GR as a Perturbative QFT: #2

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Review:

- Need $T_{\mu\nu}$ as source of gravity
  ↖ Noether current of ~~space time translation~~ invariance
- To obtain this gauge spacetime translation
- Equivalence principle and coordinate changes
- the metric as a field
- Symmetry

$$\left.\begin{array}{l} dx'^{\mu} = J^{\mu}{}_{\nu}\, dx^{\nu} \\ g'_{\mu\nu} = J^{-1}{}_{\mu}{}^{\alpha}\, J^{-1}{}_{\nu}{}^{\beta}\, g_{\alpha\beta} \end{array}\right] \cancel{x}$$

- Scalar field invariant Lagrangian

$$S_m = \int d^4x \sqrt{-g}\, \frac{1}{2}\left[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m^2\phi^2 \right]$$

- First success – $T_{\mu\nu}$ as source

✓
$$\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{\sqrt{-g}}{2}\, T_{\mu\nu}$$

- Equation of motion and preview of second success

QM
$$\left[D^2 + m^2\right]\phi = 0 \xrightarrow{NR} i\frac{\partial}{\partial t}\psi = \left[-\frac{\nabla^2}{2m} + \underbrace{m\,\phi_g}\right]\psi(x,t)$$

$m_g = m_I$

$$\dot{\vec{p}} = -i\left[H, \vec{p}\right] = -m\vec{\nabla}\phi_g = m\,\vec{a}$$

QFT
- Preview of ~~third~~ success – real QFT

$$T_{\mu\nu} \; \text{---}\!\!\wr\!\!\text{---} \; T_{\mu\nu} \xrightarrow{NR} V(r) = (\#)\,\frac{m_1 m_2}{r}$$

Covariant Derivative

$$D_\mu \mathcal{O} \Rightarrow D'_\mu \mathcal{O}' = J_\mu^{-1\,\lambda} D_\lambda \mathcal{O}'$$

Scalar field $(D^2+m^2)\phi = 0$

ϕ scalar $\quad \phi' = \phi \quad \frac{1}{\sqrt{g}}\partial_\nu(g^{\mu\nu}\sqrt{g}\,\partial_\mu)$

$D_\mu \phi = \partial_\mu \phi$

$g^{\mu\nu} D_\mu (\overset{\partial_\nu}{D_\nu} \phi)$

From GR:

$$D_\nu V_\mu = \partial_\nu V_\mu - \Gamma_{\mu\nu}{}^{\lambda} V_\lambda$$

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right]$$

checked

$$\Gamma'_{\mu\nu}{}^{\lambda} = (J^{-1})_\mu^{\mu'}(J^{-1})_\nu^{\nu'} J_{\lambda'}^{\lambda}\left(\Gamma_{\mu'\nu'}{}^{\lambda'} + (J^{-1})_\sigma^{\lambda'}\partial_{\mu'} J_{\nu'}^{\sigma}\right)$$

gauge tran

$$D_\mu T^{\alpha\beta\ldots}_{\rho\sigma\ldots} = \partial_\mu T^{\alpha\beta\ldots}_{\rho\sigma\ldots} + \Gamma^{\alpha}_{\mu\nu} T^{\nu\beta\ldots}_{\rho\sigma\ldots} + \ldots - \Gamma^{\nu}_{\mu\rho} T^{\alpha\beta\ldots}_{\nu\sigma\ldots} - \ldots\,.$$

Utiyama
* Kibble
on web page

Fermions

Lorentz inv $\qquad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

$$\Lambda^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\sigma} = \delta^{\nu}{}_{\sigma}$$

$$\partial_{\mu} = \Lambda^{\nu}{}_{\mu} \partial_{\nu}$$

Fermion $\bar{\psi}(i\underbrace{\not{\partial}}_{\gamma^{\mu}\partial_{\mu}} - m)\psi$ invariant

$$\psi \to \psi' = \underbrace{S}_{4\times 4 \text{ matrix}} \psi$$

Condition $\quad \gamma_0 S^{\dagger} \gamma_0 = S^{-1}$

$$S^{-1} \gamma^{\mu} \Lambda^{\nu}{}_{\mu} S = \gamma^{\nu}$$

Solution $\qquad \sigma_{\mu\nu} = \frac{1}{2}[\gamma_{\mu}, \gamma_{\nu}]$

$$S = \exp\left[-i \frac{\sigma_{\mu\nu}}{2} \alpha^{\mu\nu}\right] \qquad J_{\mu\nu} = \frac{\sigma_{\mu\nu}}{2}$$

where $\quad \Lambda^{\alpha}{}_{\mu} = \frac{1}{2} \mathrm{Tr}\left[S \gamma^{\alpha} S^{-1} \gamma_{\mu}\right]$

Also recall:

$$g_{\mu\nu}(x) = \eta_{ab}\, e^{a}_{\mu}(x)\, e^{b}_{\nu}(x)$$

Hidden local Lorentz symmetry

$$e'^{a}_{\mu} = \Lambda^{a}{}_{b}(x)\, e^{b}_{\mu}(x)$$

$$e'^{a}_{\mu} = (\Gamma^{-1})_{\mu}{}^{\nu}\, e^{a}_{\nu}(x)$$

Fermion issue

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^\mu \partial_\mu \text{-----}\right]\Psi$$

does not ← Transforms local

Solution $\bar{\Psi}\left[i\gamma^a e_a^{\ \mu} \partial_\mu \text{ ----- }\right]$

Transforms

vierbein

Local Lorentz invariance + Coord. inv

$$\Psi(x) \to \Psi'(x') = S(x)\Psi$$

$$S(x) = \exp\left[i J_{ab} \kappa^{ab}(x)\right]$$

local

Overall

$$\mathcal{L} = \bar{\Psi}\left[i\gamma^a e_a^{\ \mu} D_\mu - m\right]\Psi$$

$$D_\mu = \partial_\mu + i J_{ab} A_\mu^{ab} = \partial_\mu + i \underline{\underline{A_\mu}}$$

Invariant if $A'_\mu = S A_\mu S^{-1} - 2i(\partial_\mu S) S^{-1}$

$$\Longrightarrow D_\mu\Psi \to \underline{\underline{S(x)}}\, D_\mu \Psi$$

A_μ^{ab} – spin connection

here I use notation J_{ab}

$SO(3,1)$

$$[J_{ab}, J_{cd}] = i\left[\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right]$$

$$[J_{ab}, J_{cd}] = 2i\, f_{[ab][cd][ef]} J^{ef}$$

Field strength tensor

$$[D_\mu, D_\nu] = -i J_{ab} R^{ab}_{\mu\nu}$$

$$R^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + (A^{ac}_\mu A^{b}_{\nu c} - A^{ac}_\nu A^{b}_{\mu c})$$

Invariants

$$R^{ab}_{\mu\nu} R^{\mu\nu}_{ab} \leftarrow \cancel{R^2}$$

$$\boxed{e^\nu_b e^\mu_a R^{ab}_{\mu\nu}} \leftarrow \text{"}R\text{"}$$

Getting to GR

[Utiyama, Kibble] ✓

Two ways:

1) Metricity

$$\boxed{D_\mu e^a_\nu = 0}$$

motivated by EP
[coord: $e^a_\mu = \delta^a_\mu$
$\partial_\nu e^a_\mu = 0$]

$$D_\mu e^a_\nu = \left[\partial_\mu e^a_\nu - \Gamma^\lambda_{\mu\nu} e^a_\lambda - A^a_{b\mu} e^b_\nu\right] = 0$$

$$A^{ab}_\mu = e^{a\nu}\left[\partial_\mu e^b_\nu - \Gamma^\lambda_{\mu\nu} e^b_\lambda\right]$$ no longer independent

Then

$$e_{a\alpha} e_{b\beta} R^{ab}_{\mu\nu} = R_{\mu\nu\alpha\beta} \Rightarrow \text{Riemann tensor}$$

$\Rightarrow$ GR

2) Palatini

$$S = \int d^4x \sqrt{-g}\,(\text{const.})\, e^{\nu}_a e^{\mu}_b R^{ab}_{\mu\nu}$$

$$\frac{\delta S}{\delta A^{ab}_{\mu}} \Rightarrow D_{\mu} e^{a}_{\nu} = 0$$

i.e. no term like $R^{ab}_{\mu\nu} R^{\mu\nu}_{ab}$

Extra step:

Without fermion

Metricity $D_{\alpha} g_{\mu\nu} = 0 \Rightarrow \Gamma^{\lambda}_{\mu\nu}$ = as before

$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$

Variations

1) $\Gamma^{\lambda}_{\mu\nu} \neq \Gamma^{\lambda}_{\nu\mu} \Rightarrow$ extra d.o.f $\Rightarrow$ "torsion"

Einstein Cartan

2) $D_{\alpha} g_{\mu\nu} \neq 0$ non metricity "metrica affine"

(Gasperini book)

Gravitational action

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} - \partial_\nu \Gamma_{\mu\alpha}{}^{\beta} - \Gamma_{\mu\rho}{}^{\beta}\Gamma_{\nu\alpha}{}^{\rho} - \Gamma_{\nu\rho}{}^{\beta}\Gamma_{\mu\alpha}{}^{\rho}$$

$R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu}$ ← convention

$R = g^{\nu\alpha} R_{\nu\alpha}$ ← invariant

Action

R ✓ Einstein

R^2

$R_{\nu\alpha}R^{\nu\alpha}$

} ~~××××~~ return to these

Here

$$S_g = \int d^4x \sqrt{-g}\left[\frac{-2}{K^2} R\right] + \underbrace{\int \Sigma^\mu K_\mu}_{\text{York, Hawking Gibbons}}$$

$$S = S_g + S_m$$

Sign-convention $R_{\mu\nu\alpha}{}^{\mu}$

– unique sign <u>kinetic energy</u> > 0

Varying

$$\delta S_g = \int d^4x \sqrt{-g}\, \frac{-2}{K^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}$$

$$\delta S_m = \int d^4x \sqrt{-g}\, \frac{1}{2} T_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\delta(S_m + S_g) = 0$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{K^2}{4}T_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\Rightarrow K^2 = 32\pi G$$

Towards Feynman rules

Weak field limit

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\, h_{\mu\nu}(x) \quad \leftarrow \text{exact}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa\, h^{\mu\nu} + \frac{1}{2}\kappa^2 h^{\mu\alpha} h_\alpha{}^\nu + \dots$$

$$R_{\mu\nu} = \frac{\kappa}{2}\left[\partial_\mu \partial_\lambda h^\lambda{}_\nu + \partial_\nu \partial_\lambda h^\lambda{}_\mu - \partial_\mu \partial_\nu h^\lambda{}_\lambda - \Box h_{\mu\nu}\right] + \dots$$

$$R = \kappa\left[\partial_\mu \partial_\sigma h^{\mu\sigma} - \Box h^\lambda{}_\lambda\right]$$

Check $\left\{R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R\right\} = \frac{\kappa^2}{4} T_{\mu\nu}$

$\partial^\mu \left[\quad '' \quad \right] = 0$

Solve:

$$O_{\mu\nu\alpha\beta}\, h^{\alpha\beta} = \frac{\kappa^2}{4} T_{\mu\nu}$$

↑ Look for inverse of $O_{\mu\nu\alpha\beta}$

$$O_{\mu\nu}{}^{\alpha\beta} G_{\alpha\beta\gamma\delta}(x-y) = \frac{1}{2} I_{\mu\nu\gamma\delta}\, \delta_D^{(4)}(x-y)\,, \quad \Longleftarrow \quad I_{\mu\nu\gamma\delta} = \frac{1}{2}\left[\eta_{\mu\gamma}\eta_{\nu\delta} + \eta_{\nu\gamma} + \eta_{\mu\delta}\right]$$

where

$$O^{\mu\nu}{}_{\alpha\beta} \equiv \left(\delta_\alpha^{(\mu}\delta_\beta^{\nu)} - \eta^{\mu\nu}\eta_{\alpha\beta}\right)\Box - 2\delta^{(\mu}_{(\alpha}\partial^{\nu)}\partial_{\beta)} + \eta_{\alpha\beta}\partial^\mu\partial^\nu + \eta^{\mu\nu}\partial_\alpha\partial_\beta\,.$$

Inverse Does not exist

Gauge invariance

Small coord chang $x'^{\mu} = x^{\mu} + \xi^{\mu}(x)$

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \qquad \cancel{\alpha}\alpha$$

$$R' = R$$

Harmonic or de Donder gauge

$$\partial_\mu h^\mu{}_\nu - \frac{1}{2}\partial_\nu h^\lambda{}_\lambda = 0$$

or with $\underbrace{\bar{h}^\mu{}_\nu = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h^\lambda{}_\lambda}_{\text{notation}} \Rightarrow \partial_\mu \bar{h}^{\mu\nu} = 0$

Einstein eq: $\Box h_{\mu\nu} = -\frac{\kappa}{2}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda{}_\lambda\right)$

or $\Box \bar{h}_{\mu\nu} = \frac{-\kappa}{2} T_{\mu\nu}$ $\qquad \cancel{\alpha}$

know how to invert $\Box^{-1} \sim \frac{1}{8^2}$

$\Rightarrow$ Solve for ~~point~~ mass:

$$T_{\mu\nu} = \delta_{\mu 0}\delta_{\nu 0}\, m\, \delta^3(x)$$

$$\Rightarrow \kappa h_{\mu\nu} = \begin{pmatrix} 2\phi_g & & & \\ & 2\phi_g & & 0 \\ & 0 & 2\phi_g & \\ & & & 2\phi_g \end{pmatrix} \qquad \phi_g = -\frac{GM}{r}$$

Look at action

$$S = \int d^4x \sqrt{-g}\left[-\frac{2}{\kappa^2} R + \mathcal{L}_m\right]$$

$\partial^2 h$

$$-\frac{2\sqrt{-g}R}{\kappa^2} = -\frac{2}{\kappa^2}\left[\partial_\mu \partial_\nu h^{\mu\nu} - \Box h^\lambda{}_\lambda\right]$$

← total deriv

$$+\frac{1}{2}\left[\partial_\lambda h_{\mu\nu}\, \partial^\lambda \bar{h}^{\mu\nu} - 2\partial^\lambda \bar{h}_{\mu\lambda}\, \partial_\sigma h^{\mu\sigma}\right] + \mathcal{O}(h^3)$$

Gauge fixing

$$\mathcal{L}_{gf} = \xi\, \partial_\mu \bar{h}^{\mu\nu}\, \partial^\lambda \bar{h}_{\lambda\nu}$$

harmonic gauge $\xi = 1$

[Need ghosts - next time]

After gauge fixing:

$$\mathcal{L} = -\frac{1}{2} h_{\mu\nu}\underbrace{\left[I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right]}\Box h_{\alpha\beta} + \underbrace{\frac{\kappa}{2} h^{\mu\nu} T_{\mu\nu}}_{\text{matter}}$$

$$I_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right]$$

Read out propagator

Inverse $\left[I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right]\left[a\, I_{\alpha\beta\gamma\delta} + b\, \eta_{\alpha\beta}\eta_{\gamma\delta}\right] = I^{\mu\nu}{}_{\gamma\delta}$

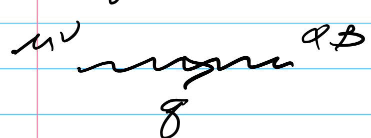
solved by $a = 1,\ b = -\frac{1}{2}$

Propagator is

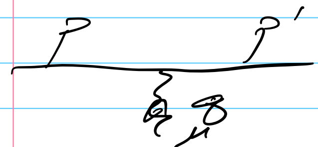
$$iD^{\alpha\beta\gamma\delta}(x) = \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot x}\, \frac{iP^{\alpha\beta\gamma\delta}}{q^2+i\epsilon}$$

$$P^{\alpha\beta\gamma\delta} = \frac{1}{2}\left[\eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\beta\gamma}\eta^{\alpha\delta} - \eta^{\alpha\beta}\eta^{\gamma\delta}\right]$$

Feynman rules



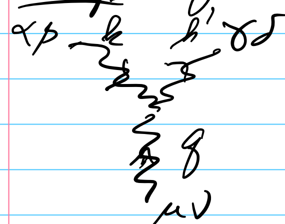
$$= i\,\frac{P^{\mu\nu\alpha\beta}}{q^2+i\epsilon}$$



$$\mathcal{L} = \frac{\kappa}{2} h_{\mu\nu} T^{\mu\nu}$$

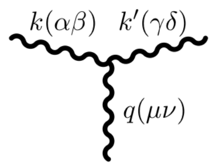
$$= -i\frac{\kappa}{2}\left[(p_\mu p'_\nu + p_\nu p'_\mu) - \eta_{\mu\nu}(p\cdot p' - m^2)\right]$$

Triple graviton



$$= -i\tau^{\mu\nu}_{\alpha\beta\gamma\delta}$$

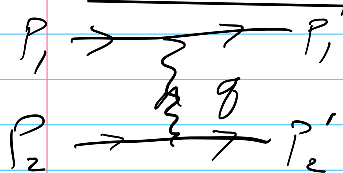
gotten by expanding $\mathcal{L}$ —



$$
\begin{aligned}
&= \frac{i\kappa}{2}\Bigg(P_{\alpha\beta,\gamma\delta}\Big[k^\mu k^\nu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\Big] \\
&+ 2q_\lambda q_\sigma\big[I^{\lambda\sigma}{}_{\alpha\beta}I^{\mu\nu}{}_{\gamma\delta} + I^{\lambda\sigma}{}_{\gamma\delta}I^{\mu\nu}{}_{\alpha\beta} - I^{\lambda\mu}{}_{\alpha\beta}I^{\sigma\nu}{}_{\gamma\delta} - I^{\sigma\nu}{}_{\alpha\beta}I^{\lambda\nu}{}_{\gamma\delta}\big] \\
&+ \big[q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\nu}{}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\mu}{}_{\alpha\beta}) \\
&- q^2(\eta_{\alpha\beta}I^{\mu\nu}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\mu\nu}{}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})\big] \\
&+ \Big[2q^\lambda\Big(I^{\sigma\nu}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\mu + I^{\sigma\mu}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\nu \\
&- I^{\sigma\nu}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu - I^{\sigma\mu}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu\Big) \\
&+ q^2(I^{\sigma\mu}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu}{}_{\alpha\delta}) + \eta^{\mu\nu}q^\lambda q_\sigma(I^{\rho\sigma}{}_{\gamma\delta}I_{\alpha\beta,\lambda\rho} + I^{\rho\sigma}{}_{\alpha\beta}I_{\gamma\delta,\lambda\rho})\Big] \\
&+ \Big[(k^2 + (k-q)^2)\Big(I^{\sigma\mu}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I^{\sigma\nu}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\Big) \\
&- k^2\eta_{\gamma\delta}I^{\mu\nu}{}_{\alpha\beta} - (k-q)^2\eta_{\alpha\beta}I^{\mu\nu}{}_{\gamma\delta}\Big]\Bigg).
\end{aligned}
$$

$\longleftarrow$ important

Gravitational scattering



$$-i\mathcal{M} = \frac{i\kappa}{2}\left[(p_1^\mu p_1'^\nu + p_1'^\mu p_1^\nu) + \cdots\right]\frac{iP_{\mu\nu\alpha\beta}}{q^2 + i\varepsilon}$$

$$\times\left[(p_2^\alpha p_2'^\beta + p_2'^\alpha p_2^\beta) + \cdots\right]$$

$$\underset{\substack{NR\\+F.T.}}{\Longrightarrow} \quad V(r) = -\frac{G m_1 m_2}{r} \quad \checkmark$$

with $\kappa^2 = 32\pi G$