# The Basics of Loop Quantum Gravity

## Discussion 3: Gravity as a Gauge Theory

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# Review question 1: What is the Ashtekar electric field?

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It is a densitized triad field

$$\tilde{E}^a_i = \sqrt{\det q}\, E^a_i,$$

that provides a sort of 'square root' of the metric

$$\tilde{E}^a_i \tilde{E}^{ib} = \det q \, q^{ab}.$$

In other words, to reconstruct the spatial metric you find the inverse of

$$q^{ab} = \tilde{E}^a_i \tilde{E}^{ib} / \det q.$$

We also organized this into a 2-form

$$E^i(x) = \tilde{E}^{ia}(x) \epsilon_{abc} dx^b \wedge dx^c.$$

# Review question 2: What is the Poynting vector in form language?

Using the same definitions as last time:

$$E = E_x dx + E_y dy + E_z dz,$$

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy,$$

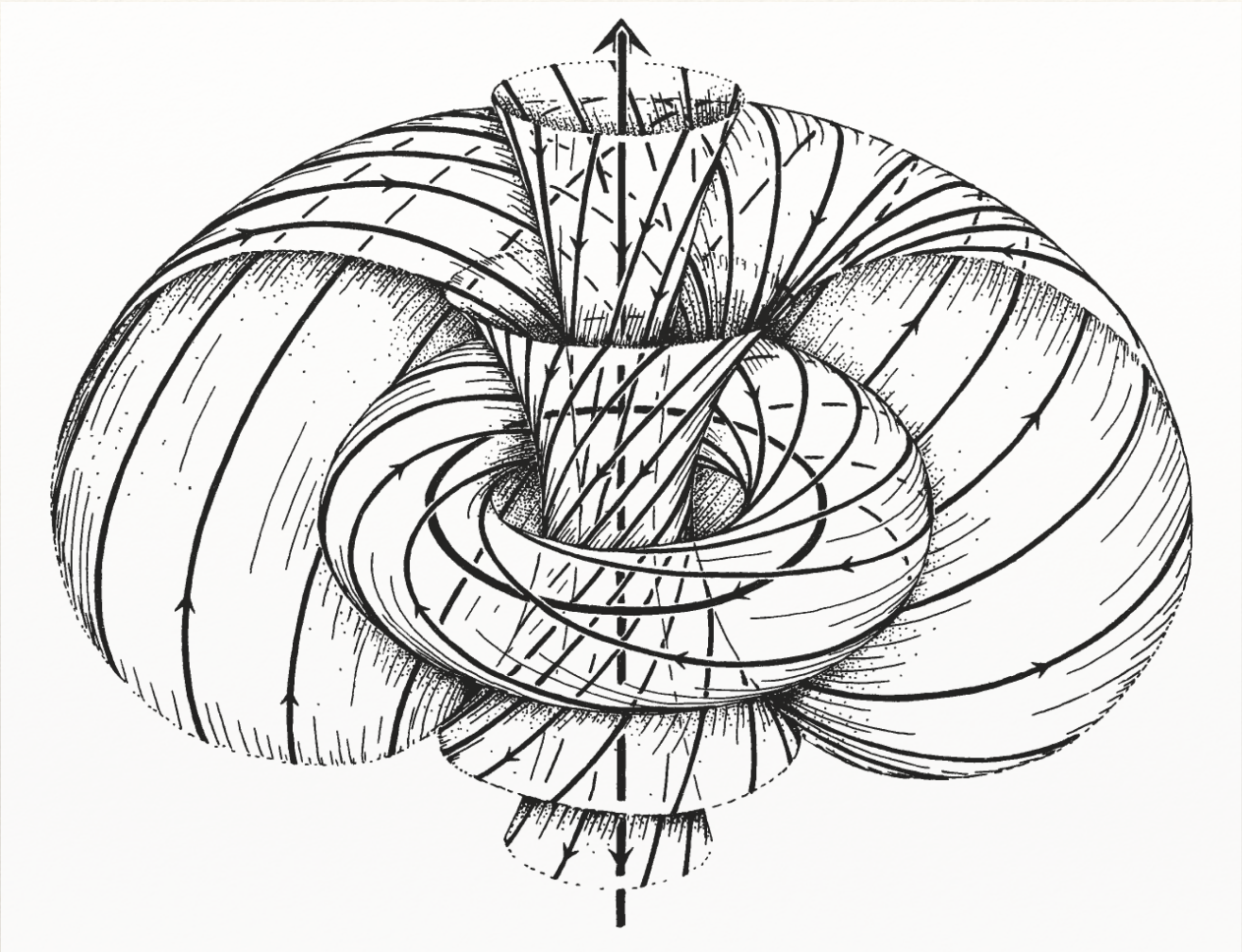
we have the Poynting 3-form $P$ or the 1-form $P_1$:

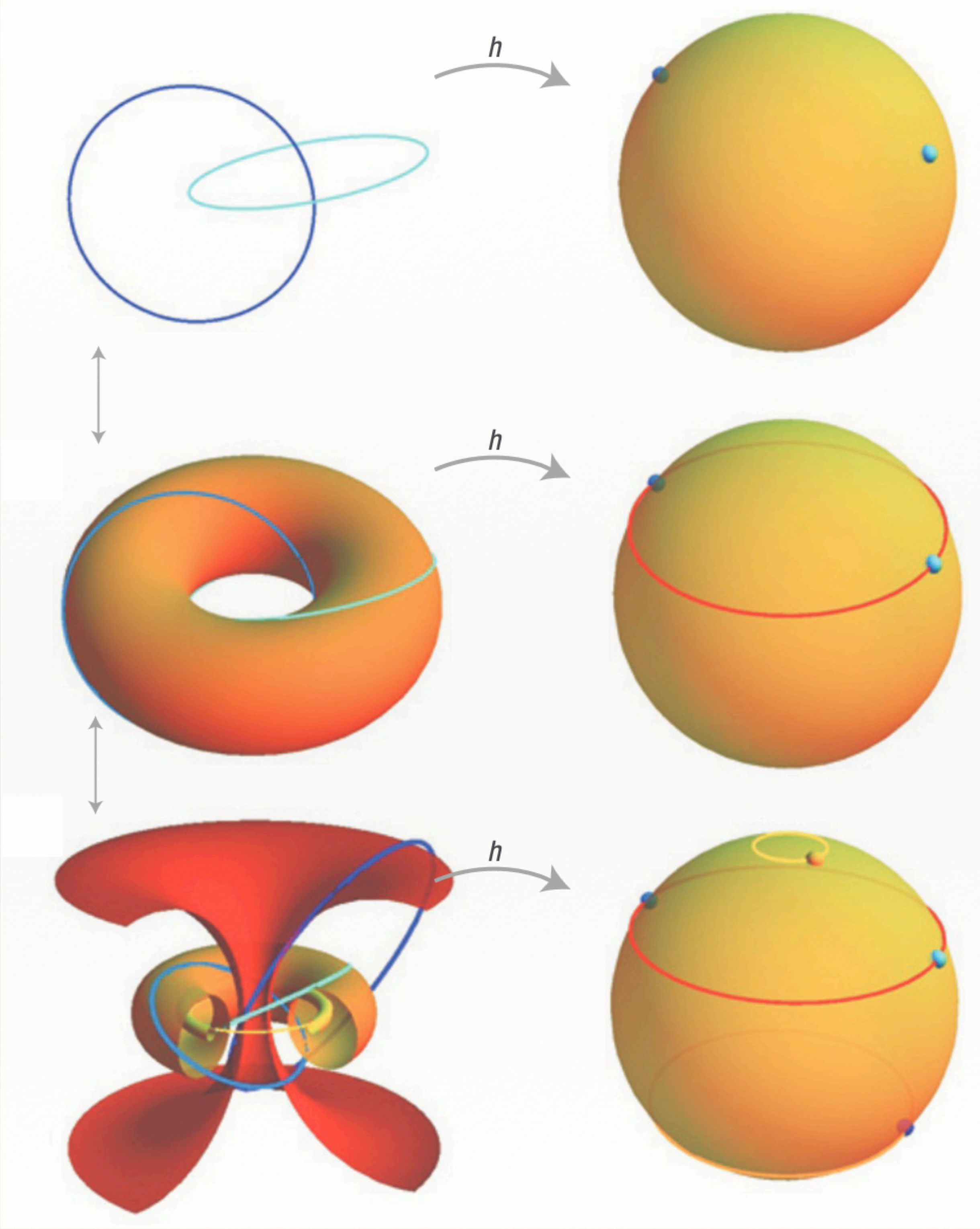
$$P = -E \wedge \star_S B, \text{ or } P_1 = -\star_S (E \wedge \star_S B).$$

Alternatively, you can view this as part of the full stress-energy and pick out components using your velocity $u$:

$$P = g(u) \wedge F(u) \wedge \star F(u).$$

# Review question 3: When can you write a physical theory as a gauge theory?






When you can cast it in terms of a "principle bundle", e.g. Hopf bundle.

[G.L. Naber, Topology, Geometry, and Gauge Fields]

# Review question 4: Now that we are using the wedge product, why not use the geometric (or Clifford) product?

Okay, let's do it. For concreteness, let's introduce everything in $\mathbb{R}^3$. Suppose $\overrightarrow{u}$ and $\overrightarrow{v} \in \mathbb{R}^3$, then

$$\overrightarrow{u}\,\overrightarrow{v} := \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \wedge \overrightarrow{v}.$$

Suppose $\{\hat{x}, \hat{y}, \hat{z}\}$ is a basis for $\mathbb{R}^3$, then

$$\hat{x}\hat{y} = 0 + \hat{x} \wedge \hat{y} = -\,\hat{y} \wedge \hat{x} = -\,\hat{y}\hat{x}, \text{ etc.}$$

Also, normalized bivectors, e.g. $B = \hat{x} \wedge \hat{y}$, satisfy:

$$BB = (\hat{x} \wedge \hat{y})(\hat{x} \wedge \hat{y}) = \hat{x}\hat{y}\hat{x}\hat{y} = -\,\hat{x}\hat{x}\hat{y}\hat{y} = -\,1$$

and

$$BBB = (BB)B = -\,B.$$

Last week I was partly enjoying the story-telling about cats, but you should know…

…there really is a ‘cat’ at the heart of quantum gravity!

I’ll explain.

In a static, weak field there is a striking relation between GR and E&M with the formulation of Newtonian gravity parallel to electrostatics:

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)$$

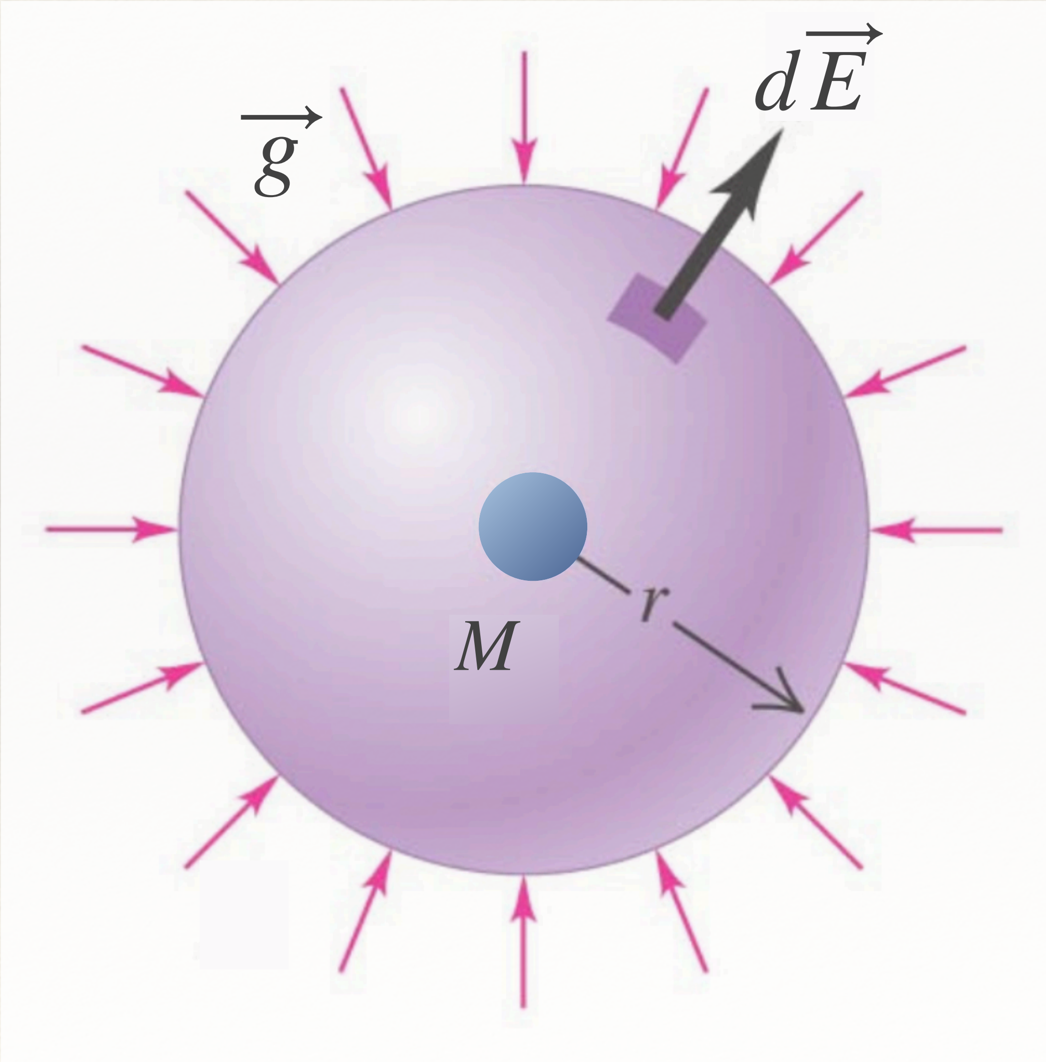
We have a gravitational Gauss' law

$$\overrightarrow{g} = -\overrightarrow{\nabla}\Phi, \quad \text{and} \quad \overrightarrow{\nabla}\cdot g = -4\pi G\mu \ .$$

As in E&M this introduces intriguing non-locality

$$\oint \overrightarrow{g}\cdot d\overrightarrow{E} = -4\pi GM$$

**Note well** the unusual notation: a small area element of the surface is denoted $d\overrightarrow{E}$, not $d\overrightarrow{A}$.



For

a region empty of mass

and

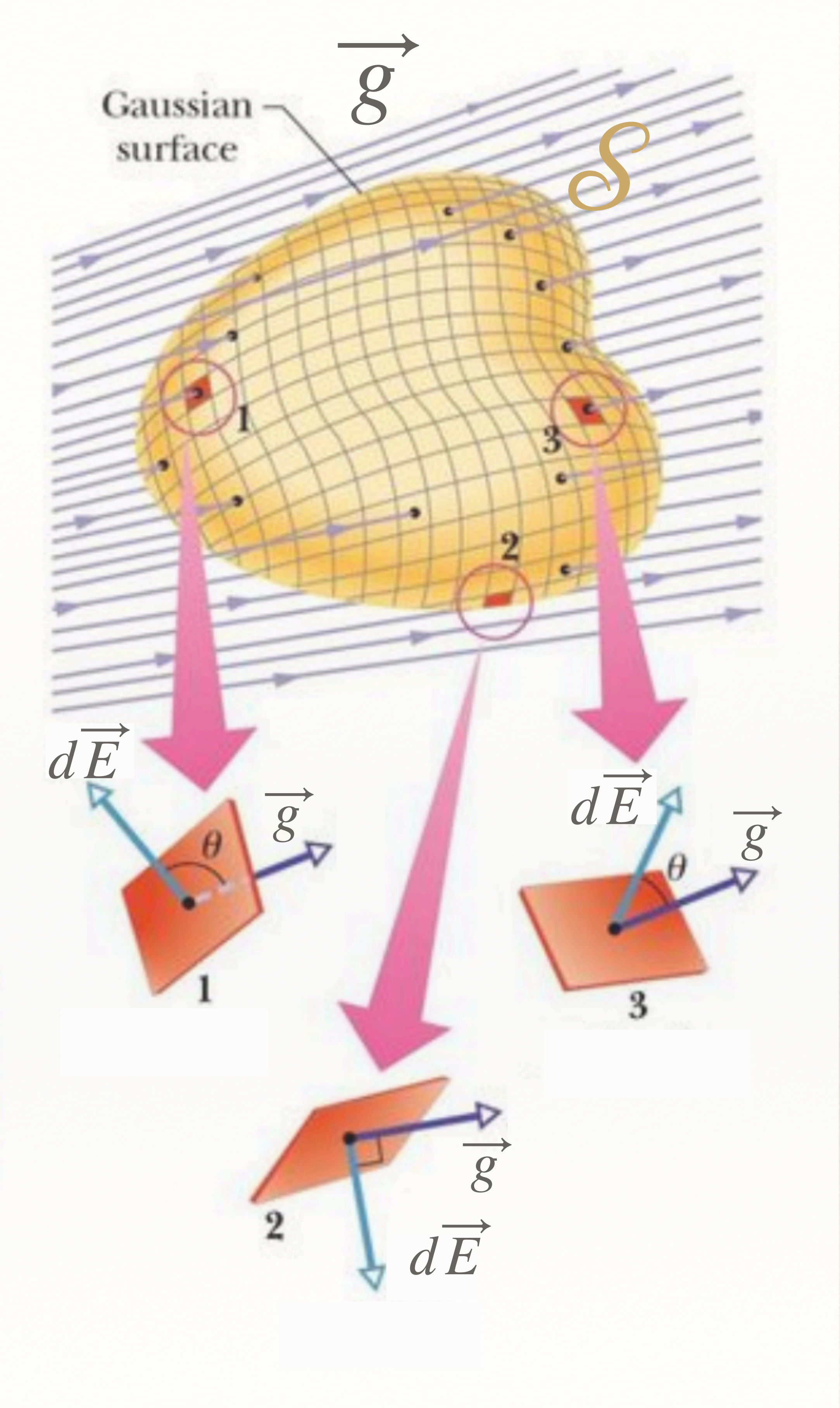
small enough that we can take the gravitational field $\vec{g}$ constant,

$$\oint_{\mathcal{S}} \vec{g} \cdot d\vec{E} = \vec{g} \cdot \oint_{\mathcal{S}} d\vec{E} = 0.$$

We've arbitrarily oriented things and so

$$\vec{E}_{\mathcal{S}} = \oint_{\mathcal{S}} d\vec{E} = 0$$

is a constraint on closed regions.



In the special case of of a spatial polyhedron

$$\vec{E}_{\mathcal{S}} = \oint_{\mathcal{S}} d\vec{E} = 0 \qquad \Longrightarrow \qquad \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0.$$

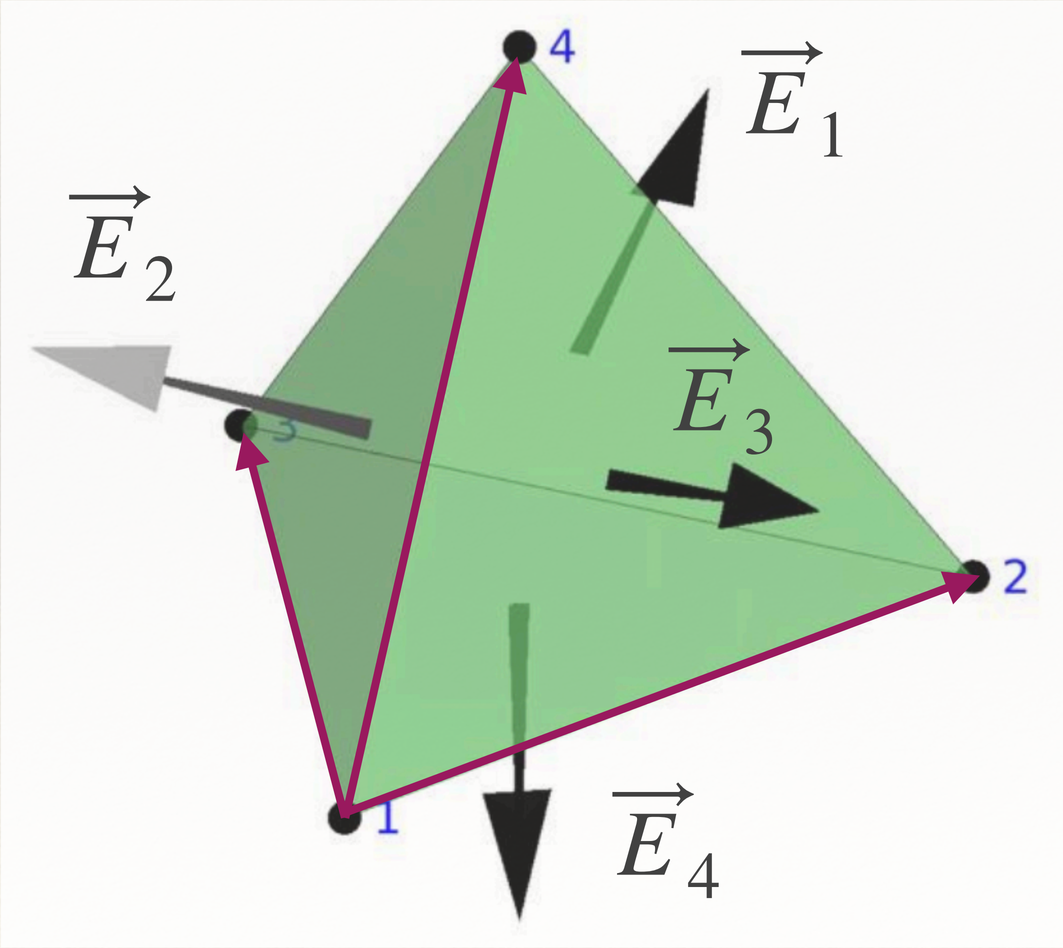
Remarkably, exactly this identity was used by Hermann Minkowski to give a complete characterization of convex polyhedra at the close of the 19th century.

As $\vec{E}_2 = \frac{1}{2}\vec{l}_{14} \times \vec{l}_{13}$, we can write

$$V = \frac{1}{6}\vec{l}_{12} \cdot (\vec{l}_{13} \times \vec{l}_{14}),$$

or equally well, *Ex. 1*,

$$V^2 = \frac{2}{9}\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3).$$






Minkowski, Nach. vd Ges. 1897.

# The electric field measures physical areas

Just as

$$\text{Length}_\gamma = \int_\gamma \left( \frac{\partial x^a}{\partial \tau} \frac{\partial x^b}{\partial \tau} q_{ab} \right)^{1/2} d\tau,$$

we have

$$\text{Area}(\mathcal{S}) = \int_\mathcal{S} \left( \frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \tau} \frac{\partial x^c}{\partial \sigma} \frac{\partial x^d}{\partial \tau} (q_{ac} q_{bd} - q_{ad} q_{bc}) \right)^{1/2} d\sigma d\tau.$$

(*Ex. 2* derive this from $d\text{Area} = |d\vec{u}||d\vec{v}|\sin\theta = |d\vec{u}||d\vec{v}|\sqrt{1-\cos^2\theta}$.)

But, then,

$$\text{Area}(\mathcal{S}) = \int_\mathcal{S} \left( \frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \tau} \frac{\partial x^c}{\partial \sigma} \frac{\partial x^d}{\partial \tau} (\epsilon_{eab} \epsilon_{fcd} \det q \, q^{ef}) \right)^{1/2} d\sigma d\tau$$

$$= \dots$$

[Ashtekar, PRL 57, 2244; Rovelli, Phys. Rev. D 47, 1703]

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$$= \int_\mathcal{S} \left( \frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \tau} \frac{\partial x^c}{\partial \sigma} \frac{\partial x^d}{\partial \tau} (\epsilon_{eab} \epsilon_{fcd} \tilde{E}^e_i \tilde{E}^{if}) \right)^{1/2} d\sigma d\tau$$

# The electric field measures physical areas

Now, compute,

$$\text{Area}(\mathcal{S}) = \int_{\mathcal{S}} \left( \frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \tau} \frac{\partial x^c}{\partial \sigma} \frac{\partial x^d}{\partial \tau} (\epsilon_{eab}\epsilon_{fcd} \tilde{E}^e_i \tilde{E}^{if}) \right)^{1/2} d\sigma d\tau$$

$$= \int_{\mathcal{S}} \left( \tilde{E}^e_i \epsilon_{eab} \frac{\partial x^a}{\partial \sigma} \frac{\partial x^b}{\partial \tau} d\sigma d\tau \; \tilde{E}^{if} \epsilon_{fcd} \frac{\partial x^c}{\partial \sigma} \frac{\partial x^d}{\partial \tau} d\sigma d\tau \right)^{1/2}$$

$$= \int_{\mathcal{S}} \left( (\tilde{E}^e_i \epsilon_{eab} dx^a dx^b)(\tilde{E}^{if} \epsilon_{fcd} dx^c dx^d) \right)^{1/2}$$

$$= \int_{\mathcal{S}} \left( E_i E^i \right)^{1/2} = \int_{\mathcal{S}} ||E|| = E_{\mathcal{S}} \, .$$

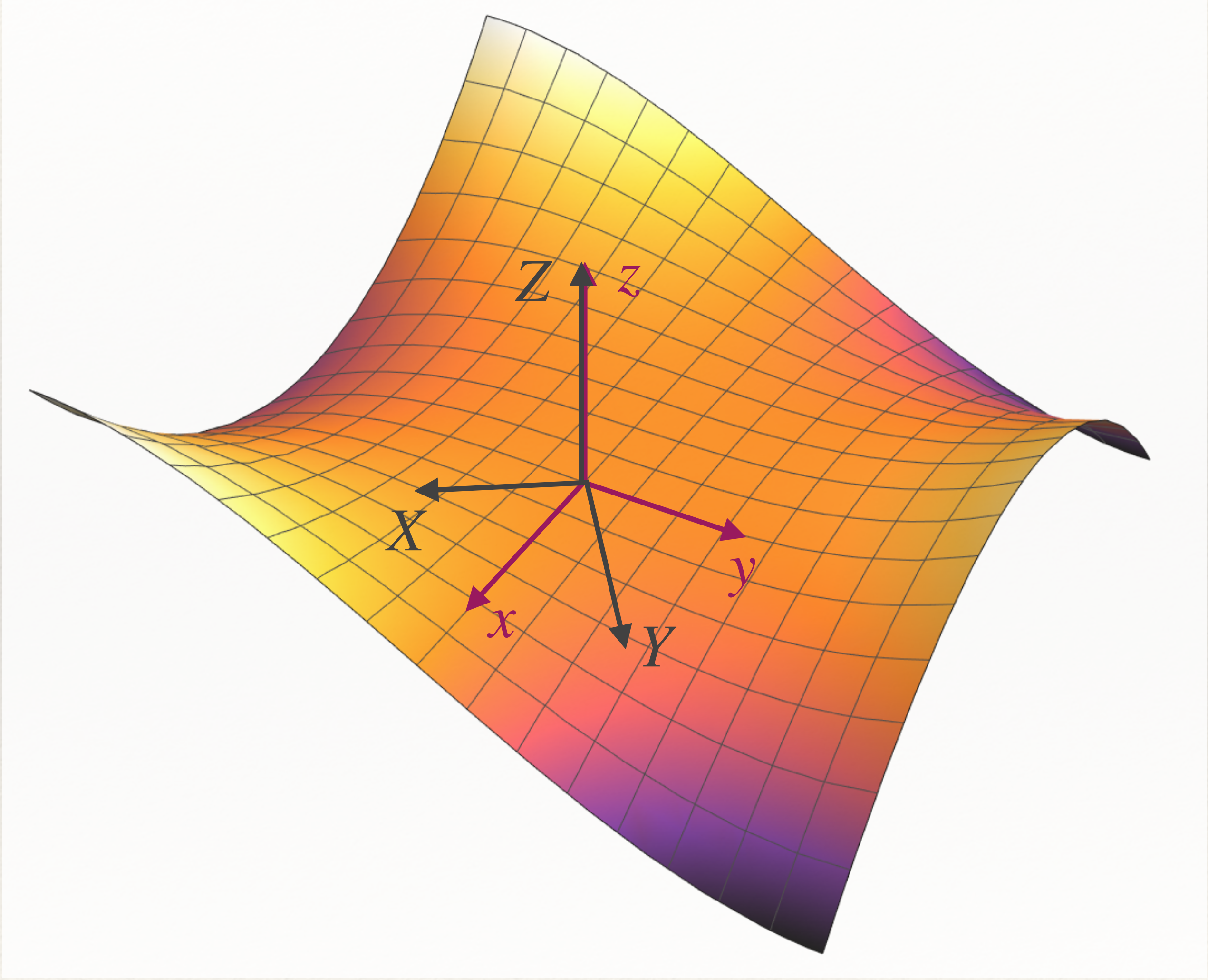
Thus, area is the norm of (Ashtekar) electric flux through $\mathcal{S}$.

# The electric fluxes provide orientations

The norm of the last slide is standard, but annoying. If we work with oriented areas and 2-forms, we don't need it!

This is a good reason to work with the flux itself:

$$E^i_{\mathcal{S}} = \int_{\mathcal{S}} E^i(x) = \int_{\mathcal{S}} \tilde{E}^{ia}(x)\epsilon_{abc}dx^b \wedge dx^c$$




The internal $i$ index gives the local inertial ($\mathbb{R}^3$) flux direction.

Take note that the internal frame needn't align with the coordinate frame: gauge freedom.

# The electric field 2-form generates rotations

The electric field 2-form is a bivector (in the local cotangent space). As such it generates rotations, because

$$
\begin{aligned}
e^{\theta B} &= 1 + \theta B + \frac{\theta^2}{2!}BB + \frac{\theta^3}{3!}BBB + \frac{\theta^4}{4!}BBBB + \cdots \\
&= 1 + \theta B - \frac{\theta^2}{2!} - \frac{\theta^3}{3!}B + \frac{\theta^4}{4!} + \cdots \\
&= \cos\theta + \sin\theta\ B\,.
\end{aligned}
$$

(*Ex. 3* confirm that to take vectors to vectors, the correct action of this 'rotor' is by conjugation, i.e. $\vec{v}\,' = e^{-B\theta/2}\vec{v}\,e^{B\theta/2}$.)

Thus, we can think of our $i$ index as labeling the components of the (dual to the) Lie algebra $\mathfrak{su}(2)^* \cong \mathfrak{so}(3)^* \cong \mathbb{R}^3$. Using Hodge ($\epsilon_{abc}$) and working with a basis $\{\tau_i\}_{i=1}^3 \in \mathfrak{su}^*(2)$:

$$\vec{E} = \tilde{E}^{ia}(x)\epsilon_{abc}dx^b \wedge dx^c \tau_i.$$

# Gauge invariance and embedding

This furnishes an interpretation of the closure

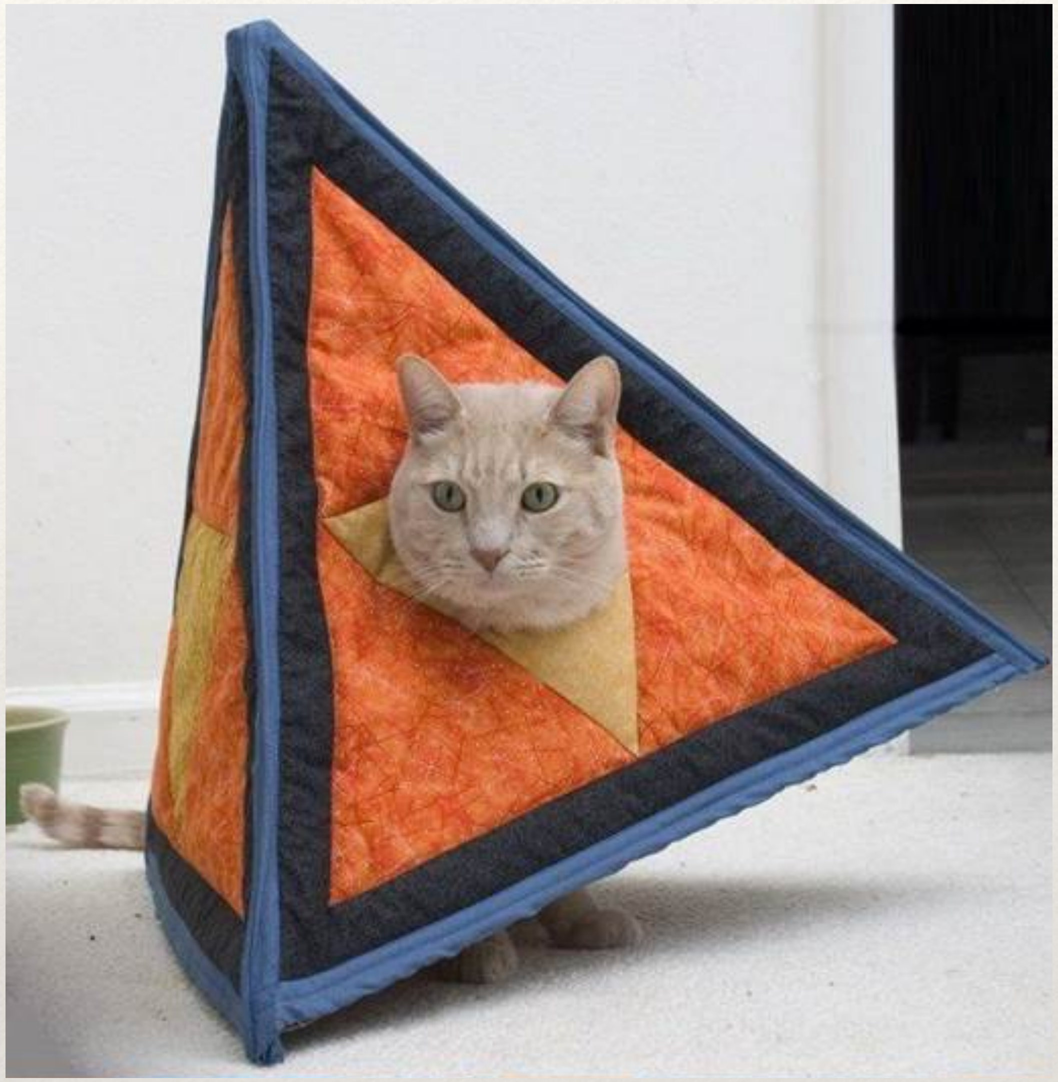
$$\vec{E}_{\mathcal{S}} = \oint_{\mathcal{S}} d\vec{E} = 0 \quad \Longrightarrow \quad \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0.$$

Vector $\vec{E}_{\mathcal{S}} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4$ generates gauge rotations:

$$R(\theta, \hat{n}) = e^{\theta(\vec{E}_1+\vec{E}_2+\vec{E}_3+\vec{E}_4)\cdot\hat{n}}$$


….these rotations of the tetrahedron change its orientation, but don't change its shape (metric geometry)!

# The ‘cat’ and the tetrahedron are the same!



The $\{\vec{E}_1, \cdots, \vec{E}_4\}$ are angular momenta and are constrained by $\vec{E}_{\text{tot}} = 0$. The gauge invariants $||\vec{E}_\ell||$ and $\vec{E}_\ell \cdot \vec{E}_m$ capture the tet’s shape (metric).

# *Today's Discussion*

1. General Relativity as a Gauge Theory Part II: the Ashtekar connection

2. Quantum Tetrahedra

3. Building Space Part I: Spin Network Motivations

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1. General Relativity as a Gauge Theory Part II: the Ashtekar connection

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# What's next?

We would like to find the variables that, together with the electric field, will make up a 'canonically conjugate' set.

If we think of the electric field as being a 'momentum variable' $p$, we want to find the 'position variable' $q$, s.t.

$$\{q, p\} = 1.$$

It turns out that the answer will be an $\mathfrak{su}(2)$-gauge potential $A$ with

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \delta_j^i \delta_a^b \delta^{(3)}(x, y),$$

and called the "Ashtekar connection". The first step is to find Poisson brackets $\{\,\cdot\,,\cdot\,\}$ for GR.

# Some basics of Symplectic & Poisson Geometry

You are certainly familiar with the fact that you can derive the Equations of Motion (EoM) of a Lagrangian theory from its action $S$…

…but, depending on your exposure to mechanics, you might not have seen that you can also use $S$ to derive the symplectic potential $\theta$, symplectic 2-form $\Omega = -d\theta$, and Poisson brackets.

We have,

$$\delta S = \int \delta L dt = \int \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right) dt = \int \left[\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)\right] dt$$

and the boundary term, gives $\theta = pdq$, so that $\Omega = dq \wedge dp$. In phase space coords $\xi^i = (q,p)$, $\Omega = -\frac{1}{2}\Omega_{ij} d\xi^i \wedge d\xi^j$, and

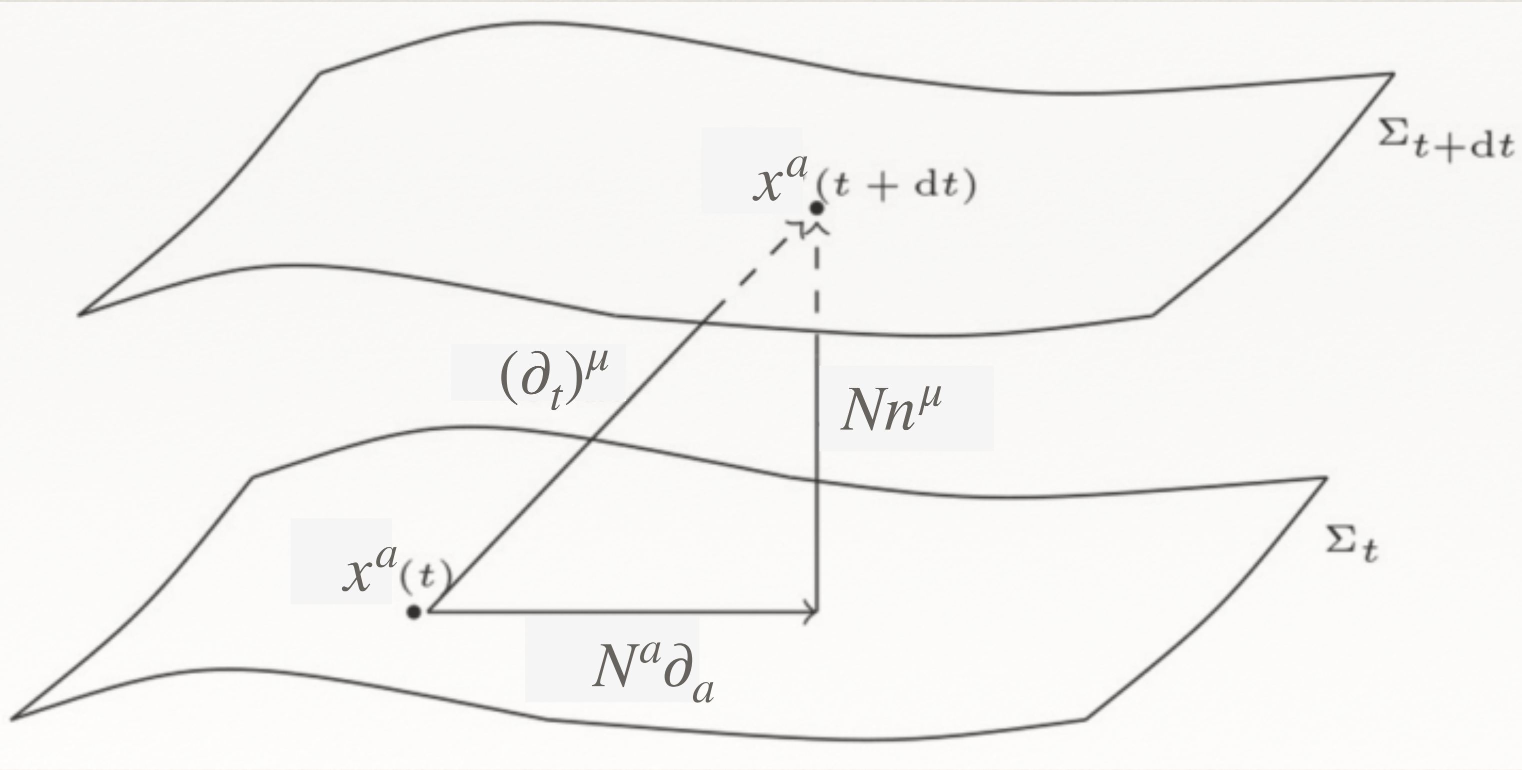
$$\{f,g\} := \partial_i f \Omega^{ij} \partial_j g = \partial_q f \partial_p g - \partial_p f \partial_q g.$$

# ADM formulation of General Relativity

Starting from the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g}\, R\,,$$

Richard Arnowitt, Stanley Deser and Charles W. Misner
(1) made a spacetime split $\mathscr{M} = \mathbb{R} \times \Sigma$ (take simpler $\partial\Sigma = 0$),
(2) worked out brackets, and (3) found the conjugate variables




(1) Let

$$n = -Ndt,$$

s.t. $n_\mu n^\mu = -1$. Then,

$q_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$ induced

$K_{\mu\nu} := \nabla_\nu n_\mu$ leads to

$$K_{ab} = \frac{1}{2N}(\dot{q}_{ab} - 2D_{(a}N_{b)}).$$

# ADM formulation of General Relativity

(2) The 4-dim Ricci scalar becomes

$$R(g) = R(q) + (K^{ab}K_{ab} - K^2) + 2\nabla_\mu(n^\mu K - n^\nu \nabla_\nu n^\mu) \; ,$$

and the metric has the block structure

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \vec{N}^2 & q_{ab}N^a \\ q_{ab}N^b & q_{ab} \end{pmatrix},$$

so that $g_{ab} = q_{ab}$, but $g^{ab} = q^{ab} + N^a N^b$. Putting it together

$$K_{ab} = \frac{1}{2N}(\dot{q}_{ab} - 2D_{(a}N_{b)}), \;\; \& \;\; \tilde{\pi}^{ab} = \partial L / \partial \dot{q}_{ab} = \sqrt{q}(K^{ab} - q^{ab}K),$$

and

$$S = \int dt d^3x \left[ \tilde{\pi}^{ab}\dot{q}_{ab} - N\tilde{S} - N^a \tilde{C}_a \right], \quad \text{with } \tilde{S} \text{ and } \tilde{C}_a \text{ constraints.}$$

# ADM formulation of General Relativity

(3) With the action in hand, they found

$$\{q_{ab}(x), \tilde{\pi}^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta^{(3)}(x,y),$$

and so in the ADM phase space it is the spatial metric and the (trace-free part) of the extrinsic curvature that are the conjugate variables. The lapse and shift are Lagrange multipliers that impose the constraints

$$S = \int dt d^3x \left[\tilde{\pi}^{ab}\dot{q}_{ab} - N\tilde{S} - N_a\tilde{C}^a\right],$$

$$\text{Spatial diffeos: } \tilde{C}^a = -2D_b\tilde{\pi}^{ab},$$

$$\text{Scalar (Hamiltonian): } \tilde{S} = q^{-\frac{1}{2}}(q_{ac}q_{bd} - \frac{1}{2}q_{ab}q_{cd})\tilde{\pi}^{ab}\tilde{\pi}^{cd} - q^{\frac{1}{2}}R(q)$$

[Compare E&M: $F^2 \to E^a\dot{A}_a + A_0\mathfrak{D}_aE^a + E^2 + B^2$,

$A_0$ acts as a Lagrange mult. that imposes the Gauss law.]

# ADM formulation of General Relativity

A key challenge in gravity is as follows:

You would like for the algebra of your constraints on the theory to have the same structure as the algebra of your gauge group and this isn't quite true.

If we smear constraints: $S(F) = \int d^3x\, \tilde{S}\, F$ & $C(\vec{G}) = \int d^3x\, \tilde{C}_a G^a$

then,

$$\{C(\vec{F}), C(\vec{G})\} = C(\mathscr{L}_{\vec{F}}\vec{G})$$
$$\{S(F), C(\vec{G})\} = S(\mathscr{L}_{\vec{G}}F)$$
$$\{S(F), S(G)\} = C(q^{ab}F\overleftrightarrow{\partial}_b G),$$

because the last depends on $q^{ab}$ these do not close to a standard Lie algebra…

# ADM formulation of General Relativity

$$\{C(\vec{F}), C(\vec{G})\} = C(\mathscr{L}_{\vec{F}}\vec{G})$$

$$\{S(F), C(\vec{G})\} = S(\mathscr{L}_{\vec{G}}F)$$

$$\{S(F), S(G)\} = C(q^{ab}F\overleftrightarrow{\partial}_b G),$$

…failure to close is concerning for quantization, where we usually represent symmetries algebraically.

More generally,

the brackets between the field variables $(q_{ab}, \tilde{\pi}^{ab})$ and the constraints $(C, S)$ are also complicated functions of $(q_{ab}, \tilde{\pi}^{ab})$: Dirac called this the hypersurface deformation algebra.

*Long Ex. 4:* Fill in more of the details of ADM.

[ADM, Phys. Rev. 116, 1959 and GRG, 1967]

# Connections, connections, connections

Our route to the Ashtekar connection will be to start from the ADM variables and to perform a canonical transformation.

To understand this approach, we will need to recall how the spin connection works. For this we return to spacetime briefly.

The spacetime covariant derivative allows us to parallel transport tensors and is usefully expressed in terms of the Levi-Civita connection:

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\sigma} v^\sigma,$$

which is uniquely determined by the two conditions:

$$\text{Metric compatibility} \quad \nabla_\mu g_{\rho\sigma} = 0,$$

$$\text{Torsion free} \quad \Gamma^\nu_{[\mu\sigma]} = 0.$$

# Connections, connections, connections

But, we have a second kind of vectors around, those that live in the Lorentz frames over every point of $\mathscr{M}$ (internal vectors)

How should we parallel transport these? A: the spin connection. The idea is

$$\mathscr{D}_\mu v^I = e^I_\nu \nabla_\mu v^\nu$$

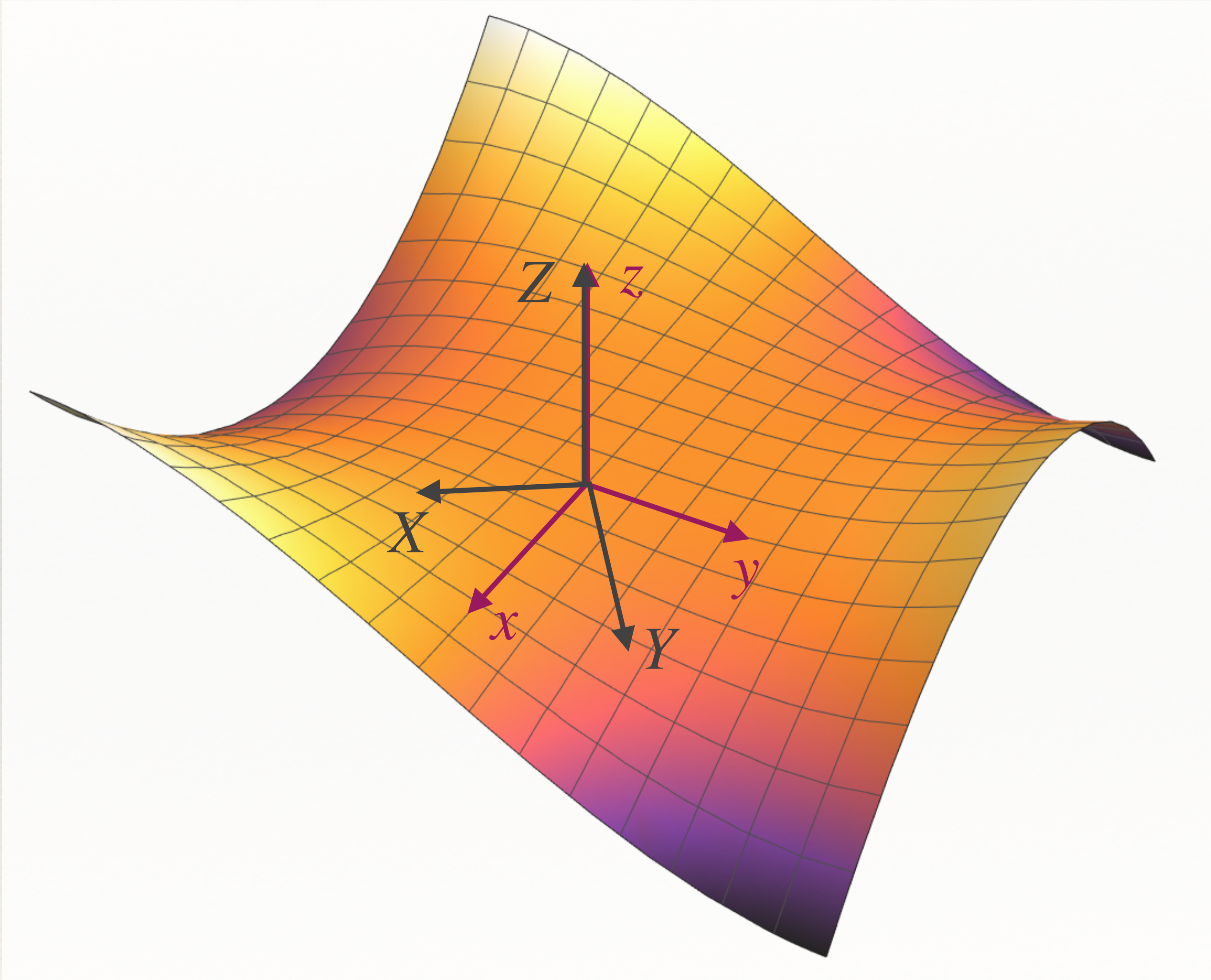
$$= \partial_\mu v^I + \omega_\mu{}^I{}_J v^J,$$

with $\omega^{IJ}_\mu$ the spin connection.

We have $\omega_\mu{}^{IJ} = e^I_\nu \nabla^{LC}_\mu e^{\nu J}$ when

$$\mathscr{D}_\mu \eta^{IJ} = 0 \iff \omega_\mu{}^{(IJ)} = 0$$

$$d_\omega e^I := de^I + \omega^{IJ} \wedge e_J = 0$$



# Spin connection split

Now let's understand the spacetime split of this connection:

$$\omega^{0i} \to \text{boosts}$$

$$\omega^{ij} \to \text{spatial rotations.}$$

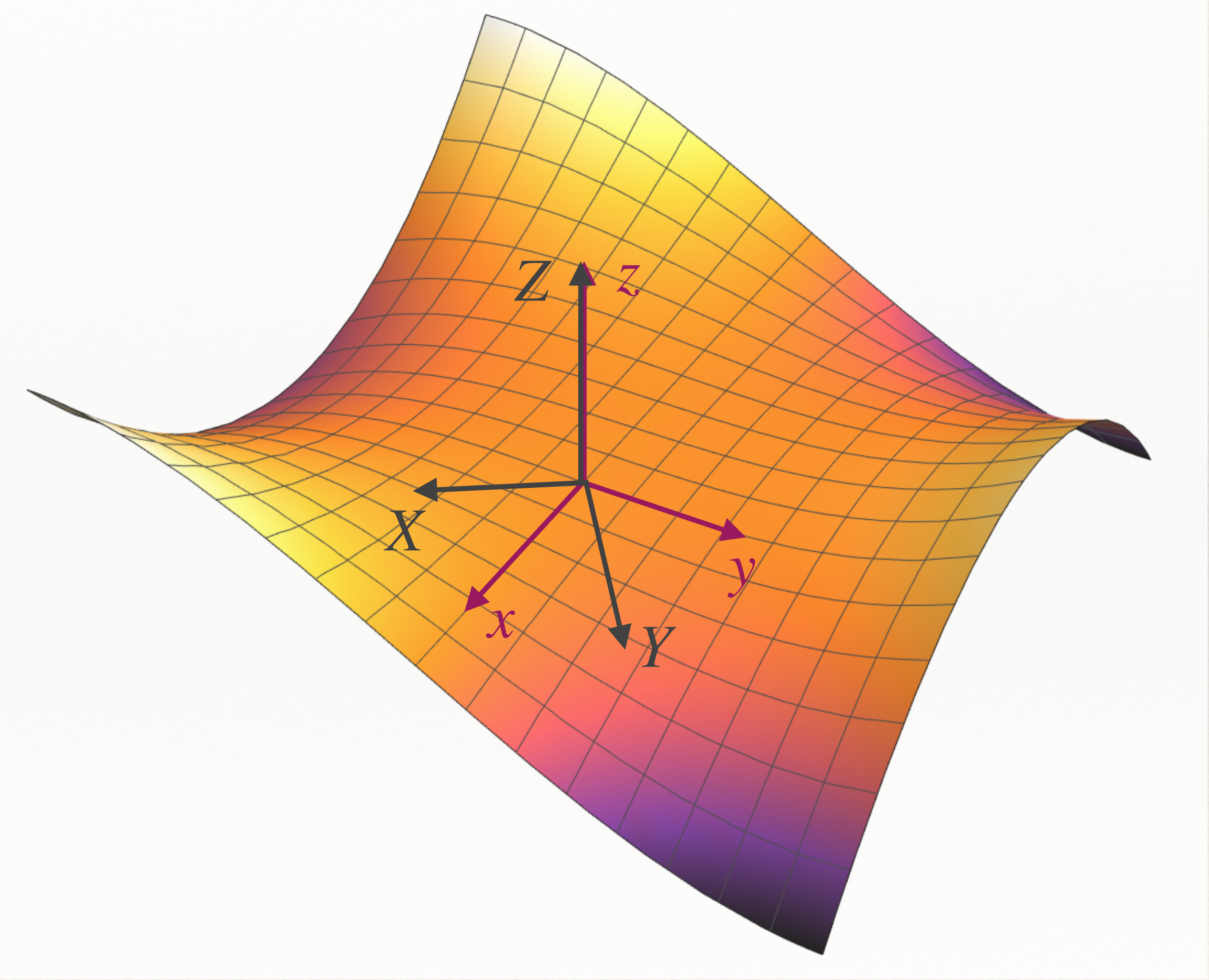
Now, define

$$\Gamma_a{}^i := \frac{1}{2}\epsilon^i{}_{jk}\omega_a{}^{jk}.$$

Just as the spacetime spin connection is determined by the tetrad, here we have

$$\Gamma_a{}^i = \Gamma_a{}^i(E),$$

is determined by the triad.



# The Ashtekar connection at last

Recall $q_{ab} = E^i_a E^j_b \delta_{ij}$ and define

$$K^i_a = K_{ab} E^{bi}.$$

The $p\dot{q}$ term of the ADM Lagrangian becomes

$$\tilde{\pi}^{ab}\dot{q}_{ab} = \sqrt{q}(K^{ab} - q^{ab}K)2\dot{E}_{i(a}E^i_{b)} = 2\tilde{E}^a_i\dot{K}^i_a + \partial_t(\,*\,).$$

Thus, $\tilde{E}$ and $K$ are conjugate variables and schematically

$$\{q, \tilde{\pi}\} = 1, \quad \{q, q\} = 0, \quad \{\tilde{\pi}, \tilde{\pi}\} = 0$$

$$\{K, \tilde{E}\} = 1, \quad \{K, K\} = 0, \quad \{\tilde{E}, \tilde{E}\} = 0.$$

Connections have the freedom that you can add any vector, so

Ashtekar connection: $A^i_a := \Gamma^i_a + \mathrm{i}\, K^i_a$, with $\mathrm{i} := \sqrt{-1}$.

Thus retaining conjugacy of $(K, E)$, and making $A$ a connection.

# A crux issue with Ashtekar's connection

You will have noticed the $i = \sqrt{-1}$ appearing in the definition.

This makes the original Ashtekar connection a complex variable. There is a good reason for this choice…

…further analysis reveals that $K^i_a = \omega_a{}^{0i}$, the boost part. And the Lorentz group has a very nice decomposition over $\mathbb{C}$:

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{su}(2,\mathbb{C}).$$

The original Ashtekar connection is the self-dual factor.

Unfortunately, to make sense of the quantum theory, you would need to be able to find the 'real parts' of your operators and no one has yet found a feasible scheme for doing so.

# The Ashtekar-Barbero connection

Instead the most common practice is to work with a real connection variable

Ashtekar-Barbero connection: $A^i_a := \Gamma^i_a + \gamma\, K^i_a,$ with $\gamma \in \mathbb{R}$.

The 'Barbero-Immirzi' parameter $\gamma$ is a new free parameter of the theory. We will see its physical meaning briefly.

Remarkably: $\{A^i_a(x), \tilde{E}^b_j(y)\} = \gamma \delta^i_j \delta^b_a \delta^{(3)}(x, y)$, $\{A, A\} = 0$, & $\{\tilde{E}, \tilde{E}\} = 0$. But, there is a tension between:

(i) Real variables

(ii) Poisson commuting connection

(iii) spacetime covariance

# Gravity as an SU(2) gauge theory

The action is now

$$S[A_a^i, E_j^b] = \frac{1}{2\gamma\kappa}\int dtd^3x\,\left[\tilde{E}_i^a \dot{A}_a^i - A_0^i \mathcal{G}_i - N\tilde{S} - N^a\tilde{C}_a\right]$$

with

| | |
|---|---|
| Gauss constraint | $\mathcal{G}_i := D_a\tilde{E}_i^a \simeq 0$ |
| Spatial diffeos | $\tilde{C}_a := \tilde{E}_i^b F_{ab}{}^i \simeq 0$ |
| Scalar constraint | $\tilde{S} := \frac{1}{2}\epsilon^{ij}{}_k \tilde{E}_i^a \tilde{E}_j^b F_{ab}{}^k \simeq 0$ |

and the field strength

$$F_{ab}^i = 2\partial_{[a}A_{b]}^i + \epsilon^i{}_{jk}A_a^i A_b^j\,.$$

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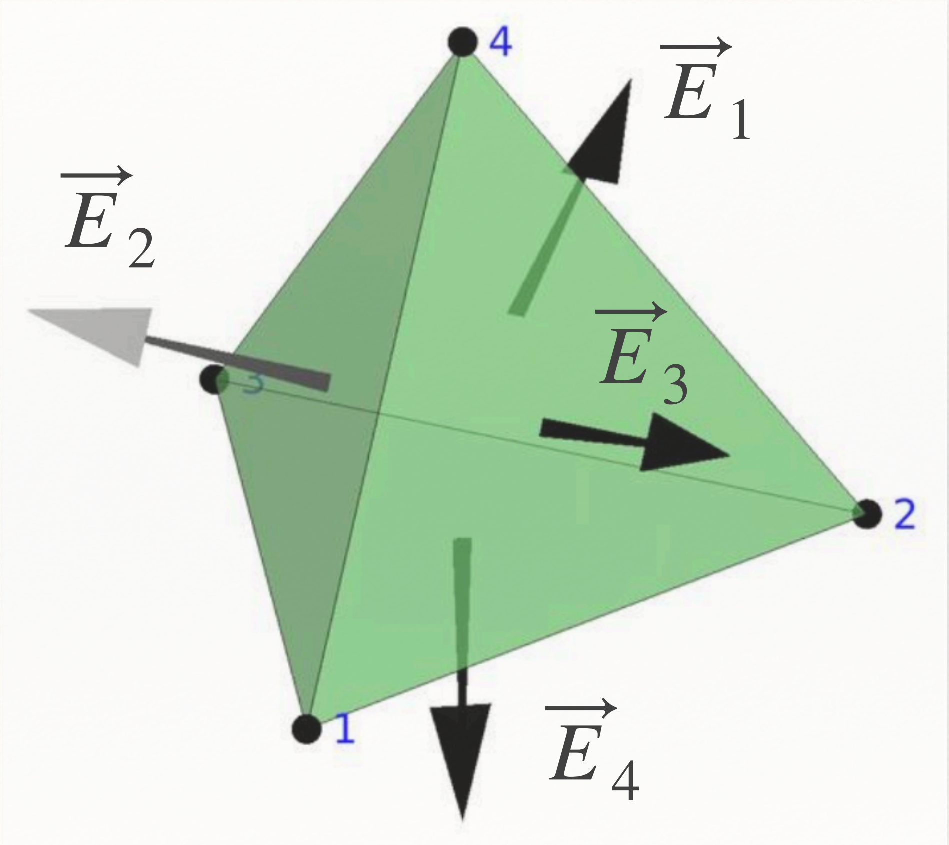
# Quantization of Geometry: Area

It's high time that we did some quantum mechanics and our tetrahedron is an ideal starting point:

$$\vec{E}_{\mathcal{S}} = \oint_{\mathcal{S}} d\vec{E} = 0 \quad \Longrightarrow \quad \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0.$$




As we have seen, each of the fluxes $\vec{E}_\ell$ can be thought of as an angular momentum vector:

Let $\mathcal{H}_{j_\ell}$ be carrier space of SU(2) irrep with basis $|j_\ell\, m_\ell\rangle$, then

$$|\hat{\mathbf{E}}_\ell|\,|j_\ell\, m_\ell\rangle = \gamma a_P \sqrt{j_\ell(j_\ell+1)}\,|j_\ell\, m_\ell\rangle$$

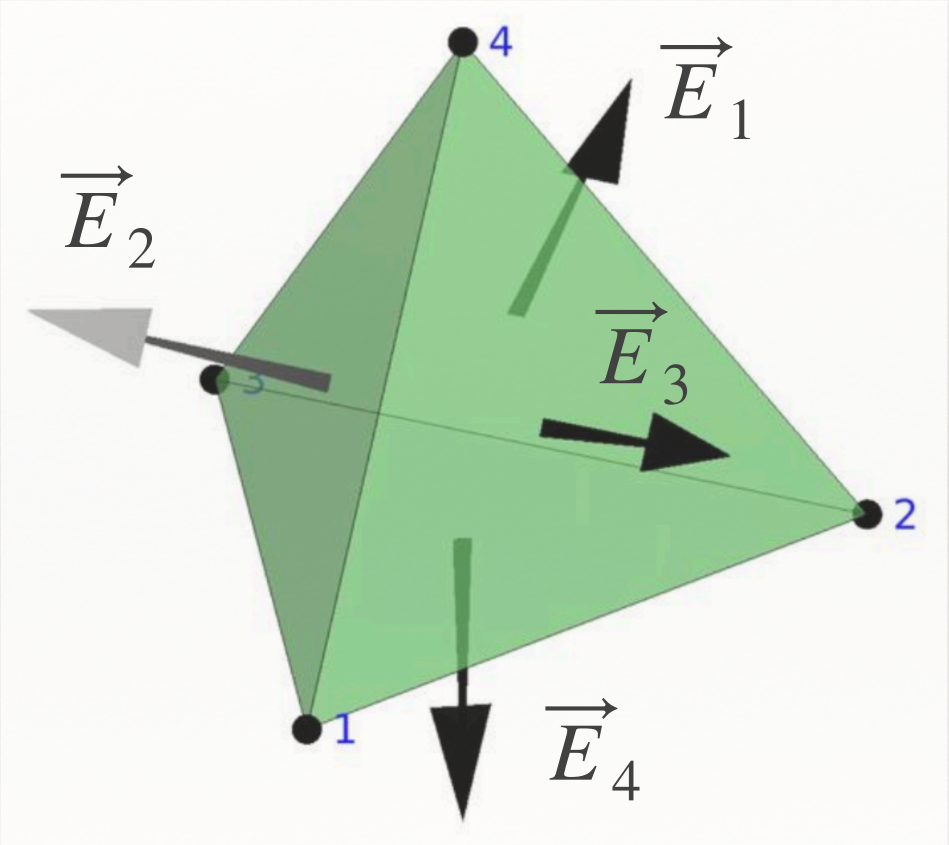
where $a_P := 8\pi\hbar G/c^3$ & Barbero-Immirzi $\gamma$ sets an 'area gap'.

# Quantization of Geometry: tetrahedra

The magnetic quantum number $m_\ell$ belies orientation dependence. This makes sense for each of the facets, but it must go away for the tet as a whole: $\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0$ encodes the rotational invariance of the tet.

To achieve this at the quantum mechanical level, we must search for rotationally invariant states of the product of the irreps:

$$|\,i\rangle \in \mathrm{Inv}(\mathscr{H}_{j_1} \otimes \mathscr{H}_{j_2} \otimes \mathscr{H}_{j_3} \otimes \mathscr{H}_{j_4}).$$




We call such an invariant state an "intertwiner" and

$$|\,i\rangle = |\,i\; j_1\, j_2\, j_3\, j_4\rangle := \sum_{m's} i^{m_1\cdots m_4}\,|\,j_1\, m_1\rangle\,|\,j_2\, m_2\rangle\,|\,j_3\, m_3\rangle\,|\,j_4\, m_4\rangle.$$

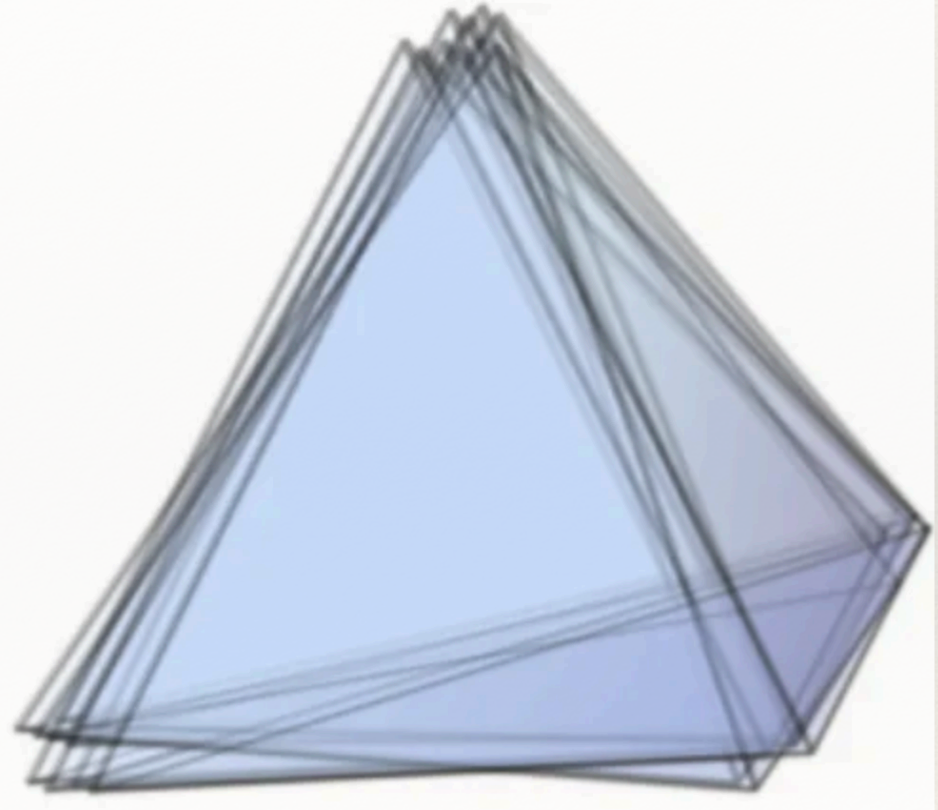
# Quantization of Geometry: tetrahedra

The classical geometry we studied at the outset suggests one way to construct an intertwiner. We saw that

$$V^2 = \frac{2}{9}\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3).$$

This is clearly a rotational invariant and if we construct the corresponding operator then its eigenvalues, $v$ say, would provide a very physical set of basis states:

$$|i\rangle = |v\, j_1\, j_2\, j_3\, j_4\rangle.$$



This highlights an important physical point—a classical tet is determined by, e.g., its 6 edge lengths. A quantum tet by only 5 parameters and hence is quantum mechanically fuzzy.

# *Today's Discussion*

1. General Relativity as a Gauge Theory Part II: the Ashtekar connection
2. Quantum Tetrahedra
3. Building Space Part I: Spin Network Motivations

# Gravity as an $SU(2) \times \mathrm{Diff}(\mathscr{M})$ gauge theory

Over the course of the discussion we have come to understand that GR is a gauge theory, but an unusual one with a gigantic gauge group: in addition to local changes of frame we have the entirety of the diffeos to consider.

Diffeos are a large part of what makes quantum gravity hard!

Observables relative: & Geometries hard to distinguish:

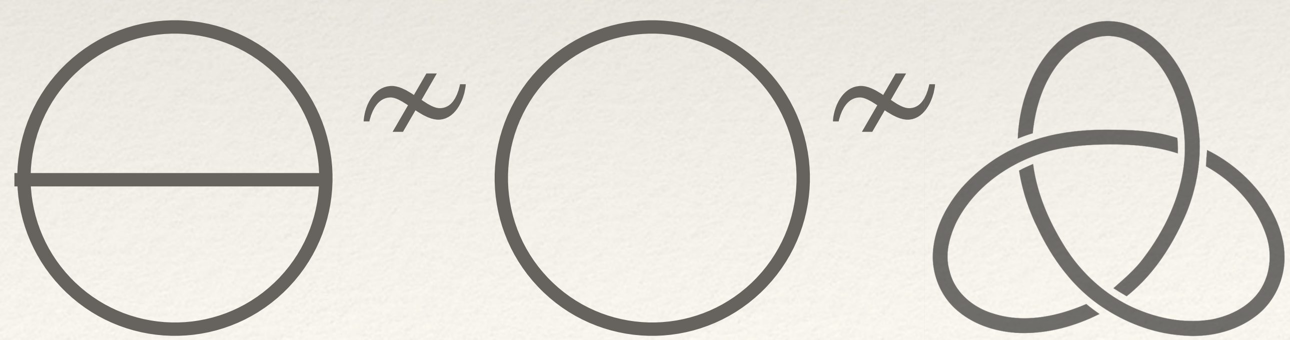
Not scalars

$$\phi(x) \to \phi'(x'),$$

but relative scalars

$$\phi(x(\psi)).$$

although



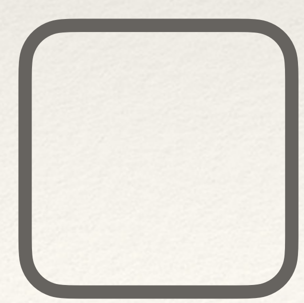
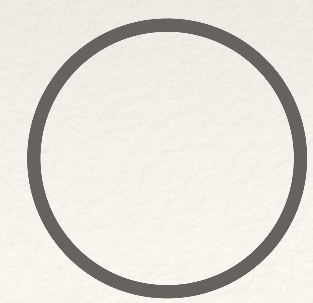
# What are the observables in a gauge theory?

| <u>Abelian</u> | <u>Non-Abelian</u> |
|---|---|
| 1. $F = (E, B)$ local | $F$ is not gauge invariant |
| 2. $A_\mu^T = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) A^\nu$ | Still possible |
| 3. $\oint A \to \oint A + \cancelto{0}{\oint d\lambda}$ | $h_\gamma(x, y) = \mathscr{P} e^{\int_x^y A}$ <br> $\to g(x) h(x, y) g^{-1}(y)$ |

Both lead to Wilson loops

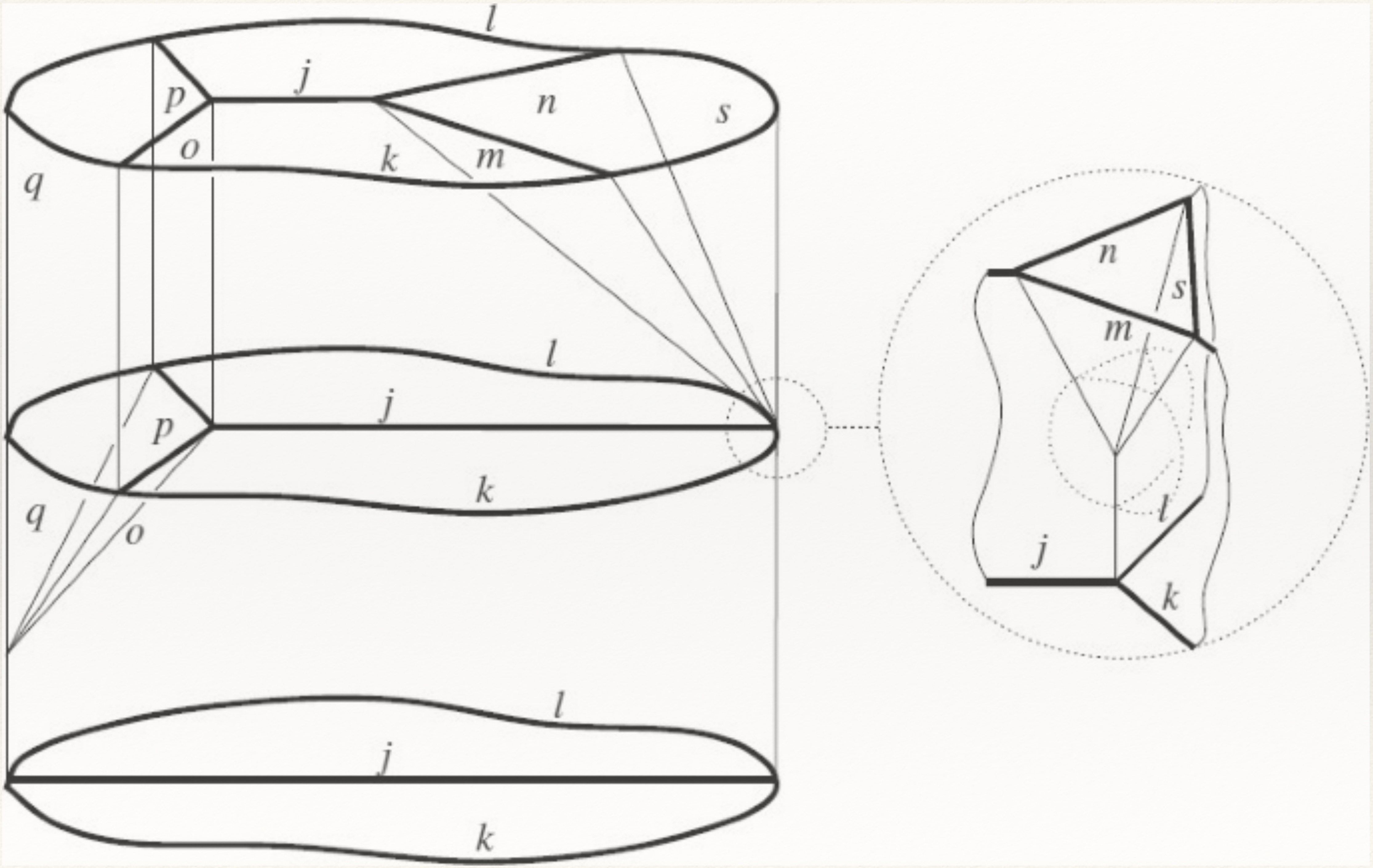
$$W(\gamma) = \mathrm{tr}\left[g(x) h(x, x) g^{-1}(x)\right] = \mathrm{tr}\left[h(x, x)\right].$$

Why aren't the $W(\gamma)$ observables used more often?

The trouble is that they distinguish  &  .

# A beautiful idea: spin networks

These two issues, each respectively from gravity and gauge theory, cancel each other out in a gauge formulation of GR!



# A beautiful idea: spin networks

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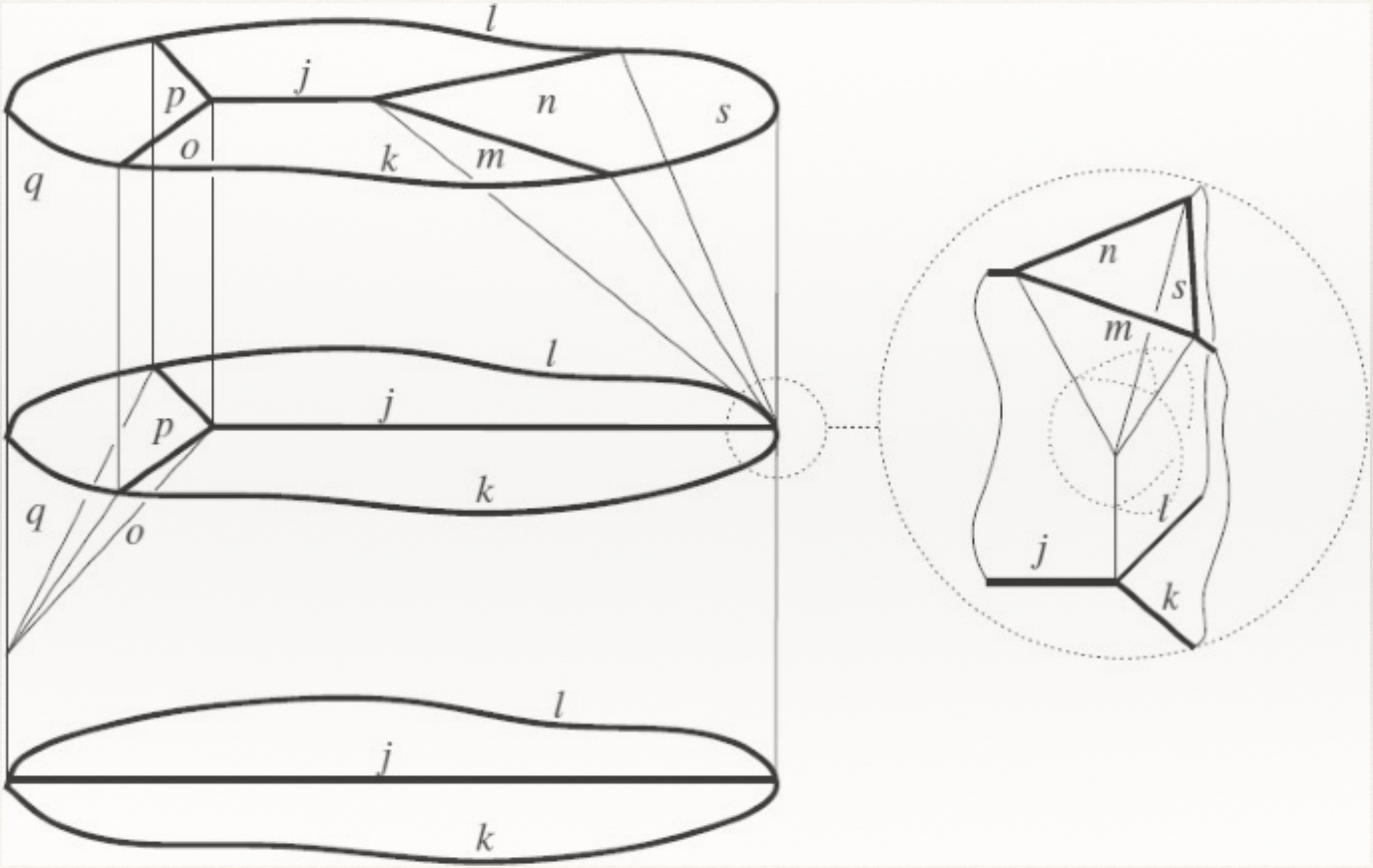
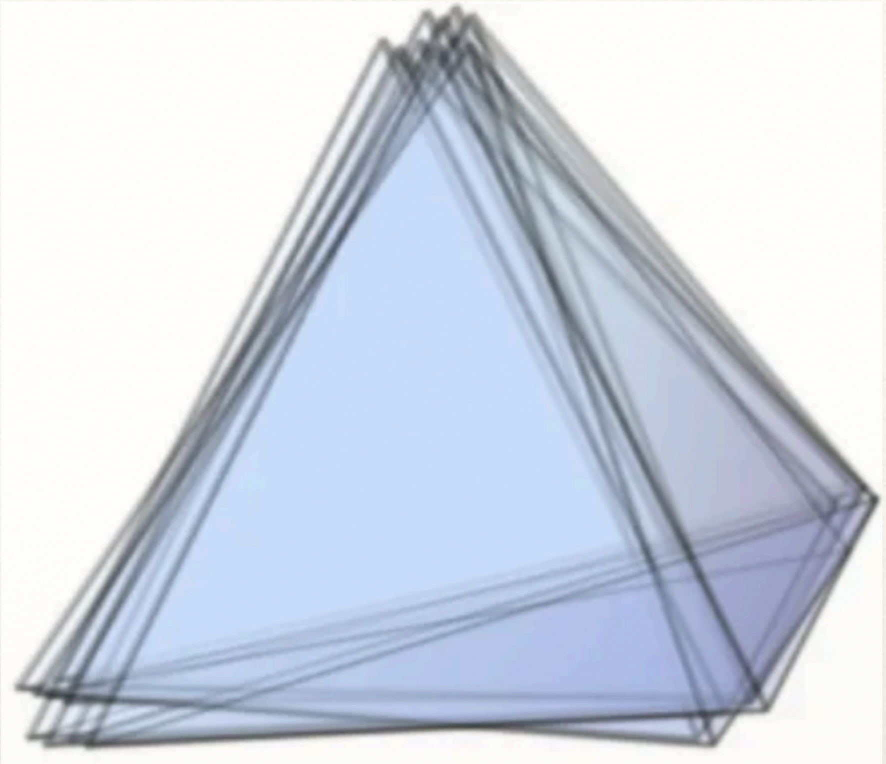
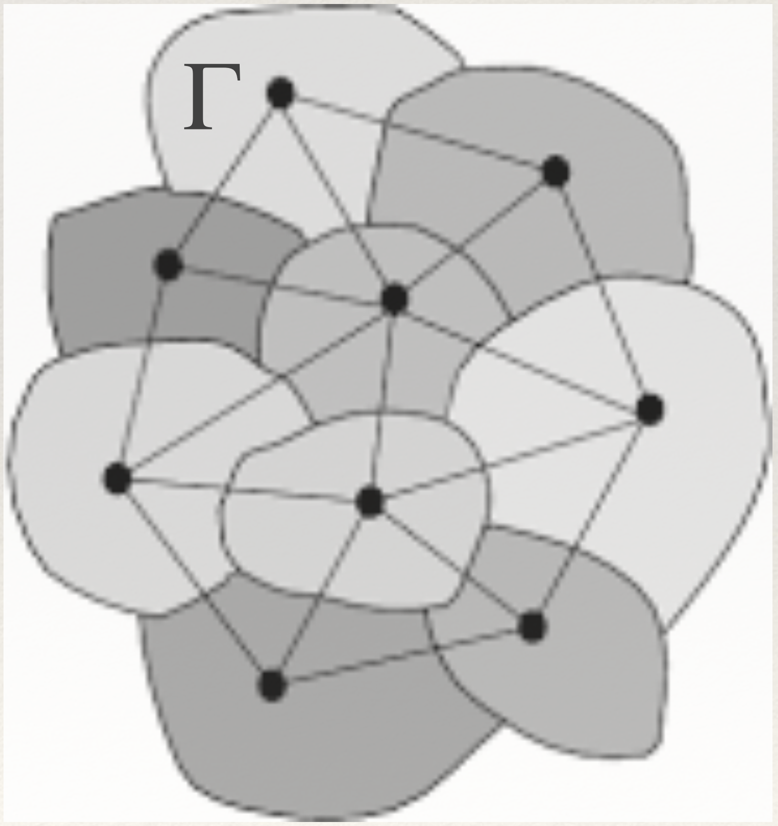


# Spin network states



A spin network is a collection of intertwiners at each node with colored links representing flux irreps of SU(2) connecting them:

$$\mathscr{H}_\Gamma = L_2[SU(2)^L/SU(2)^N].$$




Connectivity is apparent, but there is no reference to a background geometry on which they sit—they quantum mechanically manifest space.

Smaller graphs Γ can be embedded in larger graphs, representing more and more captured degrees of freedom.

Penrose, several; Rovelli & Smolin, PRD 52, 5743; Major, AJP 67, 972; …

# Thank you!

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