Lecture 3

Gauge/gravity dualities, the pp wave correspondence and spin chains

AdS/CFT: review

•From N D3-branes in two P.O.V.: extremal p-brane (gravitational) solutions vs. endpoints of open strings (Polchinski); in a decoupling limit, we obtain:

•$\mathcal{N}=4$ **SYM with gauge group** $SU(N)$**, in 't Hooft limit:** $g_{YM}^2 \to 0, N \to \infty$**, but** $\lambda = g_{YM}^2 N = 4\pi g_s N$ **fixed and large, equals string theory in** $AdS_5 \times S^5$ **background at energy** U,

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + d\vec{x}_3^2) + \sqrt{4\pi g_s N}\left(\frac{dU^2}{U^2} + d\Omega_5^2\right)\right]$$

•Is a *duality*: one side is perturbative (e.g., weak gravity), the other nonperturbative (e.g., λ large in QFT).

•**3 possible versions of AdS/CFT**:

-**Weakest**: only at $g_s \to 0$ and $g_s N$ large $\to$ string theory $\simeq$ supergravity. α' and g_s corrections might disagree.

-**Stronger**: valid at any finite $g_s N$, but only at $g_s \to 0, N \to \infty$, i.e. $\alpha'/R^2 = 1/\sqrt{4\pi g_s N}$ corrections agree, but not g_s corrections.

-**Strongest**: believed to be correct: valid at any g_{YM}^2 and N (or g_s and α'). Obs.: $\lambda = 4\pi g_s N \to 0$, N small: **Quantum Gravity!**

Gauge/gravity duality

•**Generalize**: other max. susy CFT cases: $AdS_7 \times S^4$, $AdS_4 \times S^7$ $\rightarrow$ **"gravity dual"**. Obtained also from decoupling limits (but of N M2-branes and N M5-branes in M-theory, respectively).

•Conformal invariance $\leftrightarrow$ AdS space. But we can obtain less susy by taking $AdS \times X$, e.g. by dividing by a finite group S^k/Γ.

•We can also break conformal invariance $\rightarrow$ replace AdS space.

•Theories with **mass gap**: AdS space like finite quantum mechanical box: must cut out a thin cylinder from the middle of the AdS cylinder.

•We have an UV-IR correspondence:
$E \sim U = r/\alpha' \Rightarrow$ IR in CFT $= r \rightarrow 0$ (UV) in AdS. Cut out around $r = r_{min}$.

•**Motion in** $U = r/\alpha' \rightarrow$ **Renormalization group flow** in QFT.

Minimal ingredients to simulate QCD:

- A large N quantum gauge theory ($N \to \infty$ for small g_s corrections)

- Boundary at infinity identified with flat space of QCD, but better: field theory at energy scale U corresponds to flat space at position r in the gravity dual.

- Thus $d+1$ dimensional gravity dual corresponds to d-dim. field theory plus its energy scale U.

- Since motion in U is RG flow, mass gap corresponds to minimum r of gravity dual.

- Gauge group appears in gravity dual only through N.

Map field theory/gravity dual

•Global symmetries in Mink$_d$ field th. $\leftrightarrow$ gauge symmetries in $d+1$-dim. gravity dual. $\rightarrow$ global symmetries of compact space X_m. J_μ^a couple to A_μ^a.
•P_μ Noether current: $T_{\mu\nu} \leftrightarrow$ (couples to) $g_{\mu\nu}$. So d-dim. transl. inv. $\leftrightarrow$ diffeomorphism invariance in $d+1$ dimensions.

•Open/closed coupling: $g_s = g_{YM}^2/(4\pi)$.

•Gauge invariant operators $\leftrightarrow$ (sourced by) gravity dual fields in $d+1$ dimensions: •Supergravity fields in $d+1$ dim. (reduced on X_m) $\leftrightarrow$ SYM operators (made of adjoints) ("glueballs").
•For **quarks** (fundamentals of gauge group and of some global symmetry G), introduce SYM fields for the group G in the gravity dual, coupling to G-charged, pion-like operators (made of quarks), so "SYM$\leftrightarrow$ pion fields".

•Thus: supergravity modes $\leftrightarrow$ glueballs, SYM fields $\leftrightarrow$ mesons.

•Mass spectrum of tower of glueballs = mass spectrum for wave eq. of sugra mode in gravity dual. Similar for mesons.

•Baryons: more than two fields, e.g. $B^{IJK} = \epsilon_{ijk} q^{Ii} q^{Jj} q^{Kk}$. In field theory: solitonic. $\rightarrow$ e.g. topological solitons in Skyrme model. In gravity dual: solitons: branes wrapped on cycles.

•Wave functions of states in field theory, $e^{ik\cdot x}$, correspond to gravity dual wave functions $\Phi(x, U, X_m) = e^{ik\cdot x}\Psi(U, X_m)$.

General properties for gravity duals for QCD-like, or SQCD-like theories:

- At high energy: conformal (all mass scales irrelevant). Thus, for $U \to \infty$, $AdS_5 \times X_5$, or maybe with subleading corrections to metric.

- At low energy, mass gap, so gravity dual must terminate at some U_{min}, such that "warp factor" U^2 in front of $d\vec{x}^2$ remains finite.

- For fundamental quarks, open string modes on some brane must be introduced. Couple to meson-like operators. Alternative: free probe branes, probing physics at various energy scales.

- If QCD-like theory has global symm. (like flavor, or R, symm.), gravity dual, so X_m, must have this.

•All previous examples: "top-down": system of branes in a decoupling limit gives a well-defined (heuristically derived) holographic map $QFT \to$ gravity dual. We hope this will happen for actual QCD.

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•But: gravity is always holographic (a lot of evidence for this, including black holes in general relativity). Furthermore, with AdS asymptotics, we have: a) a finite time to get to the boundary, where QFT sits. b) AdS_{d+1} symmetry = conformal $Mink_d$ symmetry = $SO(d,2)$.

•Then, also: "bottom-up". Construct gravity theory in AdS (not from brane system) and see if phenomenology fits something we want.

•AdS/CMT: phenomenological approach. "Top-down": define some known duality, see if physics of QFT matches anything. OR: "bottom-up": construct AdS theory that, given holographic map, would imply wanted properties for the (unknown!) field theory, and then calculate other properties.

Example 1: Gravity dual of Lifshitz points

•CMT: usually nonrelativistic. Construct nonrelativistic gravity dual. E.G.: "Lifshitz scaling":

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$$t \to \lambda^z t, \quad \vec{x} \to \lambda \vec{x}.$$

z=dynamical critical exponent. Model example:

$$\mathcal{L} = \int d^2x\, dt \left[(\partial_t \phi)^2 - k(\vec{\nabla}^2\phi)^2\right].$$

•Then, phenomenological gravity bgr. for Lifshitz scaling,

$$ds^2_{d+1} = R^2\left(-\frac{dt^2}{u^{2z}} + \frac{d\vec{x}^2}{u^2} + \frac{du^2}{u^2}\right)$$

(obs.: geodesically incomplete for $z \neq 1$ at $u = \infty$) has scaling invariance

$$t \to \lambda^z t, \quad \vec{x} \to \lambda \vec{x}, \quad u \to \lambda u\,,$$

with generator (Killing vector)

$$D = -i(zt\partial_t + x^i\partial_i + u\partial_u).$$

•Calculate symmetry algebra, together with M_{ij}, P_i, H (rotations, space/time translations): Lifshitz algebra!

Ex.2:Gravity dual to Galilean and Schrödinger symmetries

•Larger algebras: -*conformal Galilean algebra*: M_{ij}, P_i, H, D, but also conserved rest mass, or particle number N and Galilean boosts

$$t \to t, \quad x_i \to x_i - v_i t \ .$$

•For $z = 2$, extra generator C, special conformal generator: *Schrödinger algebra* (symmetry of the Schrödinger equation of a free particle). •AdS/CFT realization (geometrical): $d+2$-dimensional gravity dual (ξ, u extra):

$$ds^2 = R^2 \left(-\frac{dt^2}{u^{2z}} + \frac{d\vec{x}^2}{u^2} + \frac{du^2}{u^2} + \frac{2dt\, d\xi}{u^2} \right) .$$

•Not time-reversal invariant ($t \leftrightarrow -t$), nonsingular: conformal to pp wave: $ds^2 = \frac{R^2}{u^2}\left(-dt^2 u^{2(1-z)} + 2dt\, d\xi + d\vec{x}^2 + du^2\right)$.

•Invariant under scaling

$$t' = \lambda^z t, \quad \vec{x}' = \lambda \vec{x}, \quad u' = \lambda u, \quad \xi' = \lambda^{2-z} \xi \ ,$$

for generator

$$D = -i(zt\partial_t + x^i \partial_i + u\partial_u + (2-z)\xi\partial_\xi) \ .$$

Ex.3: The holographic superconductor Gubser, 2008; Hartnoll, Herzog, Horowitz, 2008

•Ingredients: a) AdS_4 background: CFT near transition point. High T_c superconductors (non-Fermi liquids) are 2+1d. b) charge transport: conserved $U(1)$ J_μ, dual to A_μ. c) Temperature, so black hole in AdS_4. d) symmetry breaking, so $\langle \mathcal{O} \rangle \neq 0$, for a complex field charged under $U(1)$. s wave superconductors $\Rightarrow$ charged scalar ψ.

•Lagrangian for gravity theory ($d = 3$)

$$\mathcal{L} = \frac{1}{2\kappa^2}\left(R + \frac{d(d-1)}{R^2}\right) - \frac{1}{4g^2}F_{\mu\nu}^2 - |(\partial_\mu - iqA_\mu)\psi|^2 - m^2\psi^2 - V(\psi) \,,$$

•Want $\psi \neq 0$ near BH horizon, for $T < T_c$, and $\psi = 0$, $T > T_c$. AdS-RN (Gubser)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{2,k}^2 \,, \quad f(r) = k - \frac{2M}{r} + \frac{Q^2}{4r^2} + \frac{r^2}{R^2}$$

$$A_0(r) \equiv \Phi(r) = \frac{Q}{r} - \frac{Q}{r_H} \,, \quad \psi = 0 \,,$$

•The scalar ψ is a probe in background. Boundary conditions

$$\psi = \frac{\psi^{(1)}}{r} + \frac{\psi^{(2)}}{r^2} + \dots \,, \quad \Phi = \mu - \frac{\rho}{r} + \dots \,,$$

but both ψ_1, ψ_2 normalizable. Then, condensates

$$\langle \mathcal{O}_i \rangle = \frac{2\Delta_i - d}{R}\psi_{(2\Delta_i - d)} = \sqrt{2}\psi^{(i)} \propto (1 - T/T_c)^{1/2}, \quad i = 1, 2.$$

The Penrose limit and pp waves

•PP waves: In flat space:

$$ds^2 = 2dx^+dx^- + H(x^+, x^i)(dx^+)^2 + \sum_i dx_i^2 \ ,$$

Linearized solution is exact! Only nontrivial Ricci is $R_{++} = -\frac{1}{2}\partial_i^2 H(x^+, x^i)$. Here: waves not localized in x^+.

•In supergravity: 11d sugra, pp wave solutions with

$$F_4 = dx^+ \wedge \phi : \quad F_{(4)+\mu_1\mu_2\mu_3} = \phi_{(3)\mu_1\mu_2\mu_3}$$
$$d\phi = 0 \ , \quad d * \phi = 0 \ , \quad -\partial_i^2 H = |\phi|^2.$$

•For $H = \sum_{ij} A_{ij} x^i x^j$, $-2\text{Tr}\, A = |\phi|^2$, we have solutions with 1/2 susy, but there is a unique solution with FULL susy,

$$H = \sum_{i,j} A_{ij} x^i x^j = -\sum_{i=1,2,3} \frac{\mu^2}{9} x_i^2 - \sum_{i=4}^{9} \frac{\mu^2}{36} x_i^2$$
$$\phi = \mu dx^1 \wedge dx^2 \wedge dx^3.$$

•In 10d IIB sugra, pp wave solutions with

$$F_5 = dx^+ \wedge (\omega + *\omega) : \quad F_{+\mu_1 \ldots \mu_4} = \omega_{\mu_1 \ldots \mu_4}; \quad F_{+\mu_5 \ldots \mu_8} = \omega_{\mu_5 \ldots \mu_8}$$
$$H = \sum_{ij} A_{ij} x^i x^j; \quad \phi = \phi_0 \ , \quad d\omega = 0 \ , \quad d * \omega = 0 \ , \quad \partial_i^2 H = -|\omega|^2.$$

•Again, solutions have 1/2 susy, but ∃! solution with full susy,

$$H = -\frac{\mu^2}{64} \sum_i x_i^2; \quad \omega = \frac{\mu}{2} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.$$

•**Penrose limit:** Penrose theorem: near a null geodesic in any metric, the spacetime becomes a pp wave. Null geodesic defined by $V = Y^i = 0, U = \tau$, we can always put the metric in form (Penrose):

$$ds^2 = dV\left(dU + \alpha dV + \sum_i \beta_i dY^i\right) + \sum_{ij} C_{ij} dY^i dY^j \, ,$$

where U, V are lightcone coords., and take the limit

$$U = u; \quad V = \frac{v}{R^2}; \quad Y^i = \frac{y^i}{R}; \quad R \to \infty \, ,$$

to obtain a pp wave (in "Rosen coordinates"; after a coordinate change, previous form: "Brinkmann coordinates")

•Interpretation of Penrose limit: boost along direction x, while taking the overall scale of metric to infinity:

$$\begin{aligned} &t' = \cosh\beta \;\; t + \sinh\beta \;\; x; \quad x' = \sinh\beta \;\; t + \cosh\beta \;\; x \Rightarrow \\ &v' \equiv x' - t' = e^{-\beta}(x - t); \quad u' \equiv x' + t' = e^{\beta}(x + t) \, , \end{aligned}$$

then scale all coordinates by $1/R$ and identify $e^\beta = R \to \infty$.

•Penrose limit of $AdS_5 \times S^5$: boost along an equator of S^5 defined by $\theta = 0$ and stay at center of AdS_5 at $\rho = 0$ (is a null geodesic).

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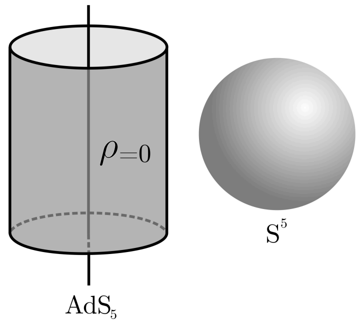

Null geodesic in $AdS_5 \times S_5$ for the Penrose limit giving the maximally supersymmetric wave. It is in the center of AdS_5, at $\rho = 0$, and on an equator of S_5, at $\theta = 0$.

$$
\begin{aligned}
ds^2 &= R^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\right) + R^2 \left(\cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\Omega_3'^2\right) \\
&\simeq R^2 \left[-\left(1+\rho^2\right) d\tau^2 + d\rho^2 + \rho^2 d\Omega_3^2\right] + R^2 \left[\left(1-\theta^2\right) d\psi^2 + d\theta^2 + \theta^2 d\Omega_3'^2\right].
\end{aligned}
$$

•Then define null coords. $\tilde{x}^{\pm} = (\tau \pm \psi)/\sqrt{2}$, and rescale to obtain the pp wave,

$$
\tilde{x}^+ = x^+; \quad \tilde{x}^- = \frac{x^-}{R^2}; \quad \rho = \frac{r}{R}; \quad \theta = \frac{y}{R} \Rightarrow
$$
$$
ds^2 = -2dx^+dx^- - \mu^2(\vec{r}^2 + \vec{y}^2)(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2.
$$

Penrose limit of AdS/CFT: large R-charge Berenstein, Maldacena, Nastase, 2002

- $E = i\partial_\tau$ in global AdS_5, and $J = -i\partial_\psi$ (for rotation $X^5 \leftrightarrow X^6$).
- But $E \leftrightarrow \Delta$ and $J \leftrightarrow U(1) \subset SU(4) = SO(6)$ R-charge rotating *fields corresponding to* $X^5 \leftrightarrow X^6$.
- Penrose limit

$$p^- = -p_+ = i\partial_{x^+} = i\partial_{\tilde{x}^+} = \frac{i}{\sqrt{2}}(\partial_\tau + \partial_\psi) = \frac{1}{\sqrt{2}}(\Delta - J)$$

$$p^+ = -p_- = i\partial_{x^-} = i\frac{\partial_{\tilde{x}^-}}{R^2} = \frac{i}{\sqrt{2}R^2}(\partial_\tau - \partial_\psi) = \frac{\Delta + J}{\sqrt{2}R^2}.$$

- Rescale p^- by $\mu\sqrt{2}$ and p^+ by $1/\mu\sqrt{2}$, so finally:

$$\frac{p^-}{\mu} = \Delta - J; \quad 2\mu p^\mu = \frac{\Delta + J}{R^2}.$$

- For string theory on pp wave, p^+, p^- finite, so as $R \to \infty$, keep $\Delta - J$ and $(\Delta + J)/R^2$ fixed, so $\Delta \simeq J \sim R^2 \to \infty$. Thus **Penrose limit is large R-charge limit in AdS/CFT!**
- In $\mathcal{N} = 4$ SYM, $\frac{R^2}{\alpha'} = \sqrt{4\pi g_s N} = \sqrt{g_{YM}^2 N}$, so for g_s fixed, we have $J \sim R^2 \sim \sqrt{N}$, so

$$\frac{J}{\sqrt{N}} = \text{fixed} \quad \text{and} \quad \frac{g_{YM}^2 N}{J^2} = \text{fixed}.$$

String quantization and Hamiltonian on pp wave

•Polyakov action on pp wave ($x^i = (\vec{r}, \vec{y})$)

$$S = -\frac{1}{2\pi\alpha'}\int_0^l d\sigma \int d\tau \frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\left[-2\partial_a x^+ \partial_b x^- - \mu^2 x_i^2 \partial_a x^+ \partial_b x^+ + \partial_a x^i \partial_b x^i\right].$$

•In conf. gauge, $\sqrt{-\gamma}\gamma^{ab} = \eta^{ab}$, light-cone gauge $x^+(\sigma,\tau) = \tau$ (rescale τ by $\alpha' p^+$), and then $l = 2\pi\alpha' p^+$,

$$S = -\frac{1}{2\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \left[\frac{1}{2}(-(\dot{x}^i)^2 + (x'^i)^2) + \frac{\mu^2}{2}x_i^2\right].$$

•The equations of motion and solutions are

$$(-\partial_\tau^2 + \partial_\sigma^2)x^i - \mu^2 x^i = 0. \Rightarrow x^i \propto e^{-i\omega_n \tau + i k_n \sigma}\ , \ \ \omega_n^2 = k_n^2 + \mu^2.$$

•In flat space $\mu = 0$, $\omega_n = k_n = n$, but now we rescaled by $\alpha' p^+$, so

$$E_n \equiv \omega_n = \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}}.$$

•Light-cone Hamilltonian $H_{\text{l.c.}} = p^-$ has no 0-modes p^i (x^i massive), so

$$H = \sum_{n\in\mathbb{Z}} N_n \omega_n\ , \quad N_n = \sum_i a_n^{i\,\dagger} a_n^i + \sum_\alpha b_n^{\alpha\dagger} b_n^\alpha .$$

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•In $\mathcal{N}=4$ SYM, $E/\mu = \Delta - J$ and $\mu p^+ \simeq J/R^2$, so

$$(\Delta - J)_n = \frac{E_n}{\mu} = \sqrt{1 + \frac{g_{YM}^2 N n^2}{J^2}}\ .$$

String states from $\mathcal{N}=4$ SYM; BMN operators

•Vacuum: $E = 0$, so $\Delta - J = 0$. Oscillators at $g_{YM} = 0$: $\Delta - J = 1$. Construct operators out of fields with $\Delta - J = 1$, on top of operator with $\Delta - J = 0$.

•Field with $\Delta = J = 1$: $Z = \Phi^5 + i\Phi^5$: unique! (charged under J). ($\bar{Z}$ has $\Delta = -J = 1$, so $\Delta - J = 2$).

•Fields with $J = 0$ and $\Delta = 1$ (so $\Delta - J = 1$): Φ^m, $m = 1, ..., 4$ and $D_\mu Z = \partial_\mu Z + [A_\mu, Z]$ (bosonic) and $\chi^a_{J=+1/2}$ (fermionic, 8 comps.; other 8: $\chi^a_{J=-1/2}$).

•Vacuum, with $\mu p^+ = J/R^2$:

$$|0,p^+\rangle = \frac{1}{\sqrt{J}N^{J/2}}\mathrm{Tr}\,[Z^J].$$

•Oscillators with $n = 0$ (BPS operators, with $\Delta - J$ indep. of g_{YM}), obtained by inserting $a^\dagger_{0,r} = \Phi^r = (D_\mu Z, \Phi^m)$ or $b^\dagger_{0,b} = \chi^a_{J=-1/2}$ in it

•Excited levels ($n \geq 1$): add also momentum wavefunction $e^{\frac{2\pi i n x}{L}}$ around the closed string, so, e.g. $a^\dagger_{n,4}$ insertion is

$$a^\dagger_{n,4}|0,p^+\rangle = \frac{1}{\sqrt{J}}\sum_{l=1}^{J}\frac{1}{\sqrt{J}N^{J/2+1/2}}\mathrm{Tr}\,[Z^l\Phi^4 Z^{J-l}]e^{\frac{2\pi i n l}{J}}.$$

•These are "BMN operators". Study "dilute gas approx.": few "impurities" among Z's.

•But we can reproduce this results from $\mathcal{N}=4$ SYM: $\exists$ small parameter $g^2_{\rm YM}N/J^2$ for this subset of (BMN) operators, even though $\lambda = g^2_{YM}N$ (nonperturbative QFT).

•Obtain strings on the pp wave: first step towards a quantum gravity description in AdS! Interactions can be introduced, but: hard.

•Goal: connect this with: perturbative SYM vs. nonperturbative strings (quantum gravity).

•Must understand HOW perturbative SYM is mapped to non-perturbative strings (the definition of nonpert. strings)

Discretized string action from $\mathcal{N}=4$ SYM

•KK states on $\mathbb{R}_t \times S^3 \leftrightarrow \mathbb{R}^4$, reduced on S^3, so constant Z: cr.op. $(b^\dagger)^i{}_j$. Similarly, for Φ, cr.op. $(a^\dagger)^i{}_j$, so states $\leftrightarrow$ operators:

$$|a_l^\dagger\rangle \equiv \text{Tr}\left[(b^\dagger)^l a^\dagger (b^\dagger)^{J-l}\right]|0\rangle.$$

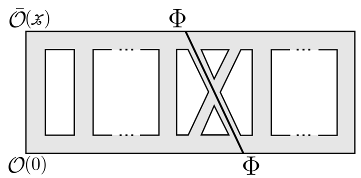
•Interacting Hamiltonian

$$H_{\text{int}} = -g_{YM}^2 \text{Tr} \sum_{I>J} \left\{[\Phi^I, \Phi^J][\Phi_I, \Phi_J]\right\} ,$$

has then term that can act on operators $\mathcal{O}$,

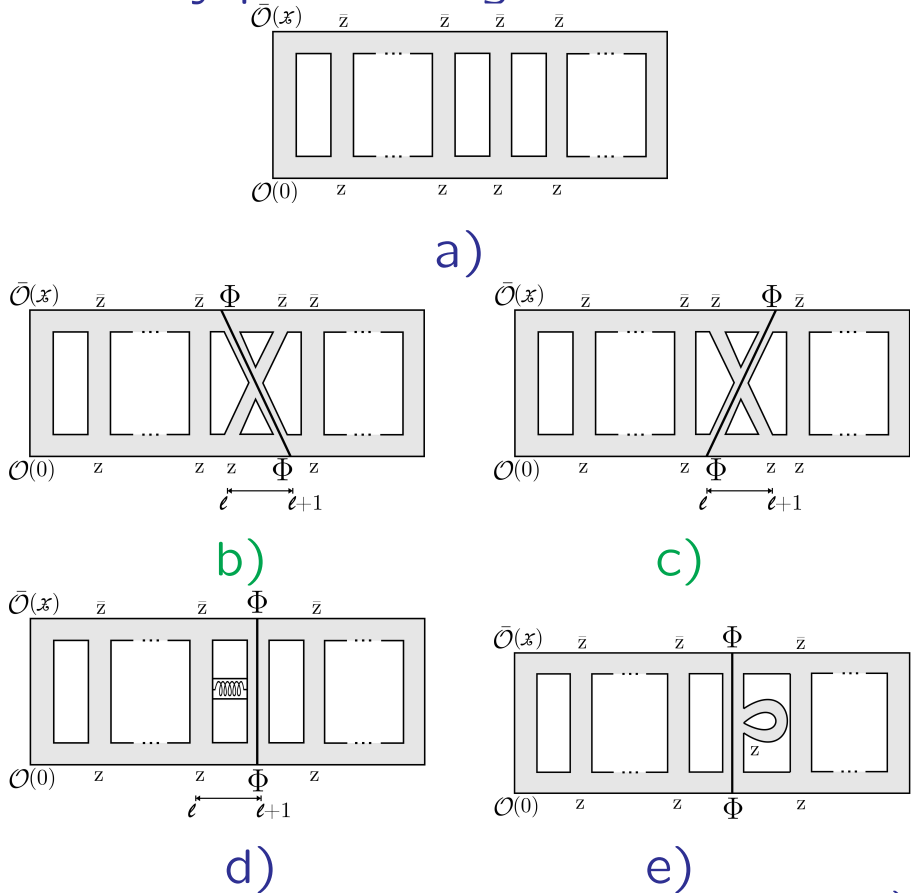
$$H_{\text{int}} = -g_{YM}^2 \text{Tr}\,\{[Z, \Phi^m][\bar{Z}, \Phi^m]\} \to 2g_{YM}^2 N[b^\dagger, \phi][b, \phi]; \quad \phi = \frac{a+a^\dagger}{\sqrt{2}} ,$$

whose action is through Feynman diagrams, as



Feynman diagram for the 2-point function of $\mathcal{O}(x)$ at one-loop.

•'t Hooft limit ⇒ only planar diagrams.



Planar Feynman diagrams for the 2-point function of $\mathcal{O}$. a) The planar tree level diagram. b) Planar one-loop Feynman diagram with hopping from $l+1$ to l. c) Planar one-loop diagram with hopping from l to $l+1$. d) One-loop planar diagram with gluon exchange e) One-loop planar diagram with scalar self-energy.

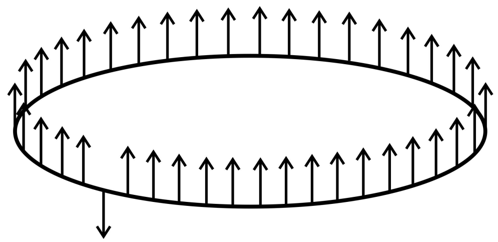
•After a calculation, find the Hamiltonian (when acting on states $\leftrightarrow$ operators in $\mathcal{N}=4$ SYM)

$$H=\sum_{j=1}^{J}\frac{a_j^\dagger a_j+a_j a_j^\dagger}{2}+\frac{\lambda}{(2\pi)^2}\sum_{j=1}^{J}(\phi_j-\phi_{j+1})^2.$$

•Continuum version of Hamiltonian = light-cone string on pp wave,

$$H=\int_0^L d\sigma\frac{1}{2}[\dot{\phi}^2+\phi'^2+\phi^2]\,,\;\; L=\frac{2\pi J}{\mu\sqrt{\lambda}}=2\pi\alpha' p^+\,,$$

so string as a discrete "spin chain" in $\mathcal{N}=4$ SYM



A periodic spin chain of the type that appears in the pp wave string theory. All spins are up, except one excitation has one spin down.

•After diagonalizing the Hamiltonian, one obtains the eigenfrequencies

$$\omega_n = \sqrt{1+4|\alpha_n|} = \sqrt{1+\frac{g_{YM}^2 N}{\pi^2}\sin^2\frac{\pi n}{J}}\ ,$$

with the corresponding Fock states (a_n is discrete Fourier tr. of a_j)

$$c^\dagger_{n,1/2}|0\rangle = \frac{a^\dagger_n \pm a^\dagger_{J-n}}{\sqrt{2}}|0\rangle = \frac{1}{\sqrt{J}}\sum_{j=1}^{J}\frac{e^{\frac{2\pi ijn}{J}} \pm e^{-\frac{2\pi ijn}{J}}}{\sqrt{2}}a^\dagger_j|0\rangle\ ,$$

•Fock states mapped to the BMN operators

$$\frac{1}{\sqrt{J}}\sum_{l=1}^{J}\frac{1}{N^{\frac{J}{2}+1}}\mathrm{Tr}\,[\Phi^1 Z^l \Phi^1 Z^{J-l}]\left[\cos\left(\frac{2\pi inl}{J}\right)\ \text{or}\ i\sin\left(\frac{2\pi inl}{J}\right)\right].$$

•Note that for $n \ll J$, both ω_n and the states match the string on pp wave. For $n \sim J$, we also have a match, but not to the pp wave (Penrose limit of $AdS_5 \times S^5$), but a different limit.
•Note that ω_n is valid *to all orders in* λ (even though the Hamiltonian was one-loop, i.e. λ^1), though only for few impurities ($M \ll J$). Why? It seems to resum all interactions.

Spin chain: $SU(2)$ sector and H_{XXX} from $\mathcal{N} = 4$ SYM

•We can construct a spin chain that is the extension of the dilute gas approx. one, in an $SU(2)$ sector with 2 scalars, corresponding to "spin up" and "spin down",

$$Z = \Phi^1 + i\Phi^2; \quad \text{and} \quad W = \Phi^3 + i\Phi^4 \; ,$$

acting on operators (and their generalizations with "magnon" momenta)

$$\mathcal{O}_\alpha^{J_1,J_2} = \text{Tr}\,[Z^{J_1} W^{J_2}] + ...(\text{permutations}).$$

•The interaction Hamiltonian in this subsector is

$$H_{\text{int}} = -g_{YM}^2 [Z, W] \text{Tr}\,[\bar{Z}, \bar{W}] \; ,$$

so the one-loop Hamiltonian is

$$H^{(1)}_{\text{planar}} = \Gamma^{(1)}_{\text{planar}} = \frac{g_{YM}^2 N}{16\pi^2} \sum_{l+1}^{L} 2\left(1 - P_{l,l+1}\right) .$$

•A more precise concept of Hamiltonian, extendable to higher loops, is of a *dilatation operator* $\mathcal{D}$, obtained by attaching Feynman diagrams to operators,

.
$$\mathcal{D} \circ \mathcal{O}_\alpha^{J_1,J_2}(x) = \sum_\beta \mathcal{D}_{\alpha\beta} \mathcal{O}_\beta^{J_1,J_2}(x) \ ,$$

•Then the dilatation operator acts on operators as spin chains as

$$\mathcal{D}^{(1)}_{\text{planar}} = \frac{g_{YM}^2 N}{8\pi^2} \sum_{l+1}^{L} \left(1\!\mathrm{l}_{l,l+1} - P_{l,l+1} \right) \ ,$$

which is the Heisenberg $XXX_{1/2}$ Hamiltonian, with $J = g_{YM}^2 N/(16\pi^2)$. Indeed, that is

$$H = -J \sum_{j=1}^{L} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} = -2J \sum_{j=1}^{L} (P_{j,j+1} - 1) \ ,$$

where we have used that on the $|\uparrow\rangle, |\downarrow\rangle$ basis on the chain, the permutation operator is $P_{ij} = \frac{1}{2} + \frac{1}{2}\vec{\sigma}_i \cdot \vec{\sigma}_j$.

Coordinate Bethe ansatz

•Denote $|x_1, ..., x_M\rangle$ state with spins down at positions $x_1, ..., x_M$ along the chain. Then the "one-magnon" eigenstate of H_{XXX} and its energy are

.

$$|\psi(p_1)\rangle = \sum_{x=1}^{L} e^{ip_1 x}|x\rangle \ , \quad E(p_1) = 8J \sin^2(p_1/2)|\psi(p_1)\rangle.$$

•The 2-magnon state is

$$\begin{aligned} |\psi(p_1, p_2)\rangle &= \sum_{1 \le x_1 < x_2 \le L} \psi(x_1, x_2)|x_1, x_2\rangle \\ \psi(x_1, x_2) &= e^{i(p_1 x_1 + p_2 x_2)} + S(p_2, p_1)e^{i(p_2 x_1 + p_1 x_2)} \ , \end{aligned}$$

where $E = E(p_1) + E(p_2)$ and the 2-body S-matrix is

$$S(p_1, p_2) = \frac{\phi(p_1) - \phi(p_2) + i}{\phi(p_1) - \phi(p_2) - i} \ , \quad \phi(p) = \frac{1}{2} \cot \frac{p}{2} \equiv u.$$

•For M magnons, in terms of $\phi(p) = u =$ rapidities (for true magnon momenta, u called Bethe roots), the energy is

$$E = \sum_{j=1}^{M} 8J \sin^2 \frac{p_j}{2} = \sum_{j=1}^{M} 2J \frac{1}{u_j^2 + 1/4}.$$