GR as a Perturbative QFT #5/6

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Effective Field Theory Day

- Download "Dynamics of the Standard Model"
  - free, open access
  - used for todays lecture
- if you want, also 6 hour lecture series Jan 2023 – on Perimeter Institute website

EFT via two examples

1) QED with heavy (top) particle

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

$$Z = \int [dA][d\psi][d\bar{\psi}]\, e^{i\int d^4x\, \mathcal{L}_{QED}(A,\psi)}$$

$$= \int [dA]_{\Lambda = m_t}\, e^{i\int d^4x\, \mathcal{L}_{eff}(A)}$$

with

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2}{240\pi^2 m_t^2} F_{\mu\nu} \underset{\uparrow_{\partial^2}}{\Box} F^{\mu\nu}$$

2) <u>Linear σ model</u>

$$\mathcal{L}_\sigma = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi}) + \frac{\mu^2}{2}(\sigma^2 + \vec{\phi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\phi}^2)^2$$

SSB $\langle \sigma \rangle = v = \sqrt{\frac{\mu^2}{\lambda}}$

$$Z = \int [d\sigma][d\vec{\phi}]\, e^{i\int d^4x\, \mathcal{L}_\sigma(\sigma, \vec{\phi})}$$

$$= \int [d\vec{\pi}]_{\Lambda = m_\sigma}\, e^{i\int d^4x\, \mathcal{L}_{eff}(\vec{\pi})}$$

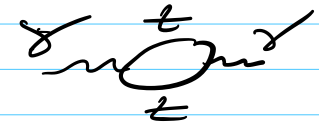
$$\mathcal{L}_{eff} = \frac{v^2}{4} \mathrm{Tr}(\partial_\mu U\, \partial^\mu U^\dagger) + \ell_1 [\mathrm{Tr}(\partial_\mu U\, \partial^\mu U^\dagger)]^2 + \ell_2 [\quad]$$

with $\ell_1 = \ldots$

$\ell_2 = \ldots$ Pauli

$$U = \exp i\frac{\vec{\tau} \cdot \vec{\pi}}{v}$$

1) QED



$$\Pi(q^2) = \frac{\alpha}{3\pi}\left[\frac{1}{\varepsilon} + \log 4\pi - \gamma - \log\frac{m^2}{\mu^2} - \frac{q^2}{5m^2}\right] \qquad (q^2 << m^2)$$
$$= \frac{\alpha}{3\pi}\left[\frac{1}{\varepsilon} + \log 4\pi - \gamma - \log\frac{-q^2}{\mu^2}\right] \qquad (q^2 >> m^2)$$

$$\frac{e_0^2}{q^2} \longrightarrow \frac{e_0^2}{q^2[1+\Pi(q^2)]}$$

QED depends on m_e ?!

Renormalization $\Longrightarrow$ measurement

$$\frac{e^2}{4\pi} = \frac{1}{137} = \frac{e_0^2}{4\pi[1+\Pi(0)]}$$ ← m_e disappear

only residual is $\frac{q^2}{m_e^2}$

Appelquist Carazzone thm – heavy physics
$\longrightarrow$ renormalized parameters
& suppressed by $1/m^2$

EFT logic –
– heavy stuff $\longrightarrow$ local (Uncertainty princ.) $\boxed{\Delta E \Delta t \sim \hbar, \ \Delta p \Delta x}$
– constant in local $\mathcal{L}$

$\Longrightarrow \mathcal{L}_{eff} = \ldots .$ (shown)

Power expansion $\int \frac{d^4q}{(2\pi)^4} (q^2)^n e^{iq\cdot x} \sim \Box^n \delta^4(x)$

but also nonlocal $\int \frac{d^4q}{(2\pi)^4} \ln q^2 \, e^{iq\cdot x} = L(x)$ ← nonlocal

EFT 1.0 – local $\mathcal{L}_{eff}$.

2) Linear σ model

a) Usual notation $\sigma = v + \tilde{\sigma}$

$$\mathcal{L} = \ldots\ldots - \lambda v \tilde{\sigma}(\tilde{\sigma}^2 + \vec{\phi}^2) - \frac{\lambda}{4}(\tilde{\sigma}^2 + \vec{\phi}^2)^2$$

b) Better notation

$$\Sigma = \sigma + i\vec{\tau}\cdot\vec{\phi} \quad \Rightarrow Tr[\Sigma\Sigma^\dagger] = 2(\sigma^2 + \vec{\phi}^2)$$

$$\mathcal{L} = \frac{1}{4} Tr(\partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger) + \frac{\mu^2}{4} Tr(\Sigma\Sigma^\dagger) - \frac{\lambda}{16}\left[Tr(\Sigma\Sigma^\dagger)\right]^2$$

shows symmetry $\Sigma \to L \Sigma R^\dagger$ SU(2) matrices

c) Best notation

$$\Sigma = (v + s)U \qquad U = e^{i\vec{\tau}\cdot\vec{\pi}/v}$$

$$\mathcal{L}_\sigma = \frac{1}{2}\left[(\partial_\mu s)^2 - m_\sigma^2 s^2\right] + \left(\frac{v+s}{4}\right)^2 Tr(\partial_\mu U\, \partial^\mu U^\dagger)$$
$$- \lambda v s^3 - \frac{\lambda}{4} s^4$$

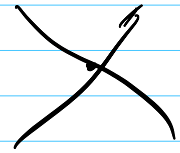
no approximation

Haag's theorem — names do not matter

$$\left.\begin{aligned} s &= \tilde{\sigma} + \ldots \\ \pi &= \phi + \ldots \end{aligned}\right\} KE \text{ same}$$

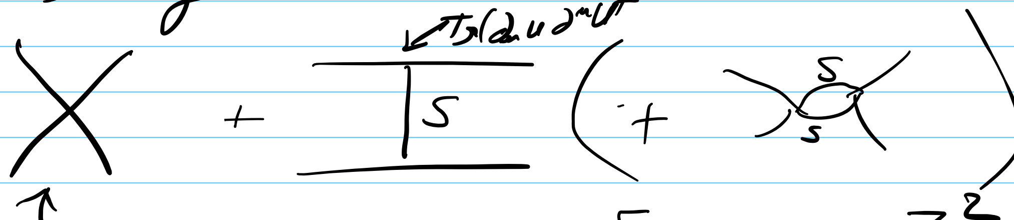
In DSM:

a)  $= \underbrace{1 - 1}_{} + E^2 + E^4$

c)  $\sim E^2$ $\qquad (\boxed{s} \sim E^4)$

Now EFT

– integrate out S



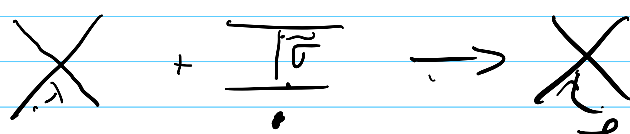
$$\mathcal{L}_{eff} = \frac{v^2}{4} Tr(\partial_\mu U \partial^\mu U^\dagger) + \frac{v^2}{8M_\sigma^2}\left[Tr\, \partial_\mu U \partial^\mu U^\dagger\right]^2$$

$$\frac{1}{q^2 - M_\sigma^2} \to \frac{-1}{M_\sigma^2}$$

local

EFT is full QFT

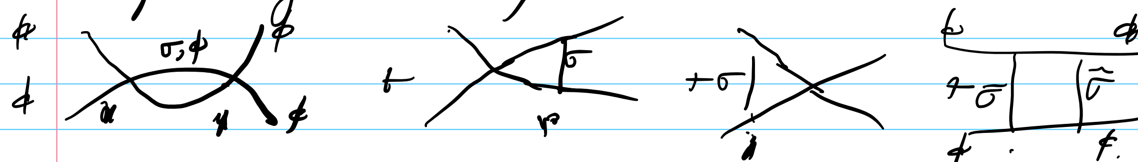
– How can loops work out



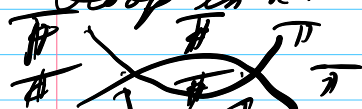
$\to$ at low E

$\mathcal{L}_{eff}$

Loops in full theory:



Loop in EFT:



$\mathcal{L}_{eff}$ $\mathcal{L}_{eff}$

At low loop energies these are the same

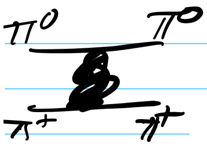
At high E they are diff. $\Rightarrow$ Look like local $\mathcal{L}$ correct difference

Example:

General local $\mathcal{L}$

ℓ_1, ℓ_2 unknown

$$\mathcal{L} = \frac{v^2}{4} \mathrm{Tr}(\partial_\mu U \partial^\mu U^+) + \ell_1 [\mathrm{Tr}(\partial_\mu U \partial^\mu U^+)]^2 + \ell_2 \mathrm{Tr}(\partial_\mu U \partial_\nu U^+) \mathrm{Tr}(\partial^\mu U \partial^\nu U^+)$$



Full theory $\rightarrow$ low E limit

$t = (p_1 - p_3)^2$
$s = (p_1 + p_2)^2$
$u = (p_1 - p_4)^2$

$$\begin{aligned} \mathcal{M}_{\text{full}} = & \frac{t}{v^2} + \left[\frac{1}{m_\sigma^2 v^2} - \frac{11}{96\pi^2 v^4}\right] t^2 \\ & - \frac{1}{144\pi^2 v^4}[s(s-u) + u(u-s)] \\ & - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{m_\sigma^2} + s(s-u)\ln\frac{-s}{m_\sigma^2} + u(u-s)\ln\frac{-u}{m_\sigma^2}\right]. \end{aligned} \tag{3.7}$$

EFT:

$$\begin{aligned} \mathcal{M}_{\text{eff}} = & \frac{t}{v^2} + \left[8\ell_1^r + 2\ell_2^r + \frac{5}{192\pi^2}\right]\frac{t^2}{v^4} \\ & + \left[2\ell_2^r + \frac{7}{576\pi^2}\right][s(s-u) + u(u-s)]/v^4 \\ & - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{\mu^2} + s(s-u)\ln\frac{-s}{\mu^2} + u(u-s)\ln\frac{-u}{\mu^2}\right], \end{aligned} \tag{3.8}$$

where we have defined[6]

$$\begin{aligned} \ell_1^r &= \ell_1 + \frac{1}{384\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right] \\ \ell_2^r &= \ell_2 + \frac{1}{192\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right]. \end{aligned} \tag{3.9}$$

Become identified with:

$$\begin{aligned} \ell_1^r &= \frac{v^2}{8m_\sigma^2} + \frac{1}{192\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{35}{6}\right] \\ \ell_2^r &= \frac{1}{384\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{11}{6}\right]. \end{aligned} \tag{3.10}$$

(close "Dynamics of Standard Model" — JFD, Golowich + Holstein)

What have we done?

Low energy DOF are Goldstone "pion"

Renormalizeable σ model
$\longrightarrow$ Unrenormalizable EFT

$\longrightarrow$ gives same prediction at low E
— 2 numbers remnant of full theory

What are predictions?
— not parameters
— structure of amplitudes
— logs — one loop prediction
— "non local"
no unknown coeff.

— other reaction
— same two constants

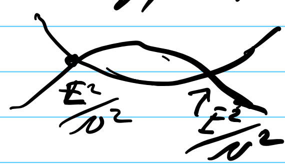
But techniques hold for other theories also
— same symmetry $\Longrightarrow$ same prediction
diff l_1, l_2

— QCD has approx $SU(2)_L \times SU(2)_R$
chiral symmetry

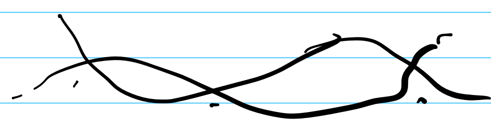
Power counting

$$\mathcal{M} = \underbrace{\frac{t}{v^2}}_{E^2} + (l_1 + \text{loop}) \underbrace{\frac{t^2}{v^4}}_{E^4} + \dots E^6 -$$

Energy expansion



$$= \frac{1}{v^4} \underbrace{A(p_i)}_{E^4}$$ only external momenta



$$\frac{1}{v^6} \quad E^6$$

Rules

$O(E^2)$ — only tree level
$O(E^4)$ — trees + 1 loop
$O(E^6)$ — trees + 2 loops
$\vdots$

vs gravity one loop $R^2 \sim E^4$

Renormalization in EFT

- <u>not</u> the important physics
- but needs to be done

Issues:

1) Same renorm for <u>all</u> processes
2) Renorm preserve symmetry?

Answer is background field method

Background field method

- renorm all processes at once

$$U = \bar{U}\, e^{i \frac{\tau \cdot \Delta}{v}}$$

$\bar{U}$ ← bkgd field; Δ ← fluctuation

$$\mathcal{L}(U) = \mathcal{L}(\bar{U}) + \mathcal{L}_1(\bar{U}, \Delta) + \mathcal{L}_2(\bar{U}, \Delta) + \dots$$

$\mathcal{L}_1$: 0 by EOM

$$\mathcal{L}_2 = \frac{1}{2} \Delta^a \left[d_\mu d^\mu + \sigma \right]_{ab} \Delta^b$$

$$d_\mu = \partial_\mu + \Gamma_\mu(\bar{U})$$

$$\sigma = \sigma(\bar{U})$$

Pull out divergence

— Feynman diagram

→ Heat kernel

$\bar{U} \to$

$$\Delta\mathcal{L} = \frac{1}{16\pi^2}\frac{1}{\varepsilon}\,\mathrm{Tr}\left[\frac{1}{12}\Gamma_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{2}\sigma^2\right]$$

$\Gamma_{\mu\nu} = [d_\mu, d_\nu]$

$$S_2^{(0)} = \int d^4x \left\{ \underline{\mathcal{L}_2(\bar{U}) - \frac{F_0^2}{2}\Delta^a \left(d_\mu d^\mu + \sigma\right)^{ab}} \Delta^b + \cdots \right\}, \tag{2.12}$$

where

$d_\mu^{ab} = \delta^{ab}\partial_\mu + \Gamma_\mu^{ab}$,

$\Gamma_\mu^{ab} = -\frac{1}{4}\,\mathrm{Tr}\left(\left[\lambda^a, \lambda^b\right]\left(\bar{U}^\dagger\partial_\mu\bar{U} +\right.\right.$ [redacted] $\left.\left. r_\mu\right)\right)$,

$\sigma^{ab} = \frac{1}{8}\,\mathrm{Tr}\left(\cancel{\{\lambda^a, \lambda^b\}}\right.$ [redacted] $\left.\chi) + \left[\lambda^a, \bar{U}^\dagger D_\mu \bar{U}\right]\left[\lambda^b, \bar{U}^\dagger D^\mu \bar{U}\right]\right).$ (2.13)

⇒ symmetry

⇒ $\bar{U} = e^{i\vec{\tau}\cdot\vec{\pi}/v}$ ← all powers

⇒ all process

Heat Kernel Method

– Read App B.1 of DSM

$$H(x,\tau) = \langle x| e^{-\tau \mathcal{D}} |x\rangle$$

$$\mathcal{D} = d_\mu d^\mu + \sigma + m^2 \qquad d_\mu = \partial_\mu + \Gamma_\mu$$

$$H(x,\tau) = \frac{1}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} \Big[\underbrace{a_0}_{1} + a_1 \tau + a_2 \tau^2\Big]$$

S.DW coeff

$$\langle x|\ln \mathcal{D}|x\rangle = \int \frac{d\tau}{\tau} H(x,\tau)$$

$$= \sum m^{d-2n}\, \Gamma(n - d/2)\, a_n(x) \quad + \text{const}$$

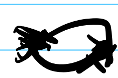
$$\Gamma(2 - d/2) \Leftrightarrow a_2(x)$$

$\Longrightarrow$ **Refer to DSM**

The physics is:

$a_0 = 1$

$a_1 = \sigma$ 

$a_2 =$  $\to$ divergence