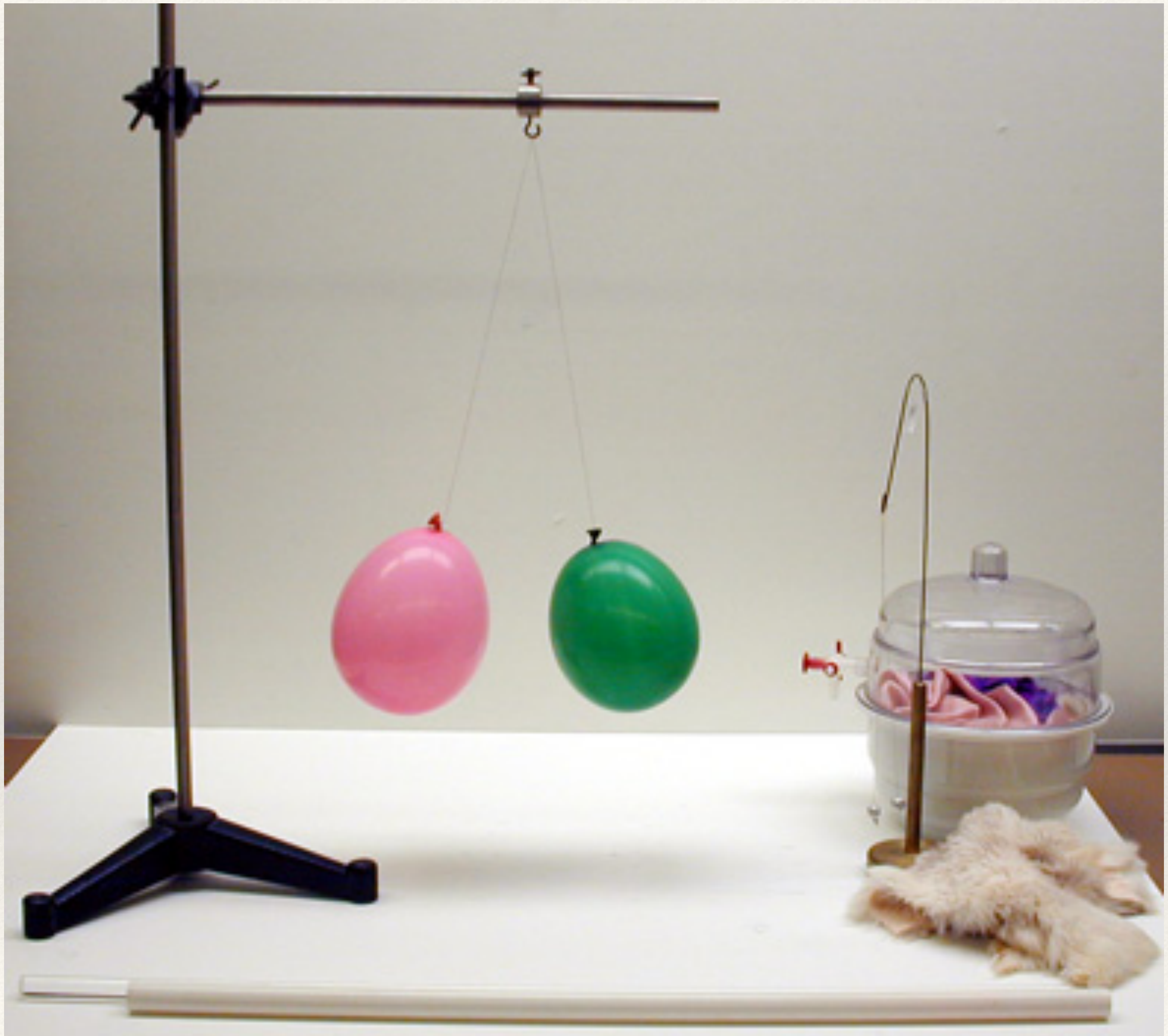
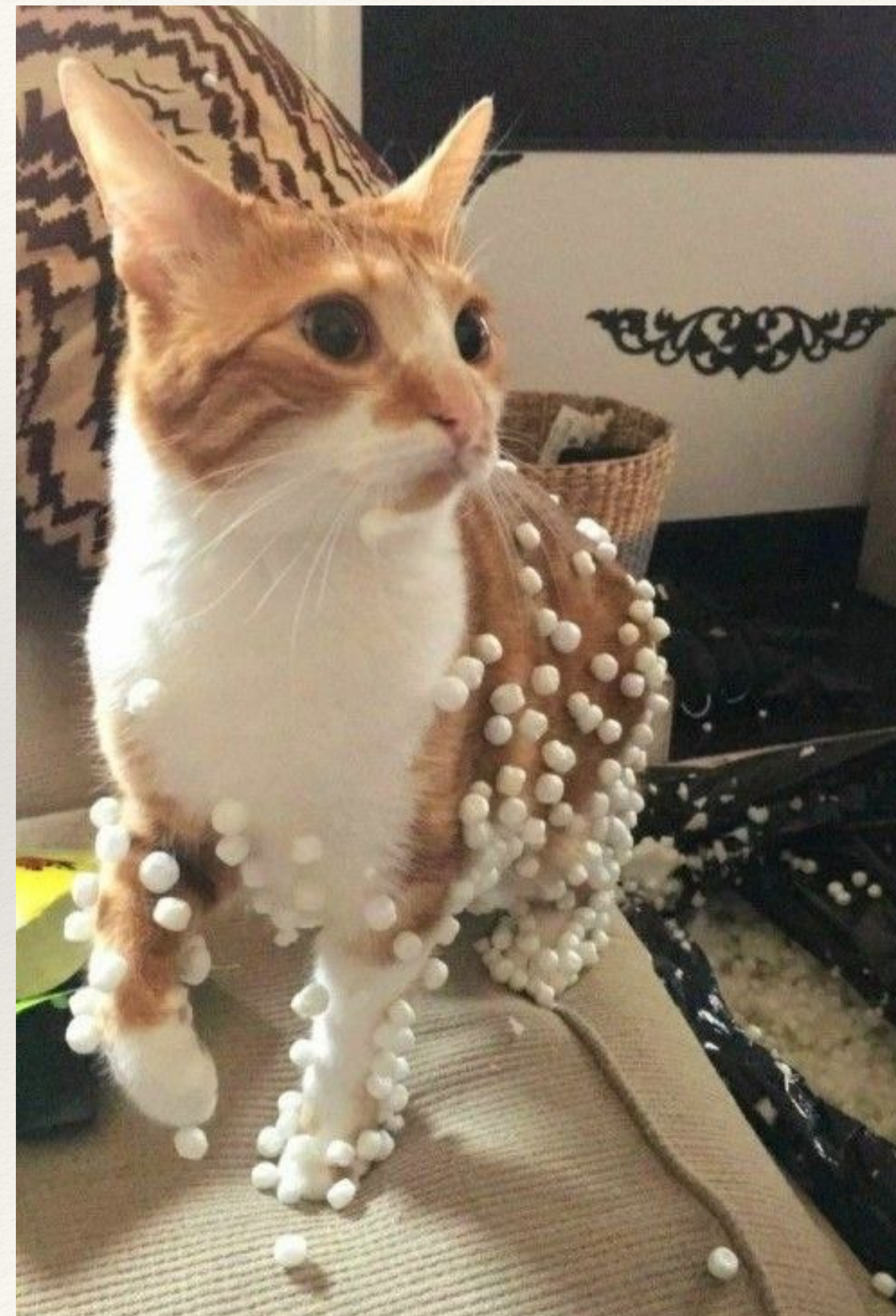
Review question 1:
What is a force?

Review question 2: But, given that, then how do two charged objects exert forces on one another?





Prologue

In 1870, Maxwell wrote in a letter to his wife that upon visiting his alma mater, Trinity College, he'd learned there was a legend that he used to toss cats from school windows to watch them acrobatically land on their padded paws…

Karin Brulliard, "Scientists just can't stop studying falling cats" Washington Post

“I had to explain that the proper method was to let the cat drop on a table or bed from about two inches, and that even then the cat lights on her feet.”

Karin Brulliard, “Scientists just can’t stop studying falling cats” Washington Post

The Physics of Somersaulting and Twisting, Cliff Frohlich
Scientific American,Vol. 242, No. 3 (March 1980), pp. 154-165

Using the images below Marey was the first person to provide an explanation, in 1894, for the question that had long vexed Natural Philosophers: How is it *physically possible* for a cat to land on its paws?

Before learning the answer, you would be right to ask, “Why is this so to physicists?”

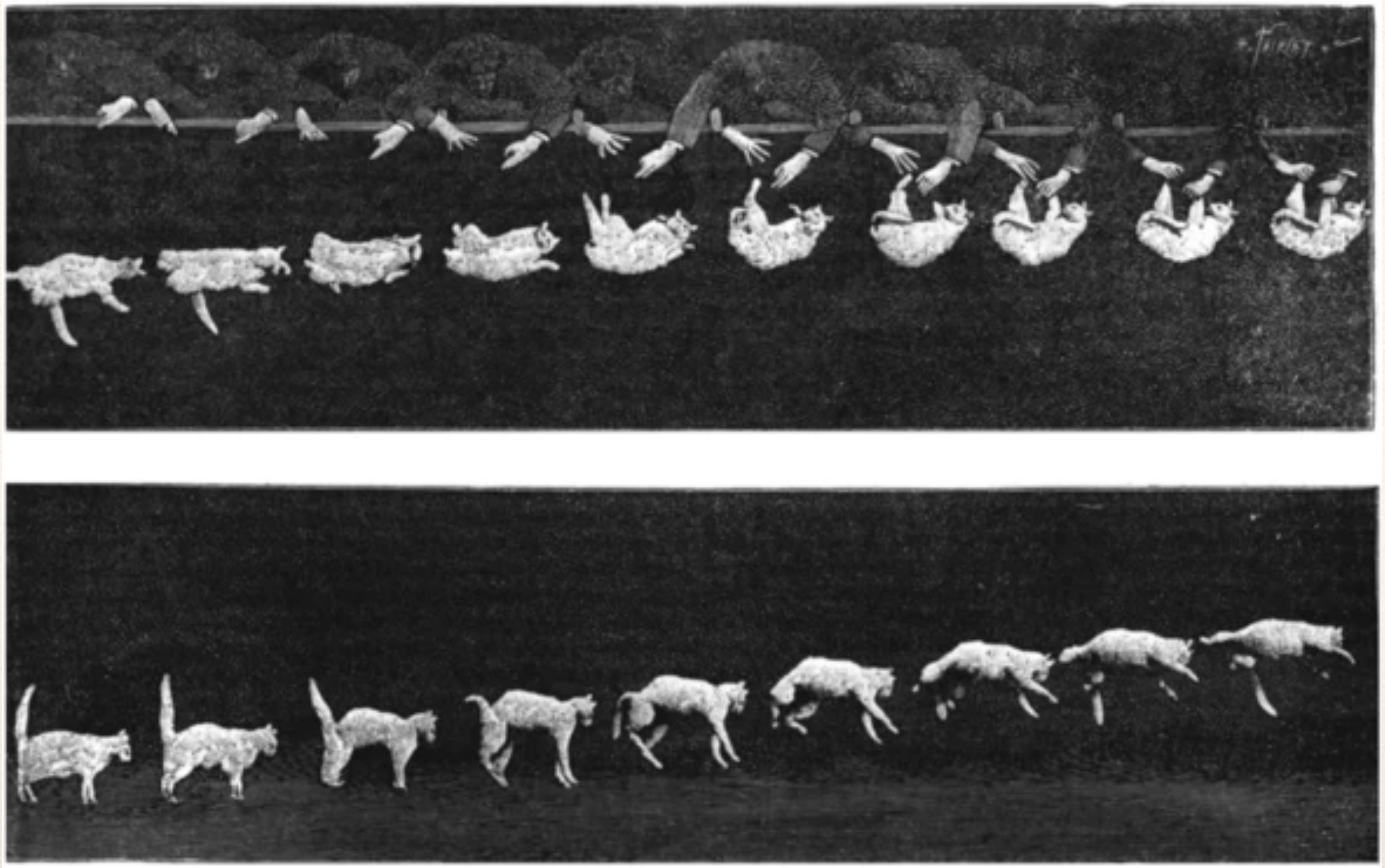


Fig. 1.—Side view of a falling cat. (The series runs from right to left.)

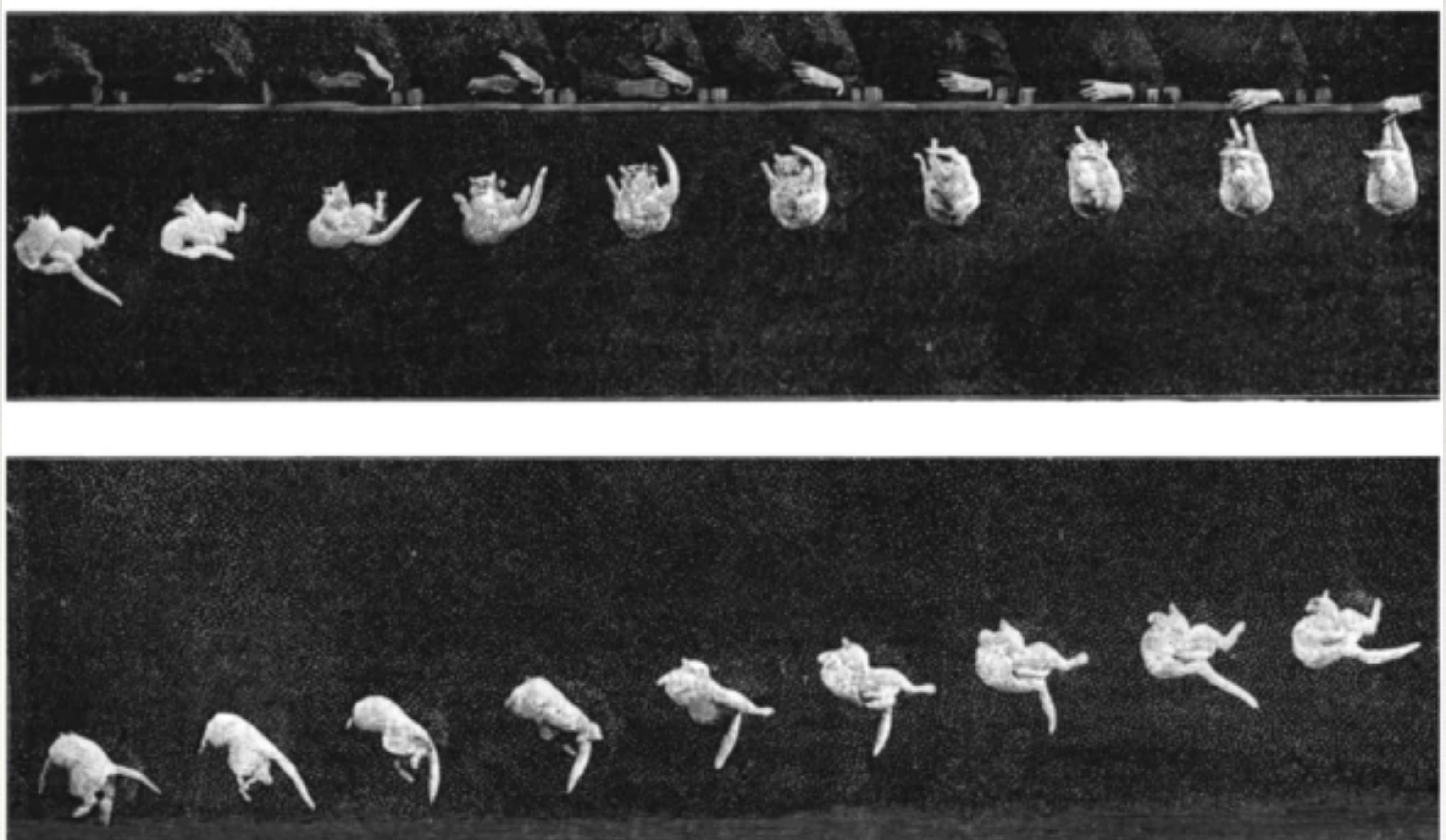


Fig. 2.—End view of a falling cat. (The series runs from right to left.)

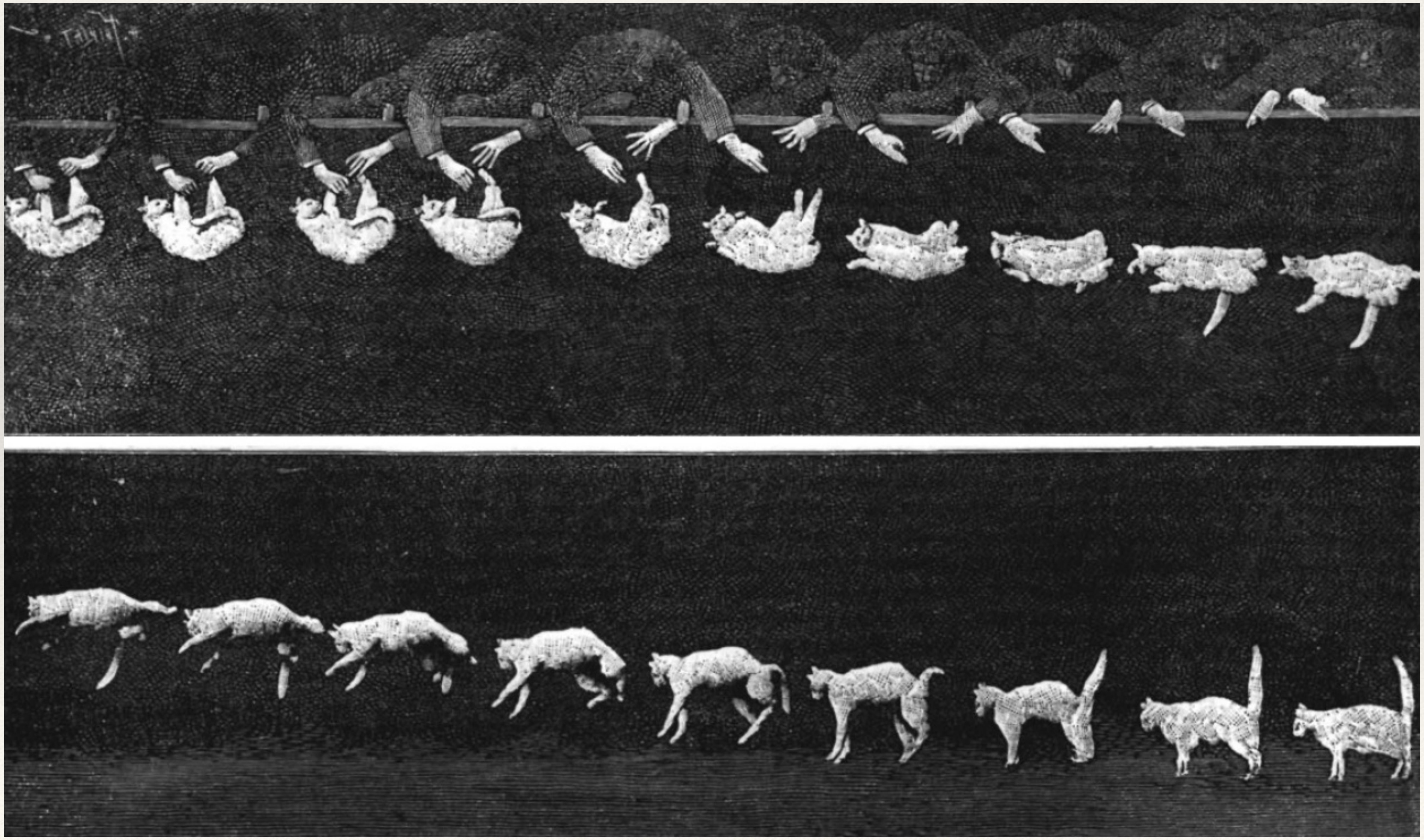
Marey, É.J (1894b). "Des mouvements que certains animaux exécutent pour retomber sur leurs pieds, lorsqu'ils sont précipités d'un lieu élevé". La Nature (in French). ***119****: 714–717.*

Marey's images of a side view…

…and a front view.

The puzzle: Done properly, the cat is released from rest with exactly zero net angular momentum, $\vec{J} = 0$.

How, then, does the cat rotate around to land on its feet?



Today's Discussion

1. Falling Cats as a Gauge Theory

2. Electromagnetism as a Gauge Theory

3. General Relativity as a Gauge Theory: Part I

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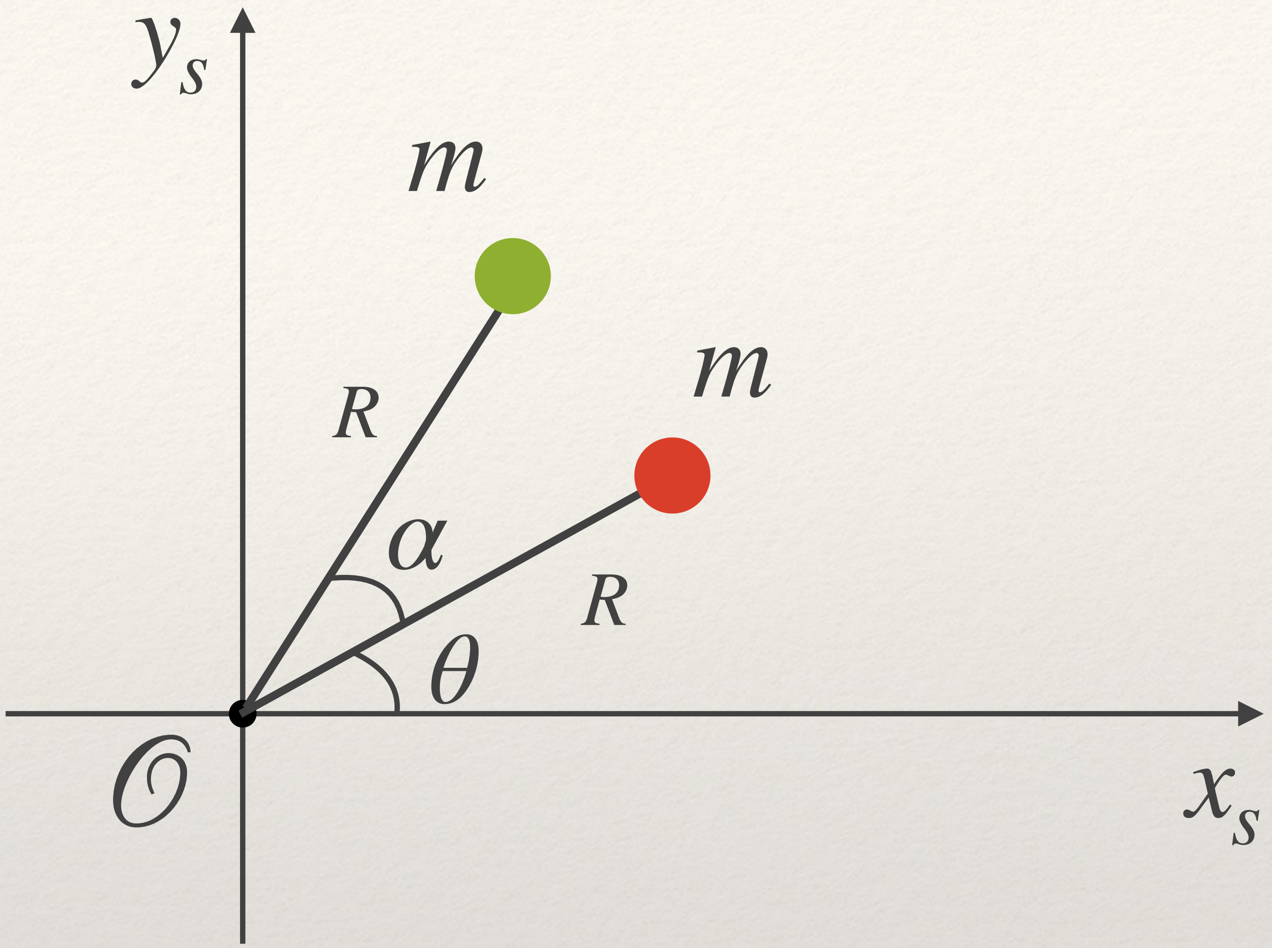
A Simplified Model of the Cat

The ‘body’ of the cat is two massless rods, length R, ending in equal masses (m).

α = shape coordinate

θ = orientational coord.



A ‘muscle’ at $\mathcal{O}$ can change α, but never generates any external torque.

Take the red and green masses distinguishable…

…then α and $(2\pi - \alpha)$ are distinct configurations, so

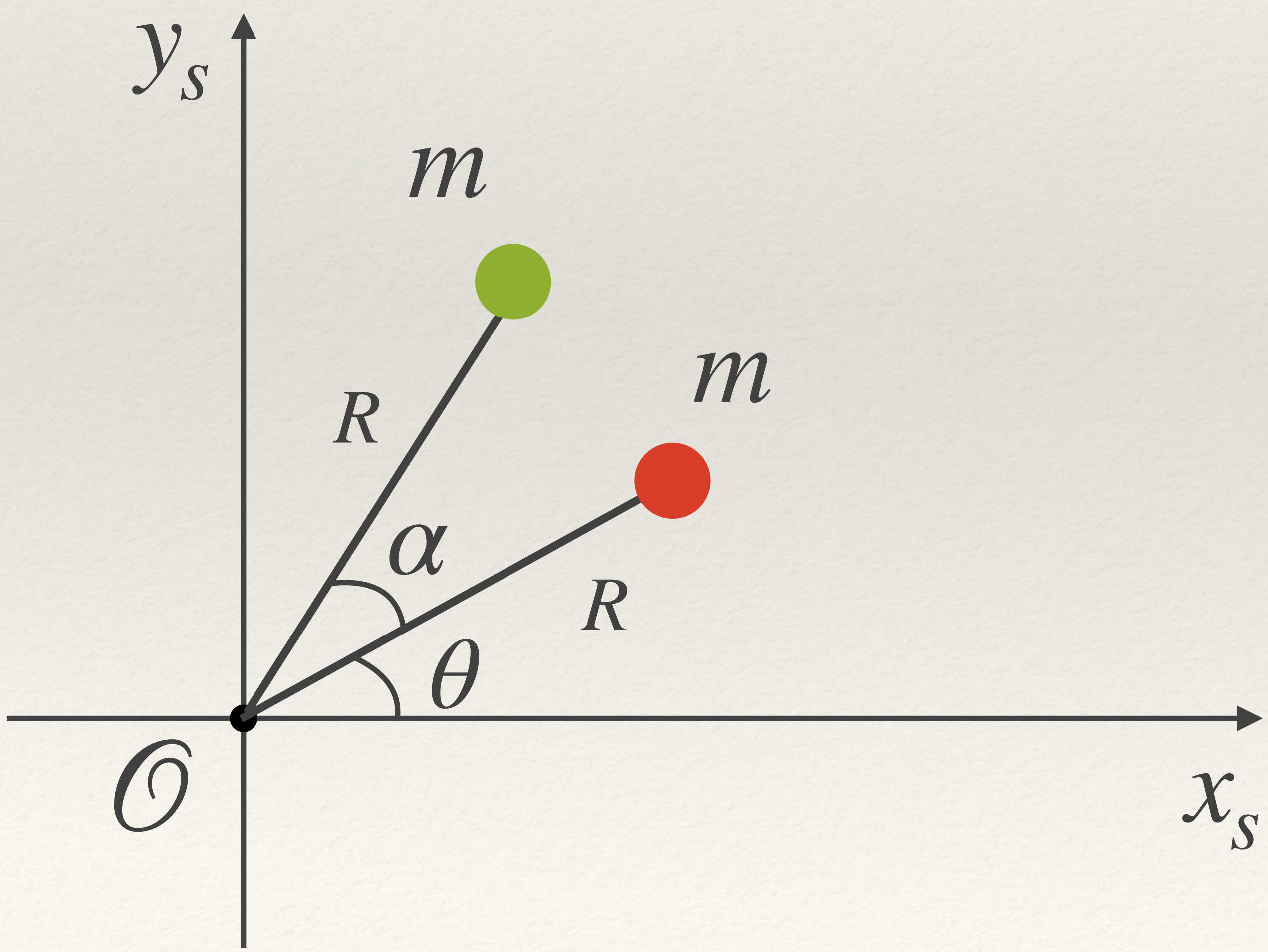
$$\alpha \in [0,2\pi) \text{ or } \alpha \in S^1.$$

[Littlejohn & Reinsch, Rev. Mod. Phys. 69, 1997]

The Cat's Angular Momentum

Exercise 1 Using your favorite definition of $\vec{J}$, and the Cartesian space coords, show: ($\dot{x} := dx/dt$, etc)

$$J_{\text{tot}} = J_z = m(x_{s1}\dot{y}_{s1} - y_{s1}\dot{x}_{s1}) + m(x_{s2}\dot{y}_{s2} - y_{s2}\dot{x}_{s2})$$
$$= \dot{\theta} + \dot{\theta} + \dot{\alpha} = 2\dot{\theta} + \dot{\alpha}$$



Inspired by the cat, we require:

$$J_{\text{tot}} = 2\dot{\theta} + \dot{\alpha} = 0.$$

Then: a change of shape ($\dot{\alpha}$) forces a change in orientation ($\dot{\theta}$) in order to maintain $J_{\text{tot}} = 0$.

The Cat Constraint

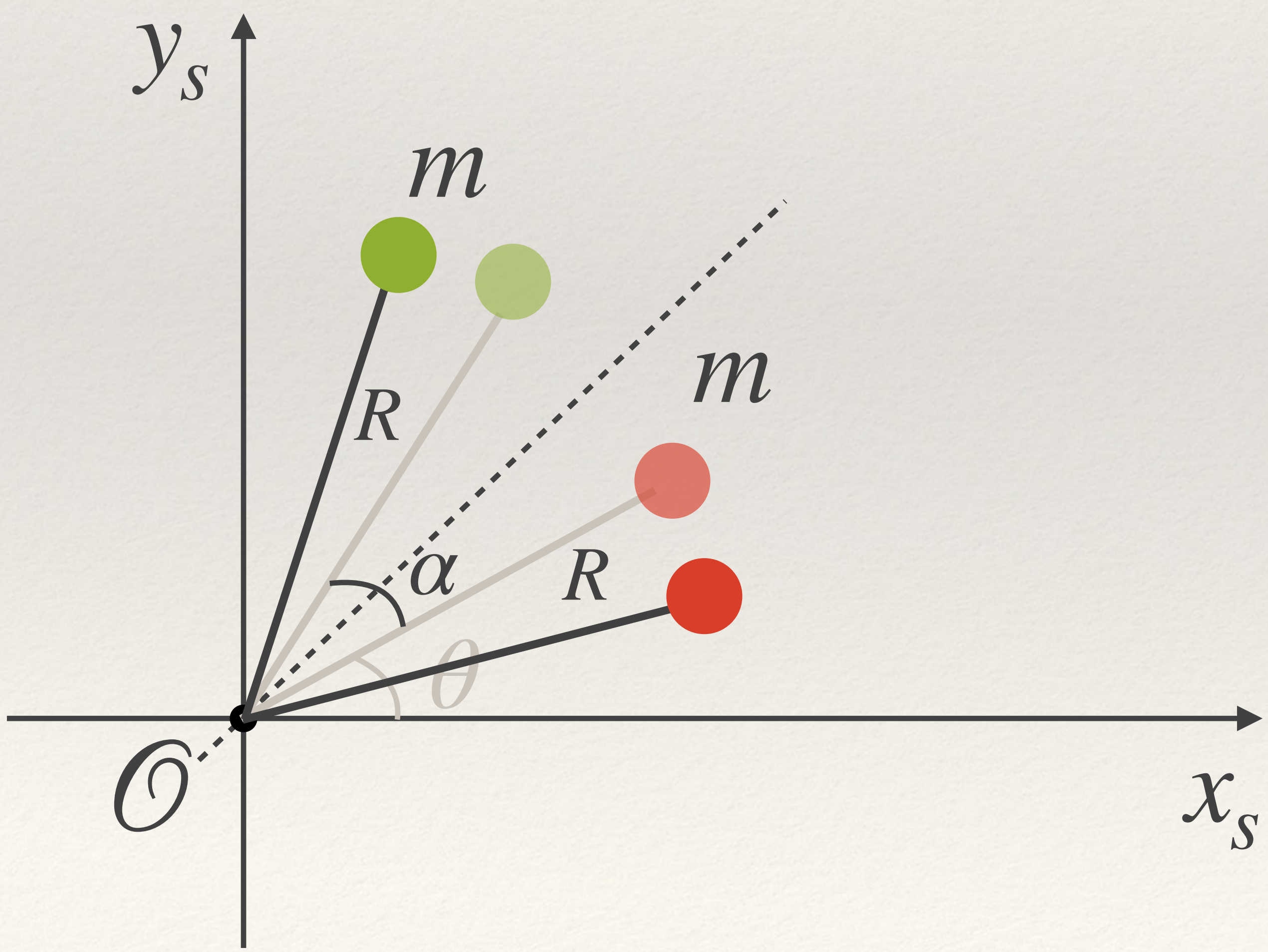
It is no coincidence that t can be removed from

$$J_{\text{tot}} = 2\dot{\theta} + \dot{\alpha} = 0 \rightsquigarrow 2d\theta + d\alpha = 0.$$

If green moves counter-clockwise(ccw) $\Delta\alpha/2$, red moves clockwise $\Delta\alpha/2$, total shape change $= \Delta\alpha$, and

$$\Delta\theta = \frac{1}{2}\Delta\alpha$$

$\therefore$ the bisector of α is fixed.



Unfortunately, this model is too simple to capture the cat's reorientation as it falls!…

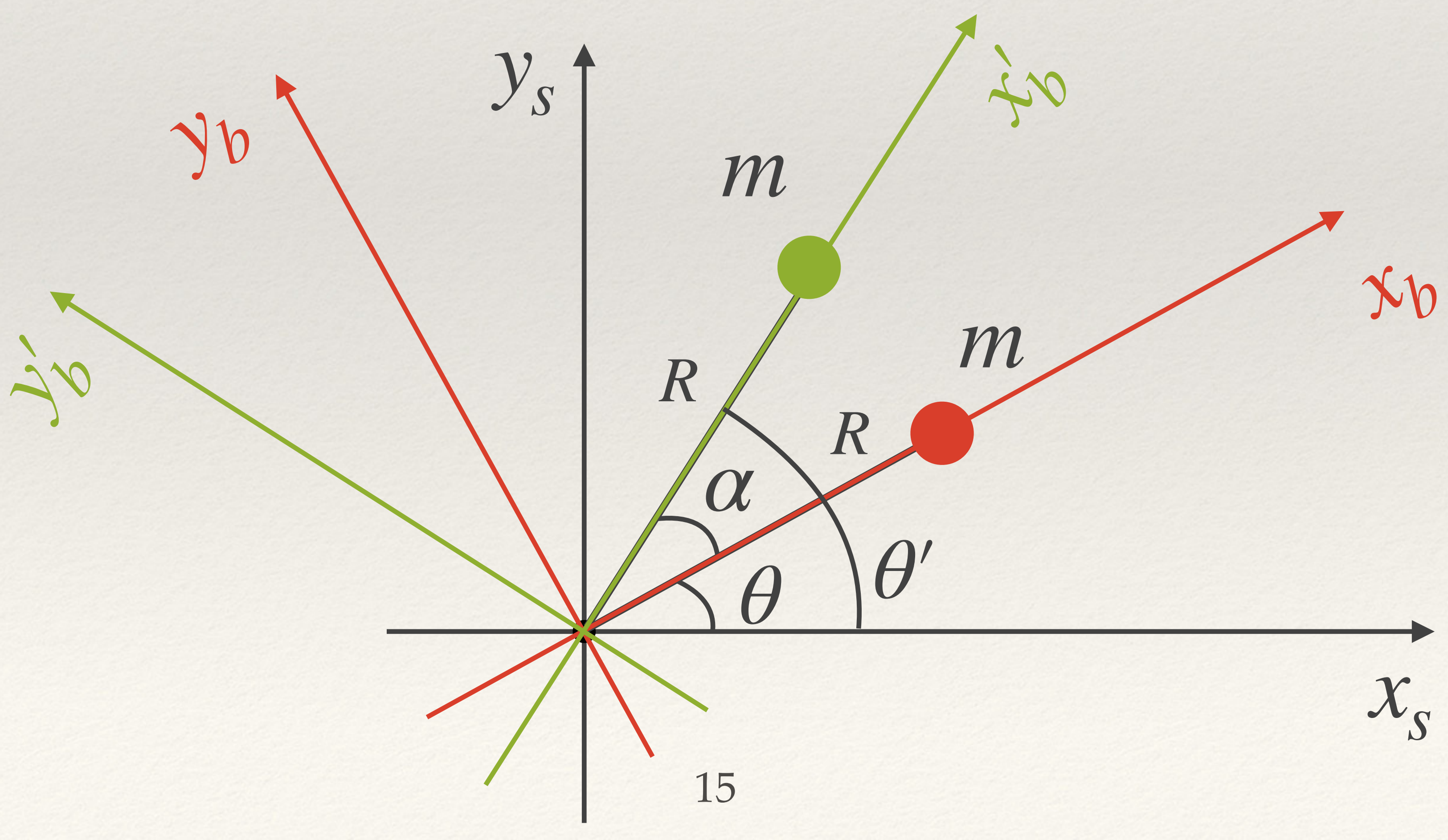
A Brief Aside on Terminology

We can also introduce different choices of 'body' axes

$$\theta' = \theta + \alpha. \quad (*)$$

Which axes you use is conventional, we call it a gauge convention and Eq. (*) is a gauge transformation.

Note that α is gauge invariant, while θ is not.



A (Less) Simplified Model of the Cat

Exercise 2 With a bit more algebra this time, show:

$$J_{\text{tot}} = (4 + 2\cos\beta)\,\dot{\theta} + (3 + 2\cos\beta)\,\dot{\alpha} + (1 + \cos\beta)\,\dot{\beta} = 0$$

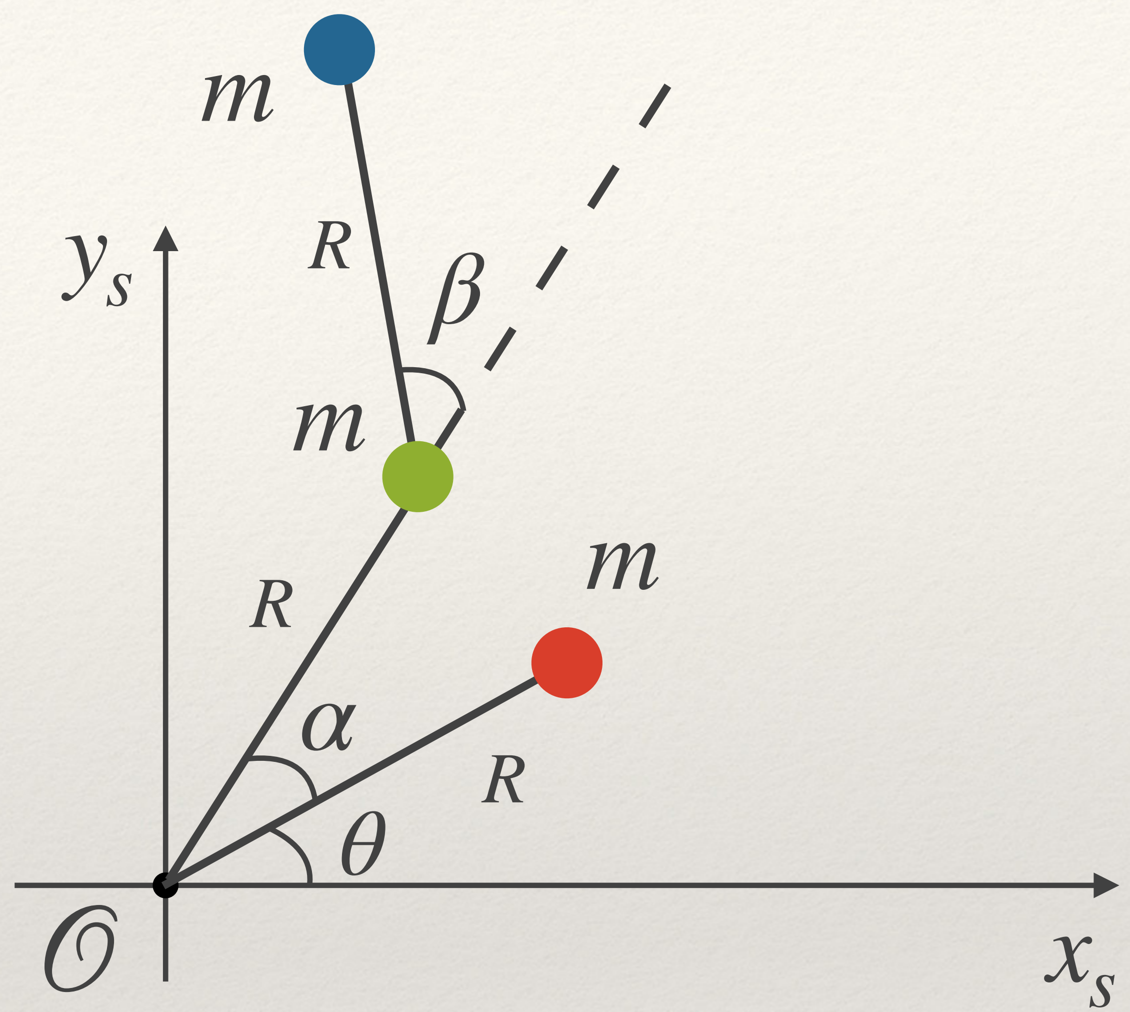
Define

$$\dot{\theta} = A_\alpha \dot{\alpha} + A_\beta \dot{\beta},$$

then,

$$A_\alpha = -\frac{3 + 2\cos\beta}{4 + 2\cos\beta},$$

$$A_\beta = -\frac{1 + \cos\beta}{4 + 2\cos\beta}.$$



How much θ changes depends on where you are in

shape space: (α, β)

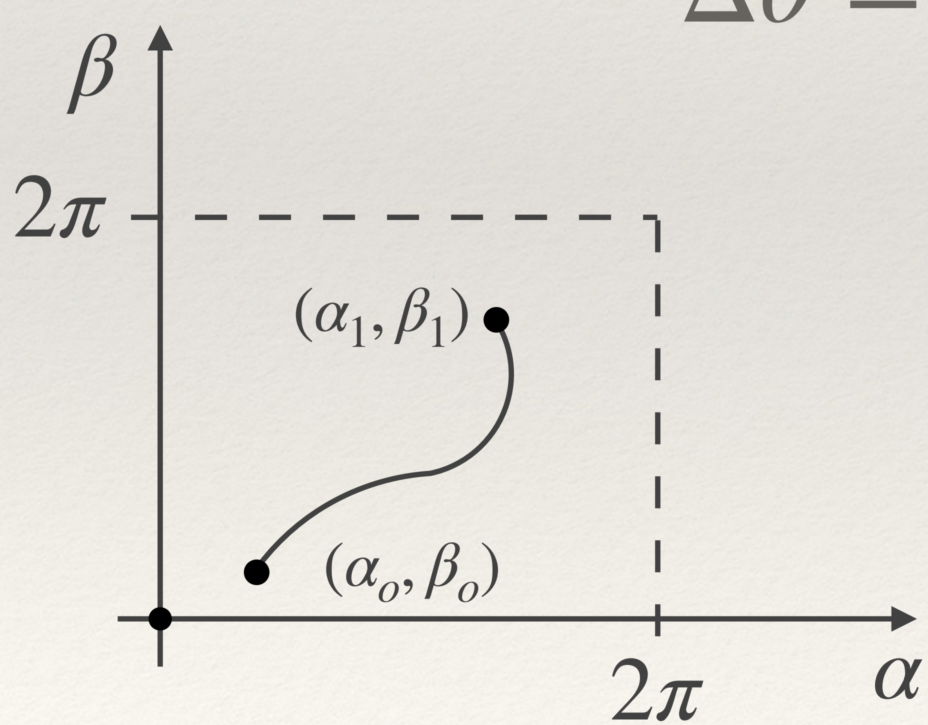
A Geometric Phase for the Cat

We have

$$\dot{\theta} = A_\alpha \dot{\alpha} + A_\beta \dot{\beta}, \quad A_\alpha = -\frac{3 + 2\cos\beta}{4 + 2\cos\beta}, \quad A_\beta = -\frac{1 + \cos\beta}{4 + 2\cos\beta}.$$

To calculate the total change in θ, call it $\Delta\theta$, we integrate

$$\Delta\theta = \int A_\alpha d\alpha + \int A_\beta d\beta.$$



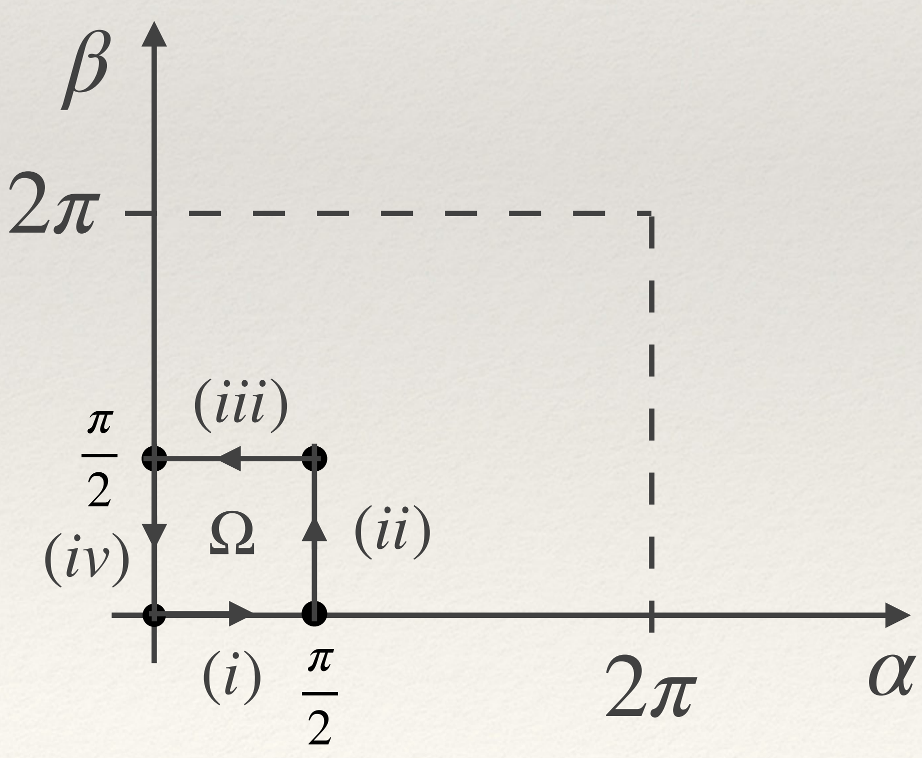
Shape Space

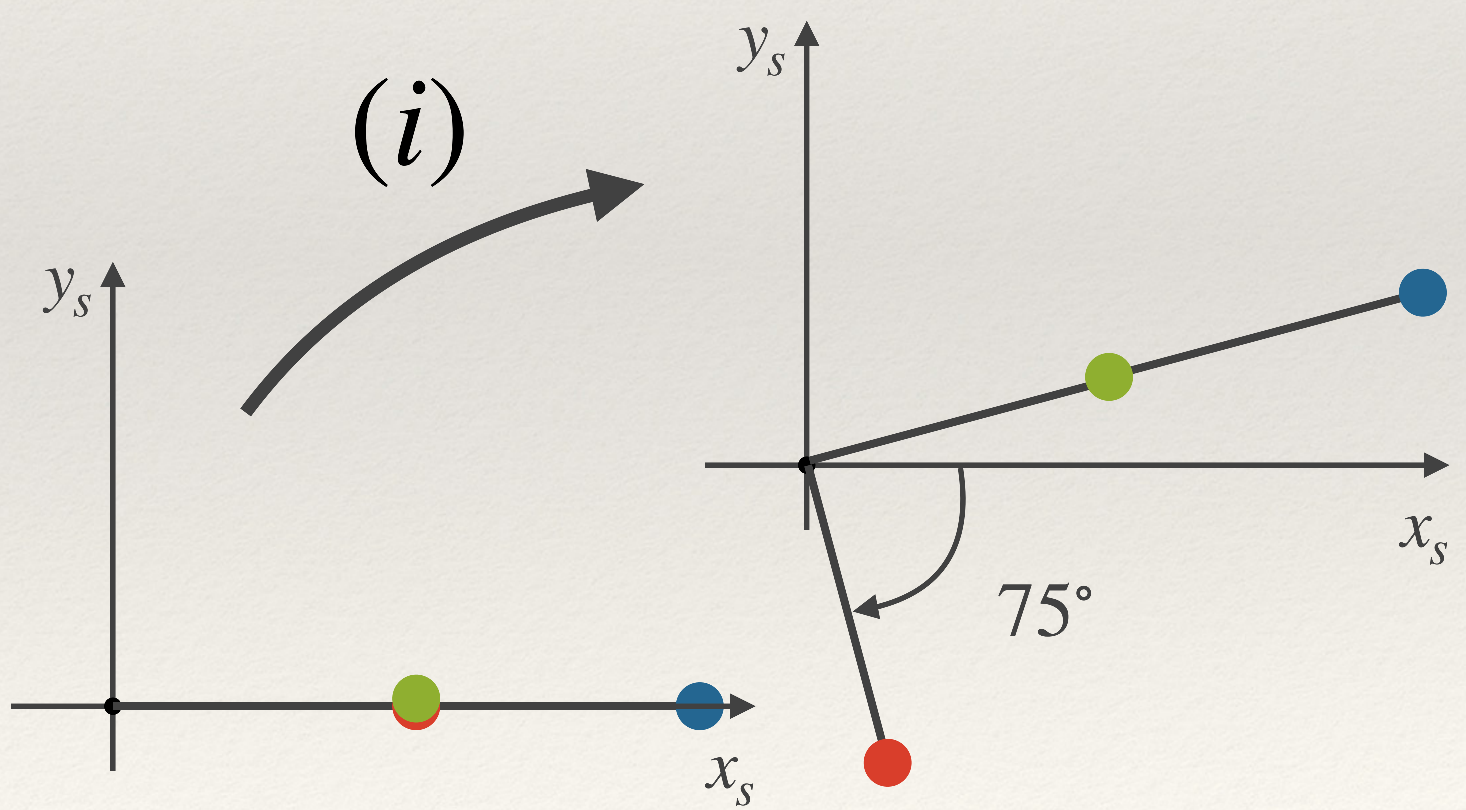
Notice that $\Delta\theta$ doesn't depend on how fast you traverse the curve: we call it a "geometric phase" or "Berry phase" in QM.

Does this model capture the Cat Trick?

Consider the closed path in shape space shown below, along path (i) we have,

$$\Delta\theta_{(i)} = \int_0^{\pi/2} A_\alpha d\alpha = \int_0^{\pi/2} - \frac{3+2\cos\beta}{4+2\cos\beta}\bigg|_{\beta=0} d\alpha = -\frac{5\pi}{12} = -75^\circ$$

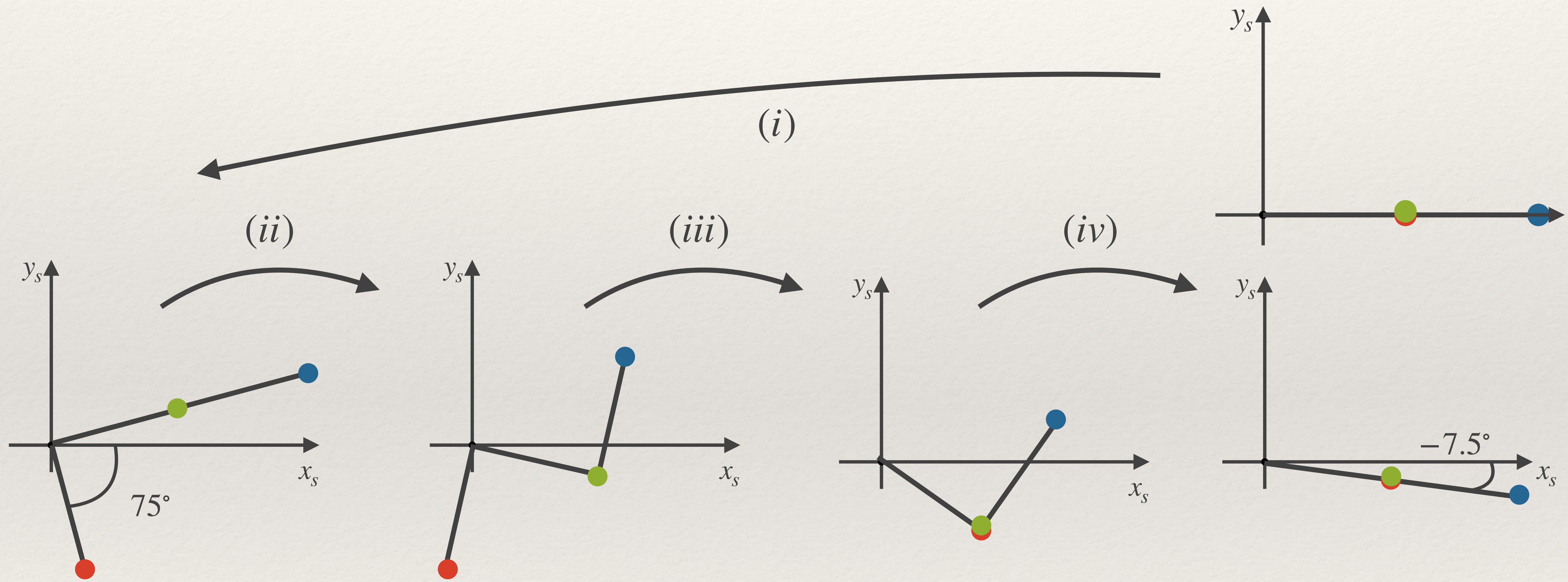




Does this model capture the Cat Trick?

Exercise 3 Prove:

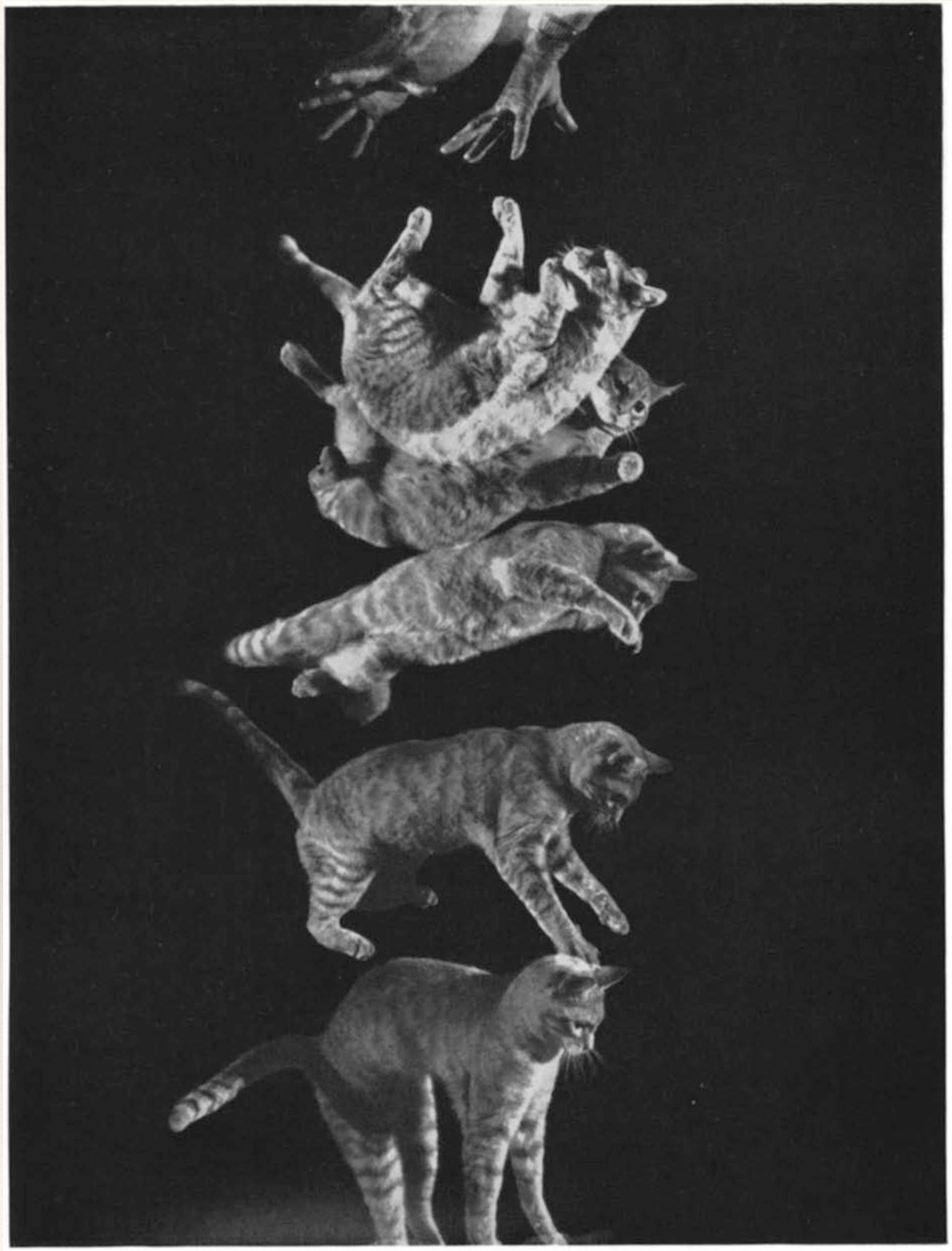
$$\Delta\theta_{(ii)} = -\frac{\pi}{4} + \frac{\pi}{6\sqrt{3}} = -27.7^\circ, \quad \Delta\theta_{(iii)} = \frac{3\pi}{8} = 67.5^\circ, \quad \Delta\theta_{(iii)} = \frac{3\pi}{8} = 27.7^\circ,$$



$$\Delta\theta_{\text{tot}} = \Delta\theta_{(i)} + \Delta\theta_{(ii)} + \Delta\theta_{(iii)} + \Delta\theta_{(iv)} = -7.5^\circ.$$

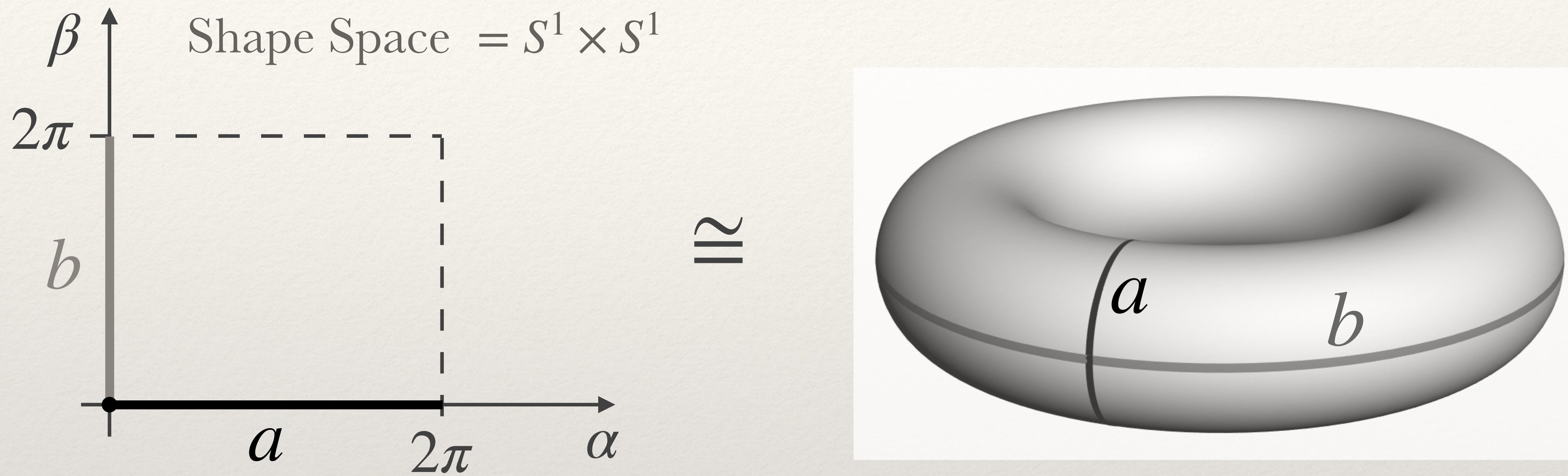
Our 'cat' has changed orientation!

A real cat has even more shape parameters and does this incredibly efficiently!



Shape Space can be and is Topologically Rich

Our shape space is a torus:



Exercise 4 Repeat the calculation of $\Delta\theta$ for the *a*- and the *b*-cycles. Draw the 'cat' before and after traversal of these cycles. [Hint: because of the topology of our shape space, these are closed paths that begin where they end.]

For many reasons differential forms are useful

Each time we worked with a cat model we found

$$d\theta = A_\alpha d\alpha + A_\beta d\beta.$$

Whatever our coordinates, say x^μ, we will generally have

$$A = A_\mu dx^\mu, \qquad \mu \in \{0,1,2,3\},$$

with A the "potential 1-form". The appearance of differential forms suggests introduction of the "field strength"

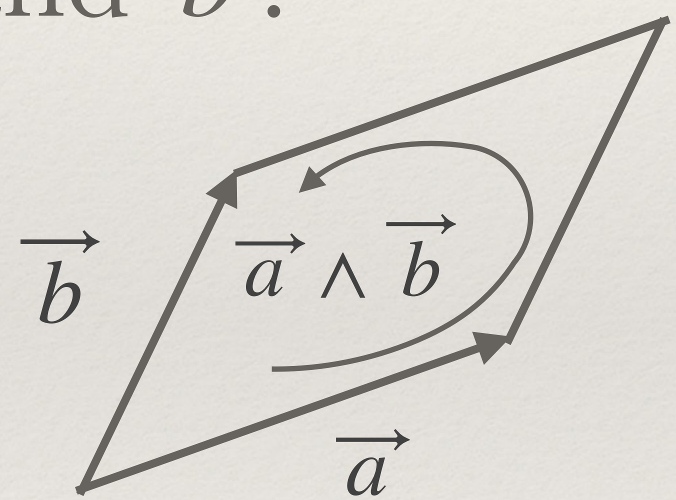
$$F := dA = \frac{1}{2}(\partial_a A_b - \partial_b A_a) dx^a \wedge dx^b, \quad a, b \in \{1,2\}.$$

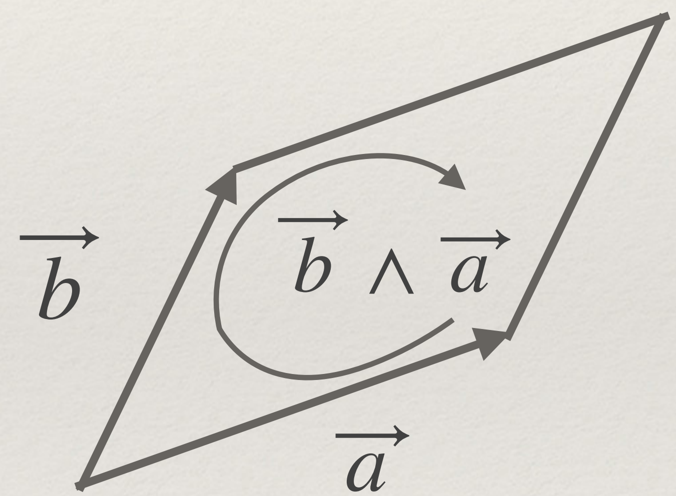
Begin asides: on the wedge product

On any vector space we can define a wedge product

$$\overrightarrow{a} \wedge \overrightarrow{b} = - \overrightarrow{b} \wedge \overrightarrow{a}.$$

We call the result a "bivector" and geometrically it is the oriented area of the parallelogram spanned by $\overrightarrow{a}$ and $\overrightarrow{b}$:





In a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of the 2D span of $\overrightarrow{a}$ and $\overrightarrow{b}$ it is

$$\overrightarrow{a} \wedge \overrightarrow{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = (a_1 b_2 - a_2 b_1)\mathbf{e}_1 \wedge \mathbf{e}_2$$

Wedges, Dets, Volume Forms and All That

You will have noticed that the wedge of the last slide has $\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ as its component. This is useful!

Do a linear trans. T on $\{\mathbf{e}_1, \mathbf{e}_2\}$ to get $\{\mathbf{f}_1, \mathbf{f}_2\}$, then

$$\mathbf{f}_1 \wedge \mathbf{f}_2 = (T_1^1 \mathbf{e}_1 + T_1^2 \mathbf{e}_2) \wedge (T_2^1 \mathbf{e}_1 + T_2^2 \mathbf{e}_2) = (\det T)\, \mathbf{e}_1 \wedge \mathbf{e}_2 \,.$$

In coordinates, the physical volume depends on the metric $g_{\mu\nu}$:

$$\text{vol} = \sqrt{|\det g_{\mu\nu}|}\, dx^1 \wedge \cdots \wedge dx^n.$$

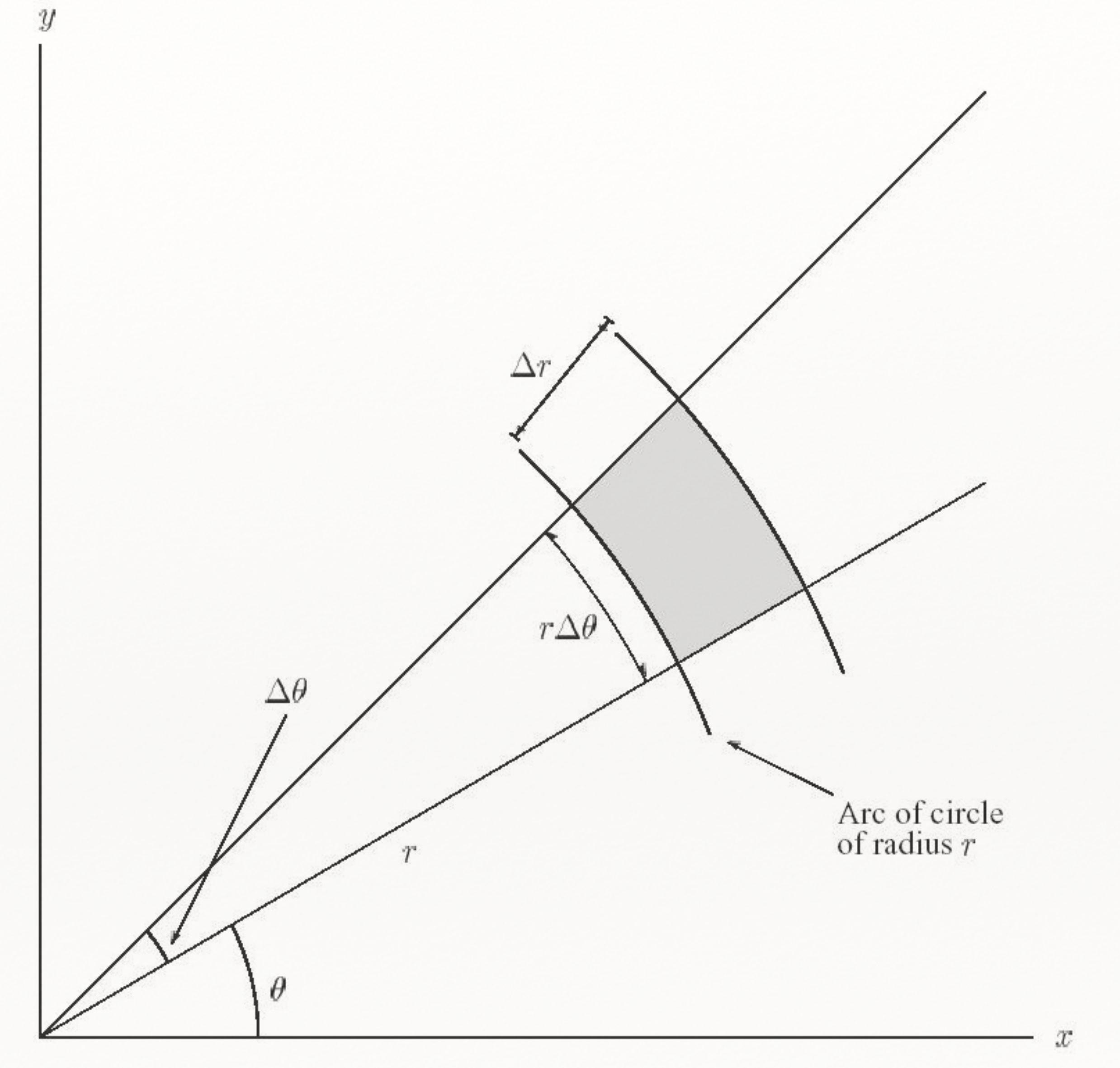
Under a coord. change $dx^{\mu'} = T^{\mu'}{}_{\nu}\, dx^{\nu}$,

$$\text{vol}' = \det T^{-1} \sqrt{|\det g_{\mu\nu}|}\, \det T\, dx^1 \wedge \cdots \wedge dx^n = \text{vol}.$$

Volume Form Example: Polar Coordinates

Polar coordinates (r, θ) for the plane

$$\text{area} = dx \wedge dy$$
$$= \sqrt{|\det g_{\mu\nu}|}\, dr \wedge d\theta$$



Here

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix},$$

and so

$$\text{area} = dx \wedge dy = r dr \wedge d\theta = dr \wedge r d\theta.$$

Converting between tensor and wedge bases

I blew past a notational subtlety: we defined

$$F := dA = \frac{1}{2}(\partial_a A_b - \partial_b A_a)dx^a \wedge dx^b, \quad a, b \in \{1,2\}.$$

What are the components of F? Usually, 'components' *means* in a tensor basis, i.e., $F = F_{ab}dx^a \otimes dx^b$.

Let's guess $F_{ab} = \partial_a A_b - \partial_b A_a$, and check

$$\begin{aligned}
\frac{1}{2}F_{ab}dx^a \wedge dx^b &= \frac{1}{2}F_{ab}(dx^a \otimes dx^b - dx^b \otimes dx^a) \\
&= \frac{1}{2}F_{ab}dx^a \otimes dx^b - \frac{1}{2}F_{ab}dx^b \otimes dx^a \\
&= \frac{1}{2}F_{ab}dx^a \otimes dx^b - \frac{1}{2}F_{ba}dx^a \otimes dx^b \\
&= \frac{1}{2}(F_{ab} - F_{ba})dx^a \otimes dx^b \\
&= F_{ab}dx^a \otimes dx^b \quad \checkmark
\end{aligned}$$

Volume Forms and Tensor Densities

In coordinates, it can be useful to break the volume form up. The Levi-Civita symbol is helpful:

$$\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} +1 & \text{if } \mu_1\mu_2\cdots\mu_n \text{ is an even permutation of } 01\cdots(n-1), \\ -1 & \text{if } \mu_1\mu_2\cdots\mu_n \text{ is an odd permutation of } 01\cdots(n-1), \\ 0 & \text{otherwise}\,. \end{cases}$$

It's called a symbol because it is not a tensor: e.g., it doesn't transform as one. However, its complete antisymmetry means that you can compute det's with it (like the wedge):

$$\tilde{\epsilon}_{\mu_1'\mu_2'\cdots\mu_n'} \det T = \tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} T^{\mu_1}{}_{\mu_1'} T^{\mu_2}{}_{\mu_2'} \cdots T^{\mu_n}{}_{\mu_n'}\,.$$

Volume Forms and Tensor Densities

For example, consider a coordinate transformation

$T^{\mu}{}_{\mu'} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}}$, then,

$$\tilde{\epsilon}_{\mu'_1\mu'_2\cdots\mu'_n} = \frac{1}{\det T}\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}T^{\mu_1}{}_{\mu'_1}T^{\mu_2}{}_{\mu'_2}\cdots T^{\mu_n}{}_{\mu'_n}$$
$$= \det\left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right|\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}T^{\mu_1}{}_{\mu'_1}T^{\mu_2}{}_{\mu'_2}\cdots T^{\mu_n}{}_{\mu'_n}\,.$$

This is almost a tensor; it only fails because of a power of $\det|\partial x^{\mu'}/\partial x^{\mu}|$ up front (called the 'density weight'); we term objects that transform with such powers "tensor densities" & denote them with the over tilde $\tilde{}$.

(Recall: Ashtekar's $\tilde{E}^a_i(x)$, densitized triad.) $\square$

Final comments on the 'cat'

The field strength

$$F := dA = \frac{1}{2}(\partial_a A_b - \partial_b A_a) dx^a \wedge dx^b, \quad a, b \in \{1,2\},$$

gives us another way to compute $\Delta\theta_{\text{tot}}$:

$$\Delta\theta_{\text{tot}} = \int_0^{\pi/2} \int_0^{\pi/2} F = \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{\partial A_\beta}{\partial \alpha} - \frac{\partial A_\alpha}{\partial \beta} \right) d\alpha \wedge d\beta \,.$$

Why does this work? It's due to Stokes' theorem:

$$\Delta\theta_{\text{tot}} = \iint_\Omega F = \iint_\Omega dA = \oint_{\partial\Omega} A.$$

Gauge Invariance

For our 2nd 'cat', body axes (x_b', y_b') (slide 15) can be rich, e.g. pick x_b' aligned with the blue mass and

$$\theta' = \theta + \lambda(\alpha, \beta).$$

[*Ex. 5* Prove this slide]

Once again

$$d\theta' = A'_\alpha d\alpha + A'_\beta d\beta,$$

where $A'_\alpha = A_\alpha + \partial_\alpha \lambda$ and $A'_\beta = A_\beta + \partial_\beta \lambda$, but(!)

$$F' = \partial_\alpha A'_\beta - \partial_\beta A'_\alpha = F + \partial_\alpha \partial_\beta \lambda - \partial_\beta \partial_\alpha \lambda = F.$$

Or, more succinctly, if $A \to A + d\lambda$, then

$$F' = d(A + d\lambda) = dA + d^2\lambda = F.$$

Today's Discussion

1. Falling Cats as a Gauge Theory

2. Electromagnetism as a Gauge Theory

3. General Relativity as a Gauge Theory: Part I

I draw from the wonderful book *Gauge Fields, Knots, and Gravity*, by J. Baez and J. P. Muniain, World Scientific, 1994 in this section.

The form language in E&M

Experience with calculations in E&M highlights electric and magnetic fluxes, suggesting again

$$E = E_x dx + E_y dy + E_z dz,$$

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$

Why these forms? Ans: *Ex. 6* Confirm that

dE has 2-form components $\vec{\nabla} \times \vec{E}$ and

dB has a single 3-form component $\vec{\nabla} \cdot \vec{B}$

Thus, two of the **static** Maxwell Eqns. are

$$dE = 0 \quad \text{and} \quad dB = 0.$$

The form language in E&M

Just as with the cat, we can collect these forms into a field strength:

$$F = B + E \wedge dt = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu,$$

where

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}, \text{ and we have } dF = 0.$$

[From now on $c = 1$, Heaviside-Lorentz units, and frequently $x^0 = t$.]

The form language in E&M

Now relax the static assumption [$E = E(x^\mu)$, $B = B(x^\mu)$] and decompose the exterior derivative into spatial and time pieces (spacetime split):

$$\begin{aligned} dB &= d_S B + dt \wedge \partial_t B \\ &= \partial_i B_I dx^i \wedge dx^I + dt \wedge \partial_t B \quad (i = 1,2,3) \\ &= (\overrightarrow{\nabla} \cdot \overrightarrow{B}) dx \wedge dy \wedge dz + dt \wedge \partial_t B, \end{aligned}$$

here I is a multi-index running over $B's$ 2-forms.

Now,

$$\begin{aligned} 0 = dF &= dB + dE \wedge dt \\ &= d_S B + dt \wedge \partial_t B + d_S E \wedge dt \end{aligned} \implies \begin{cases} d_S B = 0 \\ \partial_t B + d_S E = 0 \end{cases}$$

The Hodge dual…

…or Hodge star, $\star$, is an operation that takes a p-form to an $(n-p)$-form, in an n-dim. manifold.

For example, on flat $\mathbb{R}^3$

$$\star\, dx = dy \wedge dz, \quad \star\, dy = dz \wedge dx, \quad \star\, dz = dx \wedge dy\,.$$

The logic is that for any two p-forms ω and μ

$$\omega \wedge \star\mu = k\,\mathrm{vol},$$

with k a proportionality constant. The constant k can be fixed using the inverse metric

$$k := \langle \omega = e^1 \wedge \cdots \wedge e^p, \mu = f^1 \wedge \cdots \wedge f^p \rangle = \det[g(e^i, f^j)]$$
$$= \det[g^{\mu\nu}(e^i)_\mu (f^j)_\nu]$$

The Hodge dual…

…can also be expressed in coordinates. A close relative of the Levi-Civita symbol we already met is the Levi-Civita tensor

$$\epsilon_{\mu_1\mu_2\cdots\mu_n} = \sqrt{|g|}\,\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}.$$

In these terms,

$$(\star A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\epsilon^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}}A_{\nu_1\cdots\nu_p},$$

where the ν indices have been raised using $g^{\mu\nu}$. As the name suggests, dualizing twice gives

$$\star\star A = (-1)^{p(n-p)+s}A,$$

with s the # of minus signs in the metric signature.

The other two Maxwell equations

Ex. 7 Check that

$$(\star F)_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}.$$

Introducing $J := j_x dx + j_y dy + j_z dz - \rho dt = j - \rho dt$, the other two Maxwell equations are

$$\star d \star F = J.$$

The electromagnetic gauge potential

Just as for the cat, things simplify even more with a gauge potential

$$F = dA.$$

The 1st pair of Maxwell Eqs. become trivial

$$dF = d^2 A = 0,$$

and the 2nd pair are

$$\star\, d \star F = \star\, d \star dA = J.$$

As before, we have a gauge freedom, with

$$A \quad \text{and} \quad A' = A + d\lambda$$

giving the same F.

Temporal gauge

In Minkowski spacetime

$$A = A_0 dt + A_1 dx + A_2 dy + A_3 dz,$$

and temporal gauge is the choice $A_0 = 0$, or, more generally, if spacetime is $\mathbb{R} \times S$, $A(\partial_t) = 0$.

Then,

$$F = dA = dt \wedge \partial_t A + d_S A,$$

and

$$E = -\partial_t A, \qquad B = d_S A.$$

Next, specify **Cauchy data** (A, E) at any time $\{t\} \times S$…

Temporal gauge

…the 1st pair of Maxwell Eqs. are again trivial, but the 2nd pair constrain and evolve this data:

$$\star_S d_S \star_S E = \rho \quad \text{and} \quad -\partial_t E + \star_S d_S \star_S B = j.$$

The first of these Eqs. is the analog of our $J_{\text{tot}} = 0$ condition for the cat; it constrains the given data (A, E) at any time t and is Gauss' law, $\overrightarrow{\nabla} \cdot \overrightarrow{E} = \rho$, in form language.

Using $E = -\partial_t A$ from the previous slide, we have

$$\partial_t(A, E) = (-E, \star_S d_S \star_S d_S A - j)$$

as Eqs. to evolve the initial data.

The physics of the electromagnetic potential

You may be wondering what A tells us physically.

It's a little more abstract than for the cat, but still remarkable and still an angle:

A charge q interacting with the electromagnetic field has a quantum state ψ, the phase of which is modified as it travels along a path γ, specifically

$$\psi \to e^{-\frac{i}{\hbar} q \int_\gamma A} \psi \, ;$$

the angle captured by the potential is the angle in the complex plane describing the phase of the wave function!

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2. Electromagnetism as a Gauge Theory
3. General Relativity as a Gauge Theory: Part I

Changing perspective

We are now in an excellent position to setup General Relativity as a gauge theory.

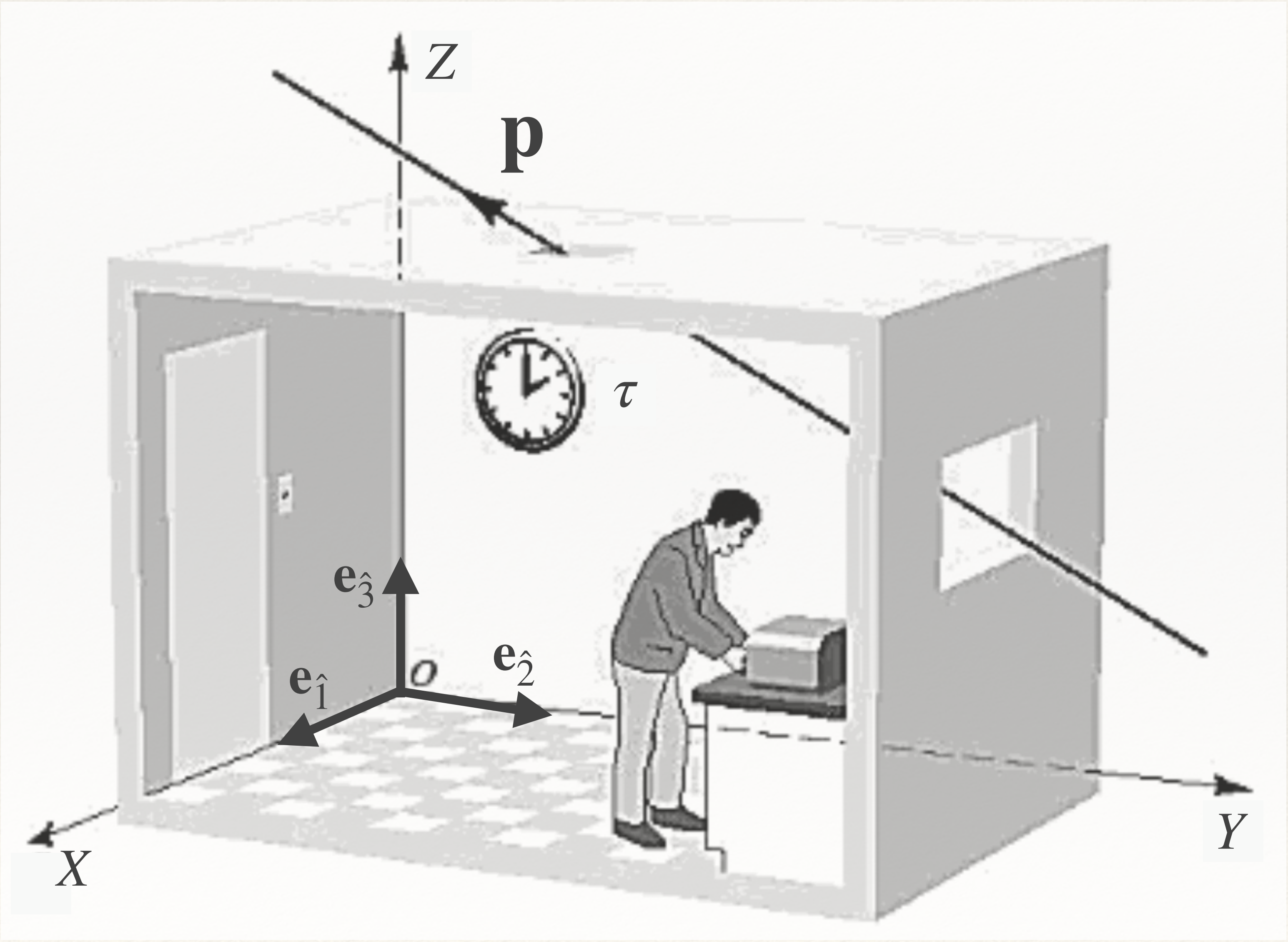
However, to do so we have to understand a somewhat surprising vantage on what the gravitational field is.

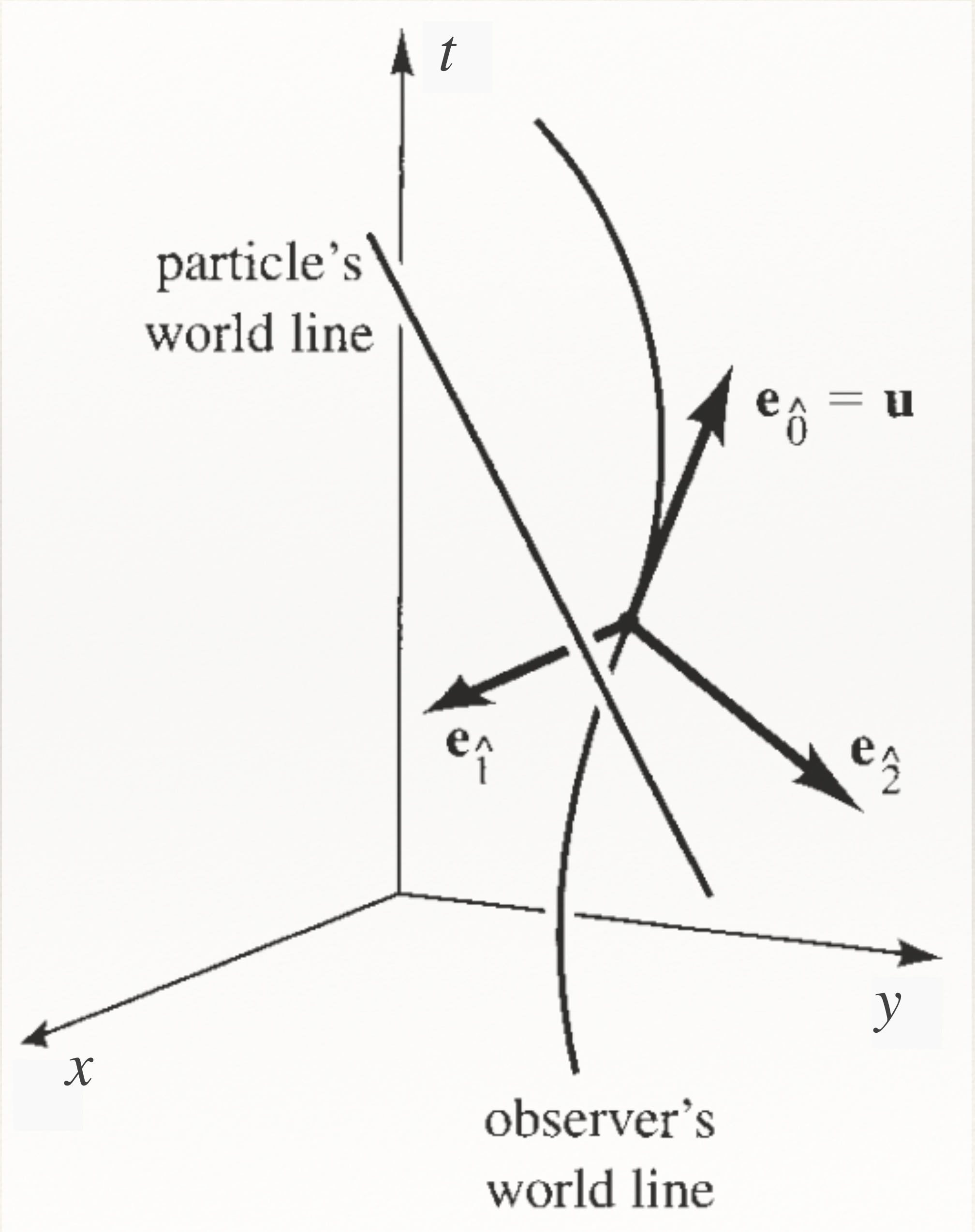
In particular, we will move away from viewing the metric $g_{\mu\nu}(x)$ as the gravitational field; observations and the equivalence principle will drive the shift in perspective.

What an observer measures

We have recently lost the great Jim Hartle. His book *Gravity* has a nice treatment of observations:

$$E = -\mathbf{p}\cdot\mathbf{u}_{\rm obs} = -\mathbf{p}\cdot\mathbf{e}_{\hat{0}}$$





The gravitational field

Spacetime curves and varies from point to point. Generally, there is no privileged coordinate choice throughout, so we work with arbitrary labels of points x^μ.

However, Einstein's great insight was that there is always a local, freely falling frame in which the effects of gravity are erased. Call the coordinates of this local frame X^I. Find $X^I(x)$ at each pt P.

Expand: $$X^I(x) \approx \left.\frac{\partial X^I}{\partial x^\mu}\right|_{x=x(P)} x^\mu := e^I_\mu(x_P)x^\mu$$

The gravitational field

In this cotetrad description,

$$e^I = e^I_\mu(x)dx^\mu, \quad I \in \{0,1,2,3\},$$

the gravitational field translates between or 'solders' the orthonormal and coordinate frames: as a 1-form it acts on the coordinate basis ∂_μ via

$$e^I(\partial_\mu) = e^I_\mu.$$

Of course, the inner product of basis vectors is

$$(\partial_\mu)\cdot(\partial_\nu) := g(\partial_\mu, \partial_\nu) = g_{\mu\nu}.$$

In an orthonormal frame we contract components

$$(\partial_\mu)\cdot(\partial_\nu) = \eta_{IJ}e^I(\partial_\mu)e^J(\partial_\nu) = \eta_{IJ}e^I_\mu e^J_\nu = g_{\mu\nu}\ !$$

The tetrad…

… is just the inverse of the cotetrad

$$e_I^\mu$$

and describes the coordinate components (μ) of an orthonormal frame of vectors. This time we have

$$g_{\mu\nu} e_I^\mu e_J^\nu = \eta_{IJ}.$$

We refer to the orthonormal frame as the internal space (I, J indices) and spacetime (μ, ν) indices. Of course, the internal metric η_{IJ} is invariant under Lorentz transformations and any frame so related provides another valid orthonormal frame.

Spacetime split

As we did in electromagnetism, we now make a split of spacetime into space and time. This is because we are initially going to develop a Hamiltonian formalism for GR.

Along a spatial slice, our frame becomes a triad

E_i^a, $a = 1,2,3$ for space, $i = 1,2,3$ internal space.

Ashtekar's electric field

And, finally we arrive at Ashtekar's electric field

$$\tilde{E}_i^a = \sqrt{\det q_{ab}} E_i^a,$$

here q_{ab} is the spatial metric on a spatial slice. The associated two-form will be one of a pair of central canonically conjugated fields

$$\tilde{E}^i(x) = \tilde{E}^{ia}(x)\epsilon_{abc} dx^b \wedge dx^c.$$

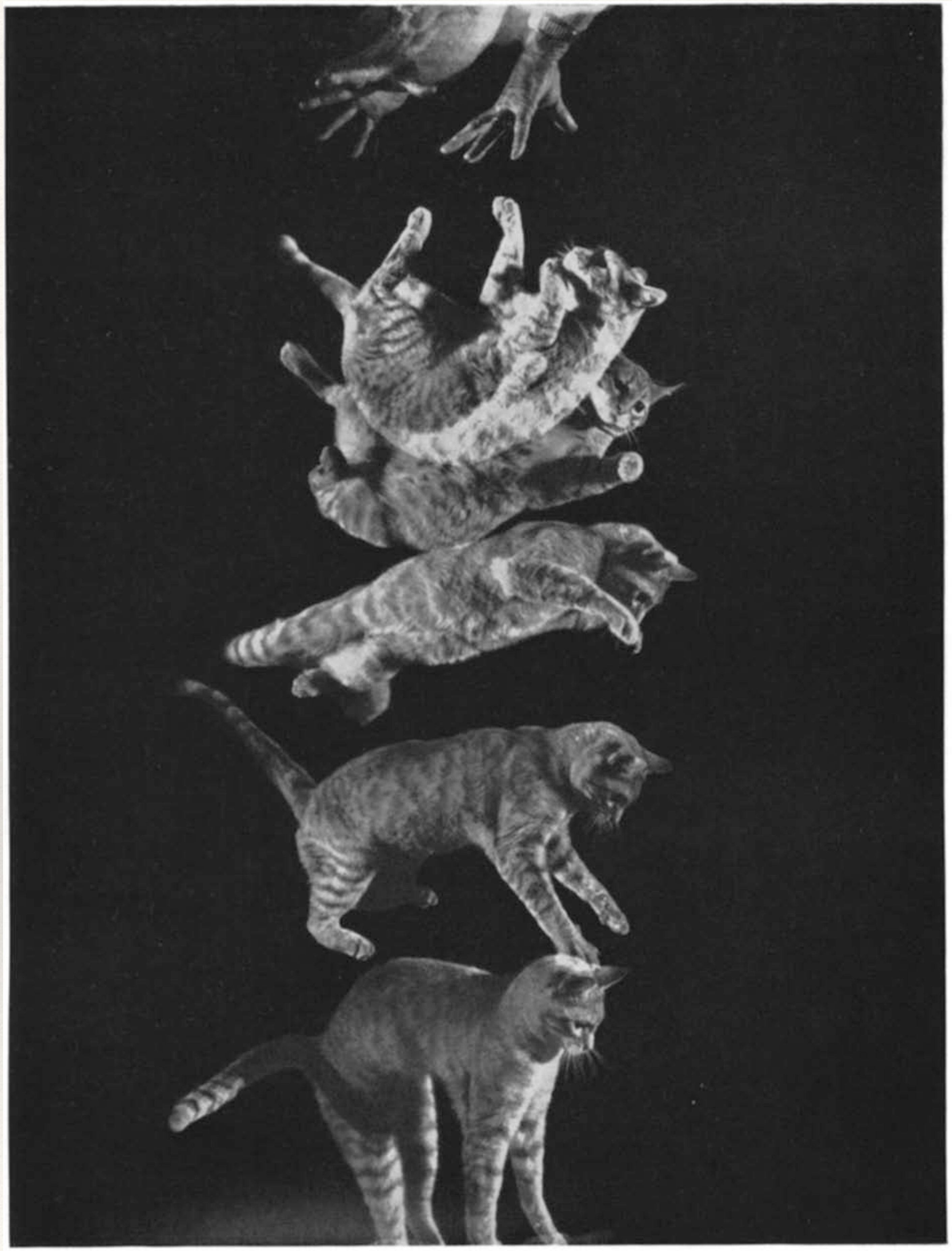
Next time…

Ashtekar's electric field

$$\tilde{E}^i(x) = \tilde{E}^{ia}(x)\epsilon_{abc}dx^b \wedge dx^c.$$

Next time we will develop this as an $SU(2)$ gauge theory and find the associated connection A.

Thank you!



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