GR as a Perturbative QFT

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Last time:

Developed GR as QFT
- gauging spacetime translations
- pathway included fermions & spin connection
- Field theory description rather than geometry

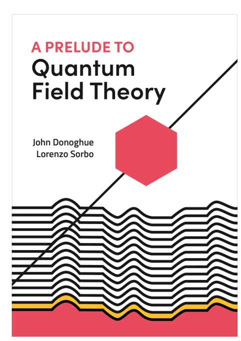
First look at Feynman rules
- weak field expansion
- gauge fixing (harmonic gauge)
- propagators and vertices

Today

- Background Field Method
- FDFP gauge fixing and ghosts

On webpage:

- Path Integral section of "Prelude to QFT"
- use PI appendix of DSM
- Abbott on BFM ✓



Path Integral Quantization

Canonical quant. $\left[\phi(x), \pi(x')\right]_{ET} = i\hbar\, \delta^3(x-x')$;
– not too useful for A_μ, $g_{\mu\nu}$

$$Z[J] = \int [d\phi]\, e^{i\int d^4x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + J\phi\right]}$$

– classical min of $\mathcal{L}$
– quantum from PI

Three results:

1)
$$Z_0[J] = \int [d\phi]\, e^{i\int d^4x \left[\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) + J\phi\right]}$$
$$= Z_0[0]\, e^{-i\int d^4x\, d^4y\, J(x)\, D_F(x-y)\, J(y)}$$

$\Longrightarrow$ Pert. Th.

2) $$Z = \int [d\phi]\, e^{i\int d^4x\, \phi\, \mathcal{O}\, \phi}$$
$$= N\left[\det \mathcal{O}\right]^{-1/2} = e^{-\frac{i}{2}\operatorname{Tr}(\ln \mathcal{O})}$$

B.F. $\rightarrow$

3) $$Z = \int [d\psi\, d\bar\psi]\, e^{i\int d^4x\, \bar\psi\, \mathcal{O}\, \psi}$$
$$= N\left[\det \mathcal{O}\right]$$ $\longleftarrow$ ghost story

Background Field Method

- expanding with a bkgd. field

Example: QED with massless scalar

$$\mathcal{L} = (D_\mu \phi)^* (D^\mu \phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$D_\mu = \partial_\mu + ieA_\mu \leftarrow$ background field

Transform

$$\mathcal{L}_\phi = -\phi^* D^\mu D_\mu \phi = -\phi^* \mathcal{O} \phi$$

acts on ϕ

$$D_\mu D^\mu = \Box + 2ie A_\mu \partial_\mu + ie(\partial^\mu A_\mu) - e^2 A_\mu A^\mu$$
$$= \Box + V(x)$$

Path Integral

$$\int [d\phi\, d\phi^*]\, e^{-i\int d^d x\, \phi^* \mathcal{O} \phi} = N[\det \mathcal{O}]^{-1}$$
$$= e^{-i \operatorname{Tr} \ln \mathcal{O}}$$
$$= e^{\int d^4x \langle x | \ln \mathcal{O} | x \rangle}$$

$$\ln \mathcal{O} = \ln(\Box + V) = \ln \Box \left(1 + \frac{1}{\Box} V\right)$$
$$= \ln \Box + \ln\left(1 + \frac{1}{\Box} V\right)$$
$$= \ln \Box + \frac{1}{\Box} V - \frac{1}{2} \frac{1}{\Box} V \frac{1}{\Box} V + \cdots \ldots$$

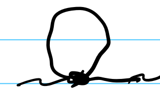
triangle

$\langle x | \ln \mathcal{O} | x \rangle$

$$\langle x | \frac{1}{\Box_{+i\epsilon}} | y \rangle = i D_F(x-y)$$

<u>First order</u> $\int d^4x \langle x | \frac{1}{\Box} V | x \rangle$

$$= \int d^4x\, i D_F(x-x)\, V(x) \sim$$



$\leftarrow$ Tadpole

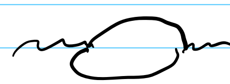
Vanishes $\quad iD(x-x) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+i\epsilon} \to 0$ in dim reg

<u>Second order</u>

$$\langle x | \frac{1}{\Box} V \frac{1}{\Box} V | x \rangle = \int \langle x | \frac{1}{\Box} V | y \rangle \langle y | \frac{1}{\Box} V | x \rangle\, d^4y$$

$$\Rightarrow \frac{1}{2} \int d^4x\, d^4y \quad V(x)\, iD_F(x-y)\, V(y)\, iD(y-x)$$

$\uparrow$ A, ∂



$=$ vac. pol

<u>Calc.</u>

Int by parts $\quad A_\mu(x) \underbrace{\left[\partial D \quad\right]^{\mu\nu}}_{M^{\mu\nu}(x-y)} A_\nu(y)$

<u>Dim reg</u> $\quad D_F\, \partial_\mu \partial_\nu D_F = \underbrace{(d\, \partial_\mu \partial_\nu - g_{\mu\nu} \Box) D_F^2}_{\text{P.V. reduction}}$

$$g^{\mu\nu} \int d^4k \frac{k_\mu k_\nu}{k^2 (k+q)^2} = g^{\mu\nu} \left(\underset{\nwarrow d}{g_{\mu\nu} A} + B\, q_\mu q_\nu \right) g^{\mu\nu}$$

$$\Rightarrow e^2 \int d^4x\, d^4y\, A_\mu(x)\, M_{\mu\nu}(x-y)\, A_\nu(y)$$

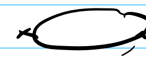
$$\uparrow \left[g_{\mu\nu} \Box - \partial_\mu \partial_\nu \right] \frac{D_F^2(x-y)}{(d-1)}$$

Calc. on web page

Then

$$\Delta \mathcal{L} = \int d^4x\, d^4y\, \frac{1}{4} F_{\mu\nu}(x) \frac{D_F^2(x-y)}{(d-1)} F_{\mu\nu}(y) \leftarrow \text{g.i.}$$

$$D_F^2(x) = \int d^4x \, e^{iq\cdot x} \, D_F^2(q)$$



bubble

$$D_F^2(q) = \frac{i}{16\pi^2}\left[\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \ln q^2/\mu^2\right]$$

$$\Rightarrow D_F^2(x-y) = F.T\left[\frac{1}{\varepsilon} \cdots \ln q^2/\mu^2\right]$$

$$= \underbrace{\frac{i}{16\pi^2}\left[\frac{1}{\varepsilon}\cdots\right]\delta^4(x-y)} + \frac{1}{16\pi^2} L(x-y)$$

F.T of log

Divergence are local

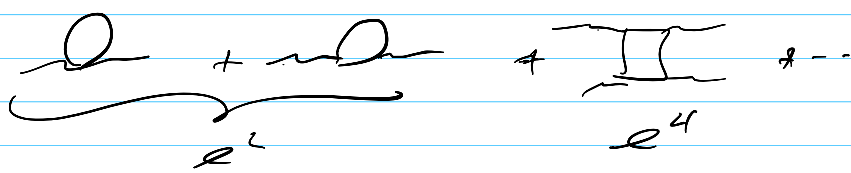
One loop effective action

Renorm

$$S = S_A + \Delta S = \int d^4x \, -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} Z_3^{-1}$$

$$+ b e^2 \int d^4x \, d^4y \, F_{\mu\nu}(x) \, L(x-y) \, F^{\mu\nu}(y)$$

Hold this   B.V



Generalize

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_N \end{pmatrix}$$

$$\mathcal{L} = \phi^* [d_\mu d^\mu + \sigma(x)] \phi$$

$$d_\mu = \partial_\mu + \Gamma_\mu(x)$$

$$\sigma \to \sigma$$

$$\Rightarrow \Delta S = \int d^4x\, d^4y\, \mathrm{Tr}\left[\Gamma_{\mu\nu}(x) \frac{D_F^2(x-y)}{4(d-1)} \Gamma^{\mu\nu} + \frac{1}{2}\sigma(x) D_F^2(x-y)\sigma(y) \right]$$

$$\Gamma_{\mu\nu} = [d_\mu, d_\nu] = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]$$

Divergences

$$\Delta S_{div} = \int d^4x\, \frac{1}{16\pi^2}\left[\frac{1}{\varepsilon} + \dots\right] \underbrace{\mathrm{Tr}\left[\frac{1}{12}\Gamma_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{2}\sigma^2\right]}_{a_2 \text{ Heat kernel coeff.}}$$

B.F. Gen Rel

Expand $\quad g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \kappa h_{\mu\nu}(x)$

$$\Gamma_{\mu\nu}^{\lambda} = \bar{\Gamma}_{\mu\nu}^{\lambda} + \underbrace{\Gamma_{\mu\nu}^{\lambda}}_{O(h)} + \underbrace{\Gamma_{\mu\nu}^{\lambda}}_{O(h^2)}$$

$$R = \bar{R} + \underbrace{R_1}_{h} + \underbrace{R}_{O(h^2)}$$

Oh!

Algebra $\Gamma_{\mu\nu}^{\ \ \lambda} = \frac{1}{2} g^{\lambda\sigma} [\partial_\mu g_{\sigma\nu} + \cdots]$

$$\underline{\Gamma}_{\mu\nu}^{\ \ \lambda} = \frac{1}{2} \bar{g}^{\lambda\sigma} [\bar{D}_\mu h_{\sigma\nu} + \bar{D}_\nu h_{\sigma\mu} - \bar{D}_\sigma h_{\mu\nu}]$$

covariant w.r.t $\bar{g}$

Gauge invariance

$$x'^\mu = x^\mu + \xi^\mu(x)$$

$$dx'^\mu = dx^\mu + \partial_\nu \xi^\mu dx^\nu$$

$$g'_{\mu\nu}(x') = (J_\mu^{-1})^\alpha g_{\alpha\beta}(x') J_\nu^{-1\,\beta}$$

$$= g_{\mu\nu}(x') - g_{\alpha\nu} \partial_\mu \xi^\alpha - g_{\beta\mu} \partial_\nu \xi^\beta$$

$$g_{\mu\nu}(x) + \xi^\alpha \partial_\alpha g_{\mu\nu}(x)$$

$$g'_{\mu\nu} = g_{\mu\nu} - g_{\alpha\nu} \partial_\mu \xi^\alpha - g_{\alpha\mu} \partial_\nu \xi^\alpha + \xi^\alpha \partial_\alpha g_{\mu\nu}$$

Expansion

$$h'_{\mu\nu} = h_{\mu\nu} + \bar{D}_\mu \xi_\nu + \bar{D}_\nu \xi_\mu$$

gen of flat space result

$$\mathcal{L} = \frac{-2}{\kappa^2} R = \sqrt{-\bar{g}} \Big[\frac{-2}{\kappa^2} \bar{R} - \frac{1}{\kappa} [h^\alpha_{\ \alpha} \bar{R} - 2 \bar{R}^\alpha_{\ \nu} h^\nu_{\ \alpha}]$$

$$+ \frac{1}{2} \bar{D}_\alpha h_{\mu\nu} \bar{D}^\alpha h^{\mu\nu} - \frac{1}{2} \bar{D}_\alpha h^\lambda_{\ \lambda} \bar{D}^\alpha h^\sigma_{\ \sigma}$$

$$+ \bar{D}_\nu h^\lambda_{\ \lambda} \bar{D}^\beta h^\nu_{\ \beta} - \bar{D}_\nu h_{\alpha\beta} \bar{D}^\alpha h^{\nu\beta}$$

$$- \bar{R} \Big[\frac{1}{2} (h^\lambda_{\ \lambda})^2 - \frac{1}{2} h^\alpha_{\ \beta} h^\beta_{\ \alpha} \Big] + h^\lambda_{\ \lambda} h^\alpha_{\ \nu} \bar{R}^\nu_{\ \alpha} + 2 h^\gamma_{\ \beta} h^\beta_{\ \alpha} \bar{R}^\alpha_{\ \nu} \Big]$$

gauge invariant

Gauge fixing

Previously $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h^\lambda{}_\lambda = 0$

$$\mathcal{L}_{gf} = \frac{1}{2}\left[\quad \right]^2$$

General.

$$C_\nu = \left[\bar{D}^\mu h_{\mu\nu} - \frac{1}{2}\bar{D}_\nu h^\lambda{}_\lambda \right]$$

$$\mathcal{L}_{gf} = \frac{1}{2} C_\nu C^\nu$$

Ghost:

Feynman & De Witt

Feynman tree theorem:

This made me investigate the entire subject in great detail to find out what the trouble is. I discovered in the process two things. First, I discovered a number of theorems, which as far as I know are new, which relate closed loop diagrams and diagrams without closed loop diagrams (I shall call the latter diagrams "trees"). The unitarity relation which I have just been describing, is one connection between a closed loop diagram and a tree; but I found a whole lot of other ones, and this gives me more tests on my machinery. So let me just tell you a little bit about this theorem, which gives other rules. It is rather interesting. As a matter of fact, I proved that if you have a diagram with rings in it there are enough theorems altogether, so that you can express any diagram with circuits completely in terms of diagrams with trees and with all momenta for tree diagrams in physically attainable regions and on the mass shell. The demonstration is remarkably easy. There are several ways of demonstra-

Feyn problem GR + YM

There is another theory, more well-known to meson physicists, called the Yang-Mills theory, and I take the one with zero mass; it is a special theory that has never been investigated in great detail. It is very analogous to gravitation; instead of the coordinate transformation group being the source of everything, it's the isotopic spin rotation group that's the source of everything. It is a non-linear theory, that's like the gravitation theory, and so forth. At the suggestion of Gell-Mann I looked at the theory of Yang-Mills with zero mass, which has a kind of gauge group and everything the same; and found exactly the same difficulty. And therefore in meson theory it was not strictly unknown difficulty, because it should have been noticed by meson physicists who had been fooling around the Yang-Mills theory. They had not noticed it because they're practical, and the Yang-Mills theory with zero mass obviously does not exist, because a zero mass field would be obvious; it would come out of nuclei right away. So they didn't take the case of zero mass and investigate it carefully. But this disease which I discovered here is a disease which exist in other theories. So at least there is one good thing: gravity isn't alone in this difficulty. This observation that Yang-Mills was also in trouble was of very great advantage to me; it made everything much easier in trying to straighten out the troubles of the preceding paragraph, for several reasons.

Well, what then, now you have the difficulty; how do you cure it? Well I tried the following idea: I assumed the tree theorem to be true, and used it in reverse. If every closed ring diagram can be expressed as trees, and if trees produce no trouble and can be computed, then all you have to do is to say that the closed loop diagram is the sum of the corresponding tree diagrams, that it should be. Finally in each tree diagram for which a graviton line has been opened, take only real transverse graviton to represent that term. This then serves as the definition of how to calculate closed-loop diagrams; the old rules, involving a propagator $1/k^2+i\varepsilon$ *etc.* being superseded. The advantage of this is, first, that it will be gauge invariant, second, it will be unitary, because unitarity is a relation between a closed diagram and an open one, and is one of the class of relations I was talking about, so there's no difficulty. And third, it's completely unique as to what the answer is; there's no arbitrary fiddling around with different gauges and so forth, in the inside ring as there was before. So that's the plan.

it no longer singular. That's the first thing; I found it out by trial and error before, when I made it gauge invariant. But then secondly, you must subtract from the answer, the result that you get by imagining that in the ring which involves only a graviton going around, instead you calculate with a different particle going around, an artificial, dopey particle is coupled to it. It's a vector particle, artificially coupled to the external field, so designed as to correct the error in this one. The forms are evidently invariant,

ghosts

Fermionic

DeWitt: Because of the interest of the tricky extra particle that you mentioned at the end, and its possible connection, perhaps, with some work of Dr Białynicki-Birula, have you got far enough on that so that you could repeat it with just a little more detail? The structure of it and what sort of an equation it satisfies, and what is its propagator? These are technical points, but they have an interest.

Feynman: Give me ten minutes. And let me show how the analysis of these tree diagrams, loop diagrams and all this other stuff is done mathematical way. Now I will show you that I too can write equations that nobody can understand. Before I do that I should like to say that there are a few properties that this result has that are interesting. First of

DeWitt formalize
Fadeev Popov gauge theories

Formalism

Logic: Insert unity into PI

$$1 = \int dx^i \, \underbrace{\delta(f(x))}_{\text{gauge fixing}} \det\left(\frac{\partial f}{\partial x}\right)$$

$$\det O = \int d\psi d\bar\psi \, e^{i\bar\psi O \psi}$$

ghost

Field theory

$$A'_\mu = A_\mu + \partial_\mu \theta$$

$$f = \partial_\mu A^\mu = 0 \,/\, F$$

$$1 = \int [d\theta(x)] \, \delta(f(A^\theta) - F) \, \det\left(\frac{\partial f}{\partial \theta}\right)$$

manipulate to gauge fixing action

The interesting factor is the determanant
- the rest is converted into usual gauge fixing
- the following on this page was not covered in the lecture
- it is also covered in modern QFT books

The gauge constraint is initially represented by the $\delta(f(A)-F)$

We can put this in a more useful form by the following trick
- Let $F = F(x)$ [OK since A in $f(A)$ is a function of x]

Then exponetiate the constraint using

$$\int [dF(x)]\, \delta(f(A) - F(x))\, e^{-\frac{i}{2}\int d^4x\, F(x)^2} = e^{-i\int d^4x\, \frac{1}{2} f(A)^2}$$

This is the usual gauge fixing Lagrangian

Recall for QED, $f(A) = \partial_\mu A^\mu$

This gives

$$e^{i\int d^4x\, \frac{1}{2}[f(A)]^2} = e^{i\int d^4x \frac{1}{2}[\partial_\mu A^\mu]^2} = e^{i\int d^4x\, \mathcal{L}_{g.f.}}$$

From this part, we recover what we previously did by hand $\rightarrow$ adding gauge fixing term to the Lagrangian

But here we still have the determanant left over and we turn to this now:

The determanent

fermion field

$$\det\left|\frac{\partial f}{\partial \theta}\right| = \int [dc\, d\bar{c}]\, e^{i\int d^4x\, \bar{c}\, \frac{\partial f}{\partial \theta}\, c}$$

new term in action

To calculate

QED

$f = (\partial_\mu A^\mu - F)$ in $\delta(f(x))$

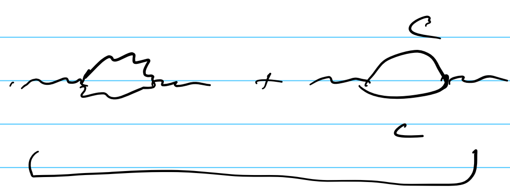
$$f^\theta = \partial_\mu(A^\mu - \partial^\mu \theta) - F$$

$$\frac{\partial f}{\partial \theta} = \Box$$

$$\mathcal{L}_{gh} = \bar{c}\, \Box\, c \quad \text{for QED}$$

For QCD

$$\mathcal{L} = \bar{c}_i \left[\delta^{ij}\Box - f^{ij}\, \partial^\mu A^k_\mu\right] c^k$$

 $\Rightarrow$ gauge invariant

Gravity

$$C_\nu = \left[\bar{D}^\mu h_{\mu\nu} - \frac{1}{2}\bar{D}_\nu h^\lambda{}_\lambda\right]$$

$$\delta(C_\nu - F_\nu)$$

$$\Longrightarrow \mathcal{L}_{gf} = \frac{1}{2} C_\nu C^\nu$$

Ghost

$$\det\left(\frac{\partial C_\nu}{\partial \xi_\mu}\right)$$

$$h_{\mu\nu} \to h_{\mu\nu} + \bar{D}_\nu \xi_\mu + \bar{D}_\mu \xi_\nu$$

$$\delta C_\mu = \bar{D}^\nu\left(\bar{D}_\nu \xi_\mu + \bar{D}_\mu \xi_\nu - \frac{2}{2}\bar{D}_\nu \bar{D}_\mu \xi^\nu\right)$$

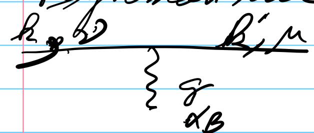
$$= \bar{D}^\nu \bar{D}_\nu \xi_\mu + \underbrace{[\bar{D}_\nu, \bar{D}_\mu]}_{R_{\mu\nu}} \xi^\mu$$

$$= \left(g_{\mu\nu}\bar{D}^2 + \bar{R}_{\mu\nu}\right)\xi^\nu$$

Ghosts η_μ

$$\mathcal{L}_{gh} = \bar{\eta}_\mu \left[g_{\mu\nu} D^2 + \bar{R}_{\mu\nu}\right]\eta^\nu$$

Feynman rule



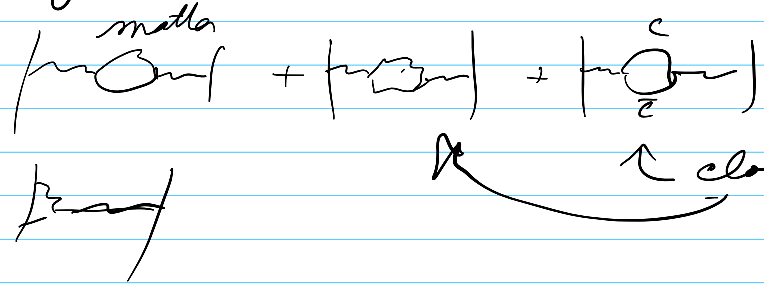
$$= -i\left[k_\alpha k_\beta \eta_{\mu\nu} + q\cdot k\,\eta_{\mu\nu}\eta_{\alpha\beta} + q_\mu k_\alpha \eta_{\nu\alpha} - q_\nu k_\alpha \eta_{\mu\beta} + q^2 \eta_{\mu\nu}\eta_{\alpha\beta}\right]$$

Summary

$$Z = \int [dh_{\mu\nu}][d\eta\, d\bar{\eta}]\; e^{i\int d^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}\bar{R} + \mathcal{L}_{gf}(h) + \mathcal{L}_{gh}(\eta,\bar{\eta},g)\right]}$$

The ghosts are used in closed loops

YM + grav



closed loop

EPFL
['t Hooft Veltman] ← ✓