GR as a Perturbative QFT #4

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Previous: QFT construction of GR

Now: Calculations
- UV properties
- transition to IR

Today: Tree + Loop calculations
W Sept 20: EFT day
F Sept 22: Using EFT techniques in pert. QG
- low energy theorems
- limitations

Review

Covariant Path Integral Quantization

$$Z = \int [dg][d\phi]\, e^{i\int d^4x \sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \ldots\right]}$$

in PI

propagator

Perturbative expansions

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$$

bkg, fluct. quantum

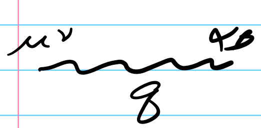
Need gauge fixing - harmonic gauge

$$\mathcal{L}_{gf} = \frac{1}{2}\left[D^\alpha h_{\mu\alpha} - \frac{1}{2}\bar{D}_\mu h^\lambda{}_\lambda\right]^2$$

and ghosts

$$\mathcal{L}_{gh} = \bar{\eta}^\mu\left[\bar{g}_{\mu\nu}\bar{D}^2 + \bar{R}_{\mu\nu}\right]\eta^\nu$$

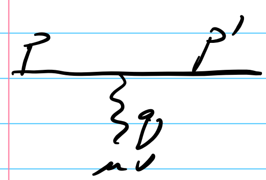
Yields Feynman rules



$$= i\,\frac{P^{\mu\nu\alpha\beta}}{q^2+i\varepsilon}$$

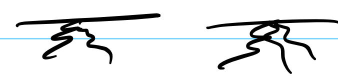
d=4

$$P_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{d-2}\eta^{\alpha\beta}\eta^{\mu\nu}\right]$$



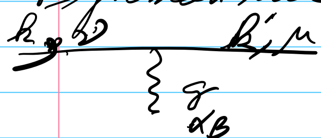
$$= -i\frac{\kappa}{2}\left[(p_\mu p'_\nu + p'_\mu p_\nu) - \eta_{\mu\nu}(p\cdot p' - m^2)\right]$$

also

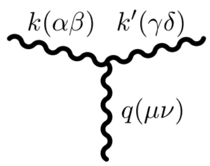


ghost:

Feynman rule

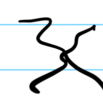


$$= -i\left[k_\alpha k_\beta \eta_{\mu\nu} + q\cdot k\,\eta_{\mu\nu}\eta_{\alpha\beta} + q_\mu k_\alpha \eta_{\nu\alpha} - q_\nu k_\alpha \eta_{\mu\beta} + q^2 \eta_{\mu\nu}\eta_{\alpha\beta}\right]$$



$$
\begin{aligned}
&= \frac{i\kappa}{2}\Bigg(P_{\alpha\beta,\gamma\delta}\Big[k^\mu k^\nu + (k-q)^\mu(k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\Big] \\
&+ 2q_\lambda q_\sigma[I^{\lambda\sigma}{}_{\alpha\beta}I^{\mu\nu}{}_{\gamma\delta} + I^{\lambda\sigma}{}_{\gamma\delta}I^{\mu\nu}{}_{\alpha\beta} - I^{\lambda\mu}{}_{\alpha\beta}I^{\sigma\nu}{}_{\gamma\delta} - I^{\sigma\nu}{}_{\alpha\beta}I^{\lambda\nu}{}_{\gamma\delta}] \\
&+ [q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\nu}{}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\mu}{}_{\alpha\beta}) \\
&- q^2(\eta_{\alpha\beta}I^{\mu\nu}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\mu\nu}{}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})] \\
&+ \Big[2q^\lambda\Big(I^{\sigma\nu}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\mu + I^{\sigma\mu}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\nu \\
&- I^{\sigma\nu}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu - I^{\sigma\mu}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu\Big) \\
&+ q^2(I^{\sigma\mu}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu}{}_{\alpha\delta}) + \eta^{\mu\nu}q^\lambda q_\sigma(I^{\rho\sigma}{}_{\gamma\delta}I_{\alpha\beta,\lambda\rho} + I^{\rho\sigma}{}_{\alpha\beta}I_{\gamma\delta,\lambda\rho})\Big] \\
&+ \Big[(k^2 + (k-q)^2)\Big(I^{\sigma\mu}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I^{\sigma\nu}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\Big) \\
&- k^2\eta_{\gamma\delta}I^{\mu\nu}{}_{\alpha\beta} - (k-q)^2\eta_{\alpha\beta}I^{\mu\nu}{}_{\gamma\delta}\Big]\Bigg).
\end{aligned}
$$

$+$



Free graviton

- 2nd quantization (like phonon)

Weak field

- $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$
- $\partial_\mu h^\mu{}_\nu - \frac{1}{2} \partial_\nu h^\lambda{}_\lambda = 0$

$\Box h_{\mu\nu} = 0$

Physical states

2 propagating polarizations
a) Lorentz group $h = \pm 2$
b) Classical grav waves

Photon polarization vectors

$\varepsilon_\mu(\lambda) = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) \qquad \lambda = \pm 1$

for $q_\mu = (q, 0, 0, q)$

$q^\mu \varepsilon_\mu = 0$

$\varepsilon^*(\lambda)\, \varepsilon(\lambda) = 1$

$\varepsilon_\mu(\lambda)\, \varepsilon^\mu(\lambda) = 0 = \varepsilon^*_\mu(-\lambda)\, \varepsilon^\mu(\lambda)$

gravity $h_{\mu\nu} = \varepsilon_{\mu\nu}\, e^{-i q\cdot x}$

$h^\lambda{}_\lambda = 0 \Leftarrow$ Lorentz scalar $\Longrightarrow \varepsilon^\lambda{}_\lambda = 0$

harmonic gauge $q^\mu \varepsilon_{\mu\nu} = 0$

Use ε_μ

$$\varepsilon_{\mu\nu}(\lambda_g = \pm 2) = \underbrace{\varepsilon_\mu(\pm 1)\,\varepsilon_\nu(\pm 1)}_{\text{product}} \leftarrow$$

$$h_{\mu\nu}(x,t) = \sum_{\lambda = ++,\, --} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_q}} \left[a(q,\lambda)\, \varepsilon_{\mu\nu}(q,\lambda)\, e^{-i q\cdot x} + h.c. \right]$$

<u>Energy in grav. wave:</u>

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$G_{\mu\nu} = G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} \quad \Leftarrow \text{powers of } h$$

define $t_{\mu\nu} = \dfrac{-1}{8\pi G} G^{(2)}_{\mu\nu}$

$$\Box h_{\mu\nu} = 8\pi G\,(T_{\mu\nu} + t_{\mu\nu})$$

$$\begin{aligned}
t_{\mu\nu} = &-\frac{1}{4} h_{\alpha\beta}\partial_\mu\partial_\nu h^{\alpha\beta} + \frac{1}{8} h \partial_\mu \partial_n u h \;\; \partial_\nu h \\
&+ \frac{1}{8}\eta_{\mu\nu}\left(h^{\alpha\beta}\Box h_{\alpha\beta} - \frac{1}{2} h\Box h\right) \\
&- \frac{1}{4}\left(h_{\mu\rho}\Box h^\rho \nu + h_{\nu\rho}\Box h^\rho \mu - h_{\mu\nu}\Box h\right) \\
&+ \frac{1}{8}\partial_\mu\partial_\nu\left(h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}hh\right) - \frac{1}{16}\eta_{\mu\nu}\Box\left(h_{\alpha\beta}h^{\alpha\beta} - \frac{1}{2}hh\right) \\
&- \frac{1}{4}\partial_\alpha\left[\partial_\nu\left(h_{\mu\beta}h^{\alpha\beta}\right) + \partial_\mu\left(h_{\nu\beta}h^{\alpha\beta}\right)\right] \\
&+ \frac{1}{2}\partial_\alpha\left[h^{\alpha\beta}(\partial_\nu h_{\mu\beta} + \partial_\mu h_{\nu\beta})\right],
\end{aligned}$$

$\leftarrow$

] vanish by EOM

] total deriv.

$h = h^\lambda{}_\lambda$

2nd Quant.

$$[a(q,\lambda), a^+(p', \lambda')] = \delta_{\lambda\lambda'}\delta^3(p-p')$$

Find

$$H = \int d^3x\, t_{00} = \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\, \omega_q \left[a^+(q,\lambda)\, a(q,\lambda) + \frac{1}{2} \right]$$

Energy state

$$|q,\lambda\rangle = a^+(q,\lambda)\,|0\rangle$$

Lore: Deser construct GR as massless spin 2
coupled to $t_{\mu\nu}$

$$\text{free field} \Rightarrow t_{\mu\nu}$$

$$\text{free} + t^{(1)}_{\mu\nu} \Rightarrow t^{(2)}_{\mu\nu}$$

$$\vdots$$

$$\Rightarrow GR$$

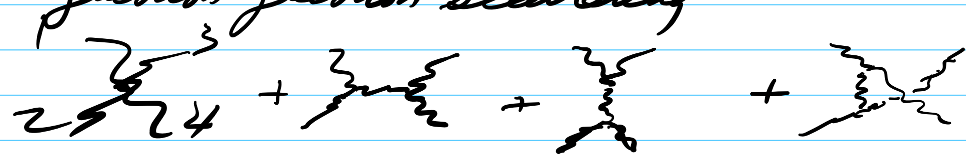
caveat $\sqrt{g} = 1$ GR also works

Gravity as Square of Gauge Theory
-amplitudes

-papers by Bern, Dixon, Elvang + Huang

On shell amplitude are remarkably simple

graviton graviton scattering



On shell $\quad M_g(1^-, 2^-, 3^+, 4^+) = \left(\frac{K}{2}\right)^2 \frac{s^3}{tu} = \left(\frac{K}{2}\right)^2 s_{12} \frac{s_{12}}{s_{23}} \frac{s_{12}}{s_{24}}$

Mandelstam $\quad s_{ij} = (p_i + p_j)^2$

Likewise for gauge theories

$$A_{gauge}(1^-, 2^-, 3^+, 4^+) = \frac{s_{12}}{s_{23}}$$

$\Rightarrow$ Feynman rules correct but clumsy

General relation!

$$M^{tree}_{grav}(1234) = s_{12}\, A^{t}_{gauge}(1234)\, A^{t}_{gauge}(1243)$$

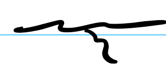
...... generalize

from KLT in string

How can it work?

Recall $\varepsilon_{\mu\nu}(++) = \varepsilon_\mu(+)\varepsilon_\nu(+)$

gauge  $f^{abc}(p-p)_\mu\, g_{\alpha\beta} + \cdots$

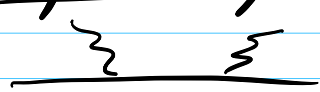
gravity  $(p)_\mu (p)_\nu$

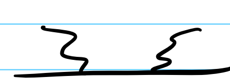
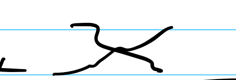
kinematic

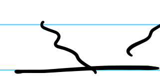
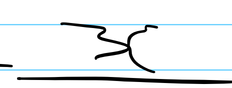
Magic!

- Bi-adjoint scalar $f^{abc} f^{cde}$
- Gauge Theories $f^{abc} p_\mu$
- gravity $p_\mu p_\nu$

Compton amplitude



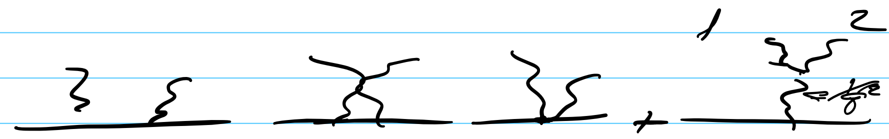
Photon scattering  +  + 

spin 1/2  + 

Low low energy theorem

- spin independent parts are universal
- linear in spin ″

Graviton

spin 0 

Universal (Gross + Jackiw)

<u>**Double copy**</u>

$$\mathrm{Amp}_{grav}(S) = \left(\frac{\kappa}{2}\right)^2 \frac{1}{\ell^4} \frac{p_1 \cdot k_1 \, p_2 \cdot k_2}{2\, k_1 \cdot k_2} \mathrm{Amp}_\gamma(S=0)\, \mathrm{Amp}_\gamma(S)$$

$\Longrightarrow$

Loop:

1) **Massless scalars**

$$p \text{———}\!\!\sim\!\! p' = i\frac{\kappa}{2}\left[(p_\mu p'_\nu + p'_\nu p_\mu) - \eta_{\mu\nu}\, p\cdot p'\right]$$

$$\Rightarrow \quad = \int\frac{d^dk}{(2\pi)^4}\,\frac{k_\mu k_\nu}{k^2+i\varepsilon} = 0$$

$$\cancel{\quad}^{\to 0} + {}^{\mu\nu}\!\!\bigcirc\!\!{}^{\alpha\beta}$$

$$\left(\frac{\kappa}{2}\right)^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^4}\,\frac{\left[k(k+q)\,k(k+q)\right]}{k^2\,(k+q)^2}$$

Dimensionally $\mu^{4-d}\, \underset{q}{E}^{d}\,\dfrac{1}{d-4}$

$$M_{\mu\nu\alpha\beta} = \kappa^2\, T_{\mu\nu\alpha\beta}\,(q^2)^2\left[\frac{1}{\varepsilon} - \ln q^2/\mu^2\right]$$

dimensionless $\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left(\eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right)_{\dots}$

$\Delta\mathcal{L} \sim R^2$

$$R = \kappa\left[\underbrace{\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu}_{q^2,\ q^2}\right]h^{\mu\nu} + \dots$$

Result is lik

$$M_{\mu\nu\alpha\beta} \Rightarrow \Delta\mathcal{L} = a\,R\left[\frac{1}{\varepsilon} - \ln\Box/\mu^2\right]R$$

$$+ b\,R_{\mu\nu}\left[\frac{1}{\varepsilon} - \ln\Box/\mu^2\right]R^{\mu\nu}$$

$$\langle \ln\Box\rangle = \ln q^2$$

$$a = \frac{1}{16\pi^2}\,\frac{1}{120}\ ,\quad b = \frac{1}{16\pi^2}\,\frac{1}{240}$$

$\Rightarrow$ get induced effect of $\mathcal{O}(R^2)$

If want to renormalize

$\mathcal{L} = c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu}$

$c_1^{ren} = c_1^{bar} + a\left[\frac{1}{\varepsilon}\right]$

c_2^{ren}

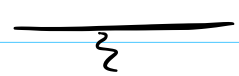
} can be renormalizing

$\uparrow$ go out & measure

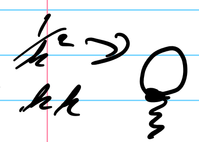
$$\mathcal{L} = \underbrace{c_1(\mu_R) R^2}_{\text{local}} + \underbrace{c\, R \ln \frac{\Box}{\mu_R^2} R}_{\text{non local}}$$

$$\underbrace{\langle x | \ln \Box | y \rangle}_{1/(x-y)^4} = \int \frac{d^4 q}{(2\pi)^4} e^{i q\cdot(x-y)} \ln q^2$$

Loop with massive field

 $= \frac{\kappa}{2}\left[\qquad \eta_{\mu\nu}(p\cdot p' - m^2)\right]$

 $\sim \frac{1}{k^2 - m^2}$

$\Rightarrow$  $\neq 0 \sim \kappa m^4$

$$\mathcal{M}_{\mu\nu\alpha\beta} = \left(\frac{\kappa}{2}\right)^2\Big[\underset{\Lambda}{I'_{\mu\nu\alpha\beta}\, m^4} + \underset{\sim R}{I_{\mu\nu\alpha\beta}\, q^2 m^2} + \underset{R^2}{I_{\mu\nu\alpha\beta}\, q^4} \Big]$$

Renorm of Λ $\quad m^4$

$$\sqrt{g}\,\Lambda = \Lambda\Big[1 + \underline{\tfrac{1}{2} h^\lambda{}_\lambda} + \tfrac{1}{8}(h^\lambda{}_\lambda)^2 - \tfrac{1}{4} h_{\sigma\lambda}h^{\sigma\lambda} + \dots$$

Linear in field 

$$\mathcal{M} = -i h^{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\,\frac{k_\mu k_\nu}{k^2 - m^2}$$

$$= -i h^{\mu\nu}\,\eta_{\mu\nu}\,\frac{m^4}{64\pi^2}\underbrace{\left[\frac{1}{\epsilon} - \ln\frac{m^2}{\mu^2} + \frac{3}{2}\right]}_{I}$$

Quadratic 
at $q=0$

$$i\,\frac{m^4}{128\pi^2}\left(h^{\mu\nu}h_{\mu\nu} - \frac{1}{2}h^2\right)\left[\frac{1}{\epsilon} \quad \ln\frac{m^2}{\mu^2} + \frac{5}{2}\right]$$

$\swarrow$ of Λ

$$\Rightarrow \delta\mathcal{L} = \delta(\sqrt{g}\Lambda) = \frac{m^4}{32\pi^2}\left[\frac{1}{\varepsilon} - \ln m^2/\mu^2 + \frac{3}{\varepsilon}\right]$$

$$\underbrace{\left[\frac{1}{2}h^\lambda{}_\lambda + \frac{1}{8}(h^\lambda{}_\lambda)^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right]}_{\text{covariant}}$$

$\Rightarrow$ renorm of Λ $\sim m^4$

<u>Cosmological constant does not run</u>:

Real running is variation with Energy of process

$\log q^2 \leftarrow$ running

<u>Mass independent</u> renorm scheme

$\left[\frac{1}{\varepsilon} - \ln q^2/\mu^2\right]$ in renorm

$\Rightarrow$ Tracking $\log \mu^2$ reveal $\ln q^2$

But here for Λ

$\left[\frac{1}{\varepsilon} - \ln m^2/\mu^2\right] \leftarrow$ does not depend on momentum

when renorm $\frac{1}{\varepsilon}$, $\ln\mu$, $\ln\mu^2$ disappear

$\Rightarrow$ no residual factor

Same of $(\delta G) \sim m^2 g^2$

G does not run either

Sociology

- many claims for running Λ, G
  - not for physical amplitudes
- abstract theory space
  - cutoff $\quad \delta\Lambda = \# \Lambda_{cutoff}^4 \quad \Lambda_{cutoff} = \text{cutoff}$

    physics independent of cutoffs

Preview: Gravity loop $\qquad$ 't Hooft Veltman

$$\Delta\mathcal{L} = \frac{1}{\varepsilon}\left[\frac{1}{120} R^2 + \frac{7}{20} R_{\mu\nu}R^{\mu\nu}\right]\frac{1}{16\pi^2}$$

R^2