Lecture 2

AdS/CFT and its nontrivial tests

•Compute charges and tensions of Dp-branes and compare with supergravity p-brane solutions (Polchinski, 1995) $\Rightarrow$ Dp-brane = extremal p-brane solution of supergravity.

.

•Open strings have "Chan-Paton factors" at endpoints $\rightarrow$ indices $\Rightarrow$ open string. $\lambda^a_{ij}|i\rangle \otimes |j\rangle$ $\Rightarrow$ massless open string state is $A^a_\mu = \alpha^\mu_{-1}\lambda^a_{ij}|i\rangle \otimes |j\rangle$ = vector in $U(N)$ gauge group for N D-branes.

•Action for a single D-brane is

$$S_p = T_p \int d^{p+1}\xi e^{-\phi}\sqrt{-\det(h_{ij} + \alpha'(F_{ij} + B_{ij}))} + \text{fermi} + \text{WZ}$$

•Static gauge: $X^i = \xi^i, i = 0, ..., p$ and $g_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$

$$\begin{aligned} h_{ij} &= \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} = \eta_{ij} + \partial_i X^m \partial_j X_m \\ B_{ij} &= \partial_u X^\mu \partial_j X^\nu B_{\mu\nu} \end{aligned}$$

•WZ term: $\int_{M_p} e^{\wedge F/2\pi} \wedge \sum_n A_n$, e.g. a term on D5 in type IIB is

$$\frac{1}{2\pi}\int_{M_6} d^6x \epsilon^{\mu_1 \dots \mu_6} A_{\mu_1} F^+_{\mu_2 \dots \mu_6}$$

•Then, for $p = 3$ and a single brane

$$S_2 = \text{const.} + \int d^3x \Big(-\frac{F_{ij}^2}{4} - \frac{1}{2}\partial_i X^m \partial^i X_m + \text{fermi} \Big)$$

•In fact, the action: "$\mathcal{N} = 4$ supersymmetric Yang-Mills" for N D3-branes.

•Fields: $\{A_i^a, X^{a[IJ]}, \Psi_\alpha^{aI}\}$, $a \in SU(N)$, $I \in SU(4)$, $[IJ] \to$ anti-symmetric of $SU(4)$: **6** representation. ($m = 1, ..., 6$: transverse to D3).

•Action

$$S_{\mathcal{N}=4SYM} = -2 \int d^4x\, tr[-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\bar{\Psi}_I \not{D} \Psi^I - \frac{1}{2} D_\mu X_{IJ} D^\mu X^{IJ}$$
$$+ig\bar{\Psi}^I[X_{IJ}, \Psi^J] - g^2[X_{IJ}, X_{KL}][X^{IJ}, X^{KL}]]$$

•**Observation:** Bosonic Nambu-Goto version $\rightarrow$ also volume spanned by worldvolume:

$$\begin{aligned} S_p &= T_p \int d^{p+1}\xi \sqrt{-\det(h_{ab})} \\ h_{ab} &= \partial_a \xi^\mu \partial_b \xi^\nu g_{\mu\nu} \end{aligned}$$

.

•In fact, strings massless fields form **spacetime supergravity multiplet**.

•Supergravity has extremal p-branes solution $\Rightarrow$ p-branes are string theory nonperturbative objects: D-branes.

•Schwarzschild solution in 4d:

$$ds^2 = -\left(1 - \frac{2mG}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{2mG}{r}} + R^2 d\Omega_2^2$$

•Reissner-Nordstrom (with charge): modify the Newtonian potential defining solution,

$$U_N(r) = -\frac{MG_N}{r} + \frac{Q^2 G_N}{4\pi\epsilon_0^2 4r^2}\,,$$

where $ds^2 = -(1+2U_N(r))dt^2 + dr^2/(1+2U_N(r)) + r^2 d\Omega_2^2$.

•In supergravity we can add charge Q_p associated with an $A_{\mu_1 \dots \mu_{p+1}}$, with source term in the action $Q_p \int d^{p+1}\xi A_{01..p+1} = \int d^D x j^{\mu_1 \dots \mu_{p+1}} A_{\mu_1 \dots \mu_p}$ giving

$$A_{01\dots p} = -\frac{C_p Q_p}{r^{D-p-3}} .$$

•The source term can be rewritten as (on the worldvolume)

$$S_s = -\frac{1}{(p+1)!} T_P \int d^{p+1}\xi \epsilon^{i_1 \dots i_{p+1}} \partial_{i_1} X^{M_1} \dots \partial_{i_{p+1}} X^{M_{p+1}} A_{M_1 \dots M_{p+1}} ,$$

•Extremal solutions $M = |Q_p|$ of sugra with action $S_D + S_s$,

$$S_D = \frac{1}{2k^2} \int d^D x \sqrt{-g} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2(d+1)!} e^{-a(d)\phi} F_{d+1}^2 \right)$$

(here ϕ is a scalar = "dilaton"), are of type

$$\begin{aligned} ds^2_{Einstein} &= e^{-\frac{\phi}{2}} ds^2_{string}; \quad H_p = 1 + \frac{\alpha_p Q_p}{|\vec{x}_\perp|^{7-p}} \\ ds^2_{string} &= H_p^{-1/2}(-dt^2 + d\vec{x}_p^2) + H_p^{1/2} d\vec{x}^2_{9-p} \\ e^{-4\phi} &= H_p^{\frac{p-3}{4}} \\ A_{01\dots p} &= -\frac{1}{2}(H_p^{-1} - 1) \end{aligned}$$

span a $(p+1)$-dimensional "worldvolume".

•Off-shell susy means that *the algebra of susy is satisfied off-shell (without the use of the eqs. of motion)*.

•The most general N-extended superalgebra in 4d, with central charges, is

$$\{Q^i_\alpha, Q^j_\beta\} = 2(C\gamma^\mu)_{\alpha\beta} P_\mu \delta^{ij} + C_{\alpha\beta} U^{ij} + (C\gamma_5)_{\alpha\beta} V^{ij} \;,$$

and must be satisfied on all fields. In 2d, for the WZ model,

$$\{Q^i_\alpha, Q^j_\beta\} = 2(C\gamma^\mu)_{\alpha\beta} P_\mu \delta^{ij} \Rightarrow [\delta_{\epsilon_1}, \delta_{\epsilon_2}] = 2\bar{\epsilon}_2 \gamma^\mu \epsilon_1 \partial_\mu .$$

•Representing the algebra with central charges and massive states using the Wigner method, we find

$$\begin{aligned} a_\alpha &= \frac{1}{\sqrt{2}}[Q^1_\alpha + \epsilon_{\alpha\dot\beta} \bar{Q}_{2\dot\beta}] \quad a^\dagger_\alpha = \frac{1}{\sqrt{2}}[\bar{Q}_{1\dot\alpha} + \epsilon_{\alpha\beta} Q^2_\beta] \\ b_\alpha &= \frac{1}{\sqrt{2}}[Q^1_\alpha - \epsilon_{\alpha\dot\beta} \bar{Q}_{2\dot\beta}] \quad a^\dagger_\alpha = \frac{1}{\sqrt{2}}[\bar{Q}_{1\dot\alpha} - \epsilon_{\alpha\beta} Q^2_\beta] \;, \end{aligned}$$

so we obtain the algebra

$$\{a_\alpha, a^\dagger_\beta\} = 2(M - Z)\delta_{\alpha\beta}; \quad \{b_\alpha, b^\dagger_\beta\} = 2(M + Z)\delta_{\alpha\beta} \Rightarrow M \geq |Z|.$$

and the rest zero, giving the **BPS bound**. Similar for supergravity.

•$\mathcal{N}=4$ SYM is obtained as $\mathcal{N}=1$ SYM in 10d reduced to 4d,

$$S_{10d,\mathcal{N}=1SYM} = (-2)\int d^{10}x \mathrm{Tr}\left[-\frac{1}{4}F^{MN}F_{MN}-\frac{1}{2}\bar{\lambda}\Gamma^M D_M\lambda\right] \Rightarrow$$

$$S_{4d,\mathcal{N}=4\ \mathrm{SYM}} = (-2)\int d^4x\ \mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}^2-\frac{1}{2}\bar{\psi}_i \not{D}\psi^i-\frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij} -g\bar{\psi}^i[\phi_{ij},\psi^j]-\frac{g^2}{4}[\phi_{ij},\phi_{kl}][\phi^{ij},\phi^{kl}]\right]$$

•Then $\mathcal{N}=4$ SYM is obtained on the worldvolume of D3-branes, in $\alpha'\to 0$ (low energy) limit.

•$\mathcal{N}=4$ susy invariance of SYM:

$$\delta A_\mu^a = \bar{\epsilon}_I\gamma_\mu\Psi^{aI}$$

$$\delta X_a^{[IJ]} = \frac{i}{2}\bar{\epsilon}^{[I}\Psi^{J]a}$$

$$\delta\Psi^{aI} = -\frac{\gamma^{\mu\nu}}{2}F_{\mu\nu}^a\epsilon^I + 2i\gamma^\mu D_\mu X^{a,[IJ]}\epsilon_J - 2gf^a{}_{bc}(X^bX^c)^{[IJ]}\epsilon_J$$

•$\mathcal{N}=4$ Super Yang-Mills = representation of conformal group, $\{A_\mu^a, \Psi_\alpha^{aI}, X_{[IJ]}^a\}$.

•beta function = 0 $\Rightarrow$ scale and conformal invariant. But $\Delta = \Delta_0 + \mathcal{O}(g)$ in general. No *infinities*, but $\exists$ finite renormalizations.

AdS/CFT in original formulation (Maldacena, 1997)

•String theory in $AdS_5 \times S^5 = \mathcal{N} = 4$ SYM with $SU(N)$ gauge group (low energy theory on N D3-branes), living at the boundary of $AdS_5 \times S^5$, involving a certain limit.

•Heuristical derivation:

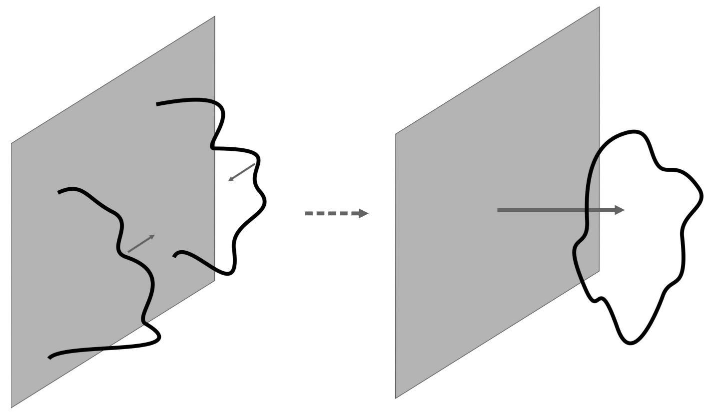
•D-branes = extremal p-branes ⇒ curve space. Solution:

$$ds^2 = H^{-1/2}(r)d\vec{x}_{||}^2 + H^{1/2}(r)(dr^2 + r^2 d\Omega_5^2)$$

$$F_5 = (1 + *)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge (dH^{-1})$$

$$H(r) = 1 + \frac{R^4}{r^4}; \quad R = 4\pi g_s N \alpha'^2; \quad Q = g_s N$$

•Add a δM → near extremal: $M = Q + \delta M$ ⇒ horizon ⇒ emits Hawking radiation: 2 open strings on D3 collide and form a closed string that peels off and goes into the bulk.



Two open strings living on a D-brane collide and form a closed string, that can then peel off and go away from the brane.

•**P.O.V. nr. 1** D3-branes = endpoints of strings. String theory gives:

-open strings on D3. Low energy ($\alpha' \to 0$) $\Rightarrow \mathcal{N} = 4$ SYM

-closed strings in bulk (all spacetime): supergravity + massive modes of string. Low energy: supergravity only.

-interactions, giving e.g. Hawking radiation as above.

$$S = S_{bulk} + S_{brane} + S_{interactions}$$

•Low energy limit, $\alpha' \to 0$, $\Rightarrow$ $S_{bulk} \to S_{supergravity}$, $S_{brane} \to S_{\mathcal{N}=4SYM}$, $S_{int} \propto \kappa_{Newton} \sim g_s \alpha'^2 \to 0$. Moreover, since Newton $\kappa_N \to 0$, $\Rightarrow$ free gravity. Thus:

•free gravity in bulk

•4d $\mathcal{N} = 4$ SYM on D3's.

•Obs: $\partial(AdS_5 \times S^5) = R^{3,1}$ or $S^3 \times R$ (4 dimensional!): S^5 shrinks to zero size at boundary.

•**P.O.V. nr. 2** D3-branes replaced by p-branes (supergravity solutions).

•Geometry has two asymptotic regions: $r \to 0$: $AdS_5 \times S^5$ and $r \to \infty$: Minkowski$_{10}$. Infinitely long throat:

•Energy at point r is

$$E_r \sim \frac{d}{d\tau} = \frac{1}{\sqrt{-g_{00}}}\frac{d}{dt} \sim \frac{1}{\sqrt{-g_{00}}}E_\infty \Rightarrow E_\infty = H^{-1/4}E_r \sim rE_r$$

•Then at $r \to 0$, for fixed E_r (energy of the throat) $E_\infty \to 0 \Rightarrow$ low energy excitations.

•At $r \to \infty$, long distance $\delta r \to \infty \Leftrightarrow E \to 0$, effective gravity coupling $GE^{D-2} \to 0 \Rightarrow$ free gravity $\to$ in the bulk.

•Compare POV 1 with POV 2. Same free gravity in the bulk $\Rightarrow$ Identify the others $\Rightarrow$

•4d $\mathcal{N} = 4$ SYM with $SU(N)$ on D3 = gravity at $r \to 0$ in D-brane background, for $\alpha' \to 0$.

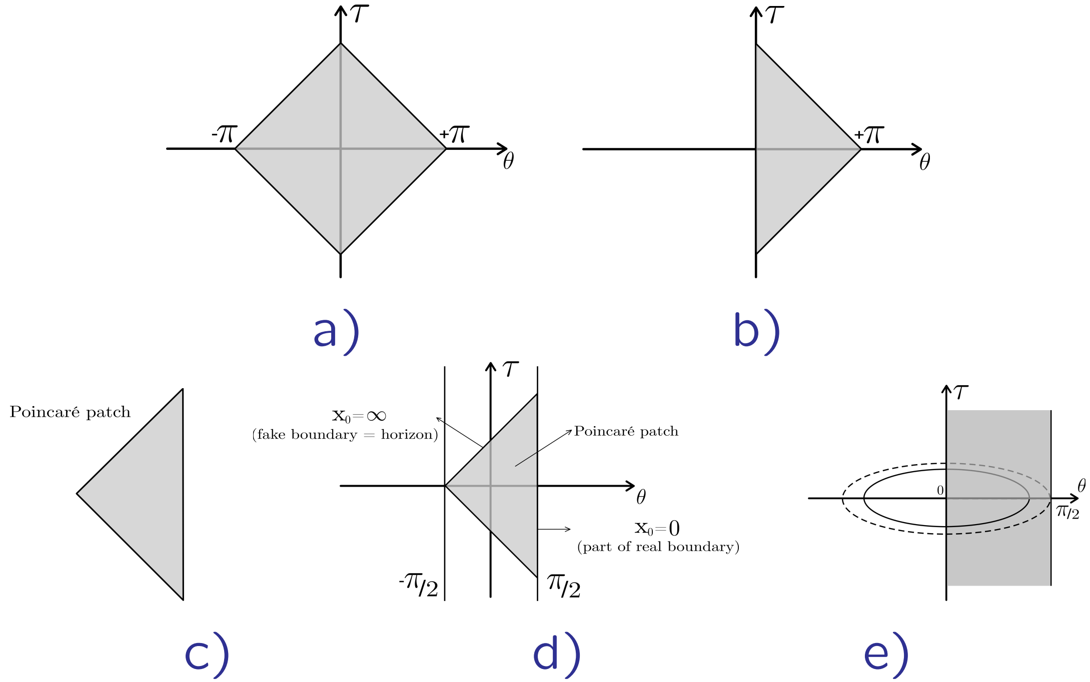
•Background for $r \to 0$, with $r/R \equiv R/x_0$.

$$ds^2 = R^2 \frac{-dt^2 + d\vec{x}_3^2 + dx_0^2}{x_0^2} + R^2 d\Omega_5^2 : AdS_5 \times S^5$$

•Then, metric is $(E_r\sqrt{\alpha'}$ fixed and E_∞ fixed; $E_\infty/(E_r\sqrt{\alpha'}) = r/\alpha' \equiv U$ fixed)

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + d\vec{x}_3^2) + \sqrt{4\pi g_s N}\left(\frac{dU^2}{U^2} + d\Omega_5^2 \right) \right]$$

•$R_{AdS}^2 = \sqrt{4\pi g_s N}$ =fixed and large (small α' corrections)



Penrose diagrams. a) Penrose diagram of 2 dimensional Minkowski space. b) Penrose diagram of 3 dimensional Minkowski space. c) Penrose diagram of the Poincaré patch of Anti-de Sitter space. d) Penrose diagram of global AdS_2 (2 dimensional Anti-de Sitter), with the Poincaré patch emphasized; $x_0 = 0$ is part of the boundary, but $x_0 = \infty$ is a fake boundary (horizon). e) Penrose diagram of global AdS_d for $d \geq 2$. It is half the Penrose diagram of AdS_2 rotated around the $\theta = 0$ axis.

Anti-de Sitter space

•d-dimensional Anti de Sitter space:

$$ds^2 = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 - dx_{d+1}^2; \qquad -x_0^2 + \sum_{i=1}^{d-1} x_i^2 - x_{d+1}^2 = -R^2$$

is explicitly invariant under $SO(d-1,2)$ by construction and $\mathcal{R} < 0$.

•**Metrics**: Poincare coordinates $(t, x_i \in R, x_0 \in R_+)$

$$ds^2 = \frac{R^2}{x_0^2}\left(-dt^2 + \sum_{i=1}^{d-2} dx_i^2 + dx_0^2\right)$$

•Up to conformal factor, same as flat space $\Rightarrow$ Penrose diagram is the same. For $d > 2$ however, we use radial coordinate $\rho > 0$ instead of spatial coordinate $x \in R \Rightarrow$ obtain half of diamond = triangle.

•We can make explicit also the exponential "warp factor"

$$ds^2 = e^{2y}\left(-dt^2 + \sum_{i=1}^{d-2} dx_i^2\right) + dy^2 \quad (x_0 = e^{-y})$$

•Even though r, x_i, x_0 are ∞ in extent, space is not complete: Infinity at $y = \infty$ is reached in finite time by a null ray:

.
$$ds^2 = 0 \;\; \Rightarrow \;\; dt^2 = e^{-2y} dy^2 \;\; \Rightarrow \;\; t = \int^{\infty} e^{-y} dy < \infty$$

•$\Rightarrow \exists$ other coordinates covering whole space: global coordinates:

$$\begin{aligned} \text{AdS}: \quad & ds_d^2 = R^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\vec{\Omega}_{d-2}^2) \\ \text{sphere}: \quad & ds_d^2 = R^2(\cos^2\rho\, dw^2 + d\rho^2 + \sin^2\rho\, d\vec{\Omega}_{d-2}^2) \end{aligned}$$

•Finally, coordinate transf. $\tan\theta = \sinh\rho \Rightarrow$

$$ds_d^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\vec{\Omega}_{d-2}^2)$$

•Here $0 \le \theta \le \pi/2$, $\tau \in R \Rightarrow$ infinite cylinder. Poincare patch: figure of revolution obtained by rotating triangle around a side, situated along the axis of the cylinder
•Boundary of cylinder still reached by light ray in finite time (and reflected back).
•AdS is somewhat like a finite box, with a boundary.

•AdS/CFT in $\alpha' \to 0, g_s \to 0$ limit: large N limit of 't Hooft, with effective coupling $\lambda = g_{YM}^2 N$, and loop counting $1/N$, so

$$\mathcal{A} \sim (g^2 N)^L N^{1-2h} ,$$

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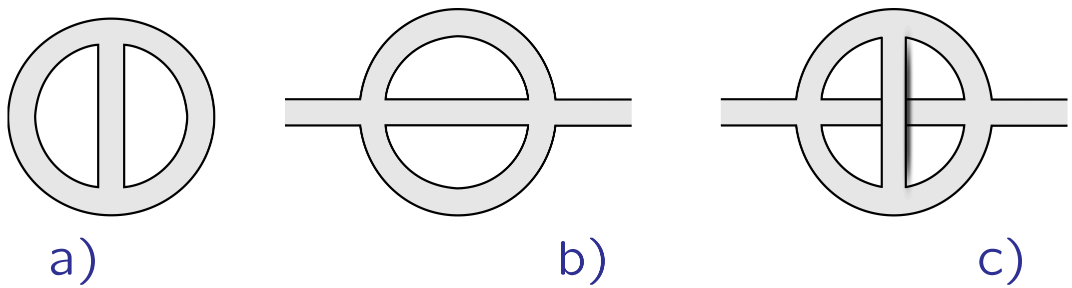


a) Planar 2-loop diagram with 2 3-point vertices b) Planar 2-loop diagram with 2 4-point vertices c) Nonplanar 3-loop diagram.

•just that now $g_{YM}^2 N = \lambda$ is fixed and large! $\Rightarrow$ nonperturbative QFT.

•Witten map: Gauge invariant operator $\mathcal{O}$ of $\mathcal{N} = 4$ SYM, with conformal dimension Δ and representation I_n of $SO(6) = SU(4)$ $\leftrightarrow$ field in AdS_5, of mass m and representation I_n of $SO(6) =$ symmetry of S^5. •Then $\phi_{(n)}^{I_n} \leftrightarrow \mathcal{O}_{(n)}^{I_n}$, with

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 R^2}$$

Witten construction

•Near boundary $x_0 = 0$ in Poincaré coords., $\Box\phi = 0 \Rightarrow \phi \to x_0^{d-\Delta}\phi_0$ and $\phi \to x_0^{\Delta}\phi_0$ (Δ= dim. of dual op.).
•Then ϕ_0= source for dual operator $\mathcal{O}$.
•Observables for $\mathcal{O} \leftrightarrow \phi$: generating functional for $\mathcal{O}$:

$$Z_{boundary} = Z_{\mathcal{O},CFT}[\phi_0] = \int \mathcal{D}[\text{SYM fields}]e^{-S_{\mathcal{N}=4\ SYM}+\int d^4x\mathcal{O}(x)\phi_0(x)}$$

•Fundamental idea: $Z_{boundary} = Z_{bulk} = Z_{string}[\phi_0]$, where ϕ_0= boundary sources. But for $\alpha' \to 0, g_s \to 0, R^4/\alpha'^2 \gg 1 \to$ string $\simeq$ classical supergravity, and $Z_{string}[\phi_0] = e^{-S_{sugra}[\phi[\phi_0]]}$.

$$\Rightarrow Z_{\mathcal{O},CFT}[\phi_0] = e^{-S_{sugra}[\phi[\phi_0]]}$$

•But in CFT, correlators are obtained by derivation:

$$\begin{aligned} < \mathcal{O}(x_1)...\mathcal{O}(x_n) > &= \frac{\delta^n}{\delta\phi_0(x_1)...\delta\phi_0(x_n)} Z_{\mathcal{O}}[\phi_0]|_{\phi_0=0} \\ &= \frac{\delta^n}{\delta\phi_0(x_1)...\delta\phi_0(x_n)} e^{-S_{sugra}[\phi[\phi_0]]}|_{\phi_0=0} \end{aligned}$$

•But: perturbative ($g^2_{YM}N \to 0$) correlators match nonperturbative ones (classical supergravity) only if there is some susy argument, otherwise different ($\exists f_i(\lambda)$).
•Exception: anomalies. Gauge anomaly $\leftrightarrow$ CS coupling:

.

$$\langle J^{ia}(x_1)J^{jb}(x_2)J^{kc}(x_3)\rangle_{\text{CFT, d}_{\text{abc}}\text{ part}} = -\left.\frac{\delta^3 S^{\text{3-pnt vertex}}_{\text{CS,sugra}}[A^a_\mu[a^d_l]]}{\delta a^a_i(x_1)\delta a^b_j(x_2)\delta a^c_k(x_3)}\right|_{a=0},$$

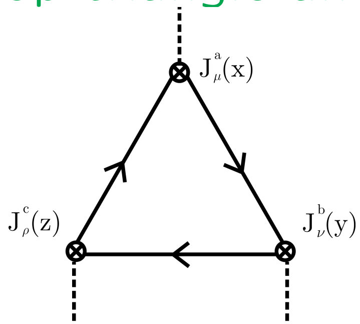
using

$$S_{\text{CS}}(A) = \frac{N^2}{18\pi^2}\text{Tr}\int_{B_5=\partial M_6}\epsilon^{\mu\nu\rho\sigma\tau}(A_\mu(\partial_\nu A_\rho)\partial_\sigma A_\tau + A^4 \text{ terms} + A^5 \text{ terms}),$$

(large λ) and find equality with the CFT result (small λ)

$$\frac{\partial}{\partial z^k}\langle J^a_i(x)J^b_j(y)J^c_k(z)\rangle_{\text{CFT,d}_{\text{abc}}} = -\frac{(N^2-1)id_{abc}}{48\pi^2}\epsilon^{ijkl}\frac{\partial}{\partial x_k}\frac{\partial}{\partial y_l}\delta(x-y)\delta(y-z),$$

coming from the one-loop triangle anomaly (which is one-loop exact!),

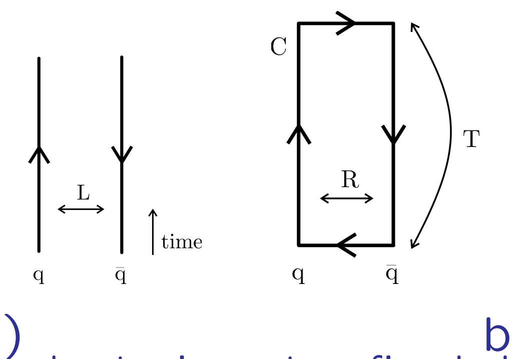


Triangle diagram contributing to the $\langle J^a_i(x)J^b_j(y)J^c_k(z)\rangle$ correlator. Chiral fermions run in the loop.

•**Wilson loops**: in QCD: very heavy quarks $q+\bar{q}$, so fixed $\rightarrow$ define contour. Observable $q\bar{q}$ potential, $V_{q\bar{q}}(L)$. Define

$$W(C) = \mathrm{Tr}\left[P\exp\left\{i\oint_C A_\mu(\xi)d\xi^\mu\right\}\right]$$

$\rightarrow$ is gauge invariant. If we take a very long rectangle in the time direction T (and short in the spatial one L),



a)Heavy quark and antiquark staying at a fixed distance L. b)Wilson loop contour C for the calculation of the quark-antiquark potential.

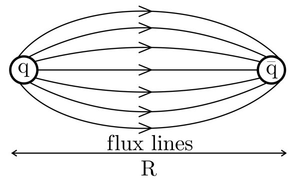
•Then from the VEV of the Wilson loop, as $T\rightarrow\infty$, extract $q\bar{q}$ potential,

$$\langle W(C)\rangle_0 \propto e^{-V_{q\bar{q}}(R)T}.$$

•Confining theory: constant force $\rightarrow$ linear potential (at strong coupling λ!),

$$V_{q\bar{q}}(R) \sim \sigma R\,,$$

. σ= QCD string tension. QCD string = flux tube of constant cross section.



Between a quark and an antiquark in QCD, flux lines are confined: they live in a flux tube.

•In string theory, this is obtained from the partition function for a classical string with boundary condition = Wilson contour,

$$\langle W[C]\rangle = Z_{\text{string}}[C] = e^{-S_{\text{string}}[C]}\,,$$

•Subtlety: susy generalized Wilson loop

$$W[C] = \frac{1}{N}\text{Tr}\ P\exp\left[\oint\left(iA_\mu\dot{x}^\mu + \theta^I X^I(x^\mu)\sqrt{\dot{x}^2}\right)d\tau\right]\,.$$

$x^\mu(\tau)$: loop, θ^I: on unit S^5. We consider only θ^I=const.: rectangular Wilson loop is 1/2 susy. (invariant under susy transf.).

•But $\langle W[C]\rangle$ is a good observable at any coupling λ. In fact, for a circular C, it can be calculated exactly (at small λ, we have usual Feynman diagrams, but in general, it can be calculated using a) matrix models; b) supersymmetric localization techniques: all in field theory). Result

$$\langle W[C]\rangle \overset{N\to\infty}{=} \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = \begin{cases} 1+\frac{\lambda}{8}+\frac{\lambda^2}{192}+\dots\,, & \lambda \ll 1 \\ \sqrt{\frac{2}{\pi}}\frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}}\left(1-\frac{3}{8\sqrt{\lambda}}+\dots\right)\,, & \lambda \gg 1 \end{cases}$$

$$\langle W[C]\rangle = \frac{1}{N} L^1_{N-1}\left(-g^2/4\right) e^{g^2/8}$$

•On the string side, $Z_{\text{string}}[C]$, calculated by $\langle W[C]\rangle$, is a good quantum gravity observable (partition function with a boundary condition on the *fixed* boundary)

•$\frac{1}{\lambda} = 1/\sqrt{g_s N} = \alpha'/R^2$ corrections correspond to α' corrections: quantum fluctuations of string; $1/N$ corrections correspond to $(g_s N)/N = g_s$ corrections: string worldsheet making loops (changing topology). In the end, this quantum gravity observable is defined and calculated from $\mathcal{N}=4$ SYM at any coupling!

•**Finite temperature**: Witten metric for AdS at finite T (limit of AdS black hole)

$$
\begin{aligned}
ds^2 &= \frac{R^2}{z^2}\left[-f(z)dt^2 + d\vec{y}^2 + \frac{dz^2}{f(z)}\right] + R^2 d\Omega_5^2 \\
f(z) &= 1 - \frac{z^4}{z_0^4}\,,
\end{aligned}
$$

with temperature $T = 1/(\pi z_0)$.

•Bekenstein-Hawking entropy of this black hole,

$$
S = \frac{A}{4G_N}\,,
$$

counts the number of effective d.o.f. in this semiclassical gravity theory (holographic: d.o.f.'s on the horizon of the black hole, not in volume), and should equal dual QFT's ($\mathcal{N} = 4$ SYM at finite T) entropy. But area of horizon, $A = \frac{R^3}{z_0^3}\int dy_1 dy_2 dy_3$ is ∞, Therefore the entropy density is

$$
s = \frac{S}{\int dy_1 dy_2 dy_3} = \frac{R^3}{4G_{N,5} z_0^3}.
$$

•But $2\kappa_N^2 = 16\pi G_{N,10} = (2\pi)^7 g_s^2 \alpha'^4$ and in $AdS_5 \times S^5$, $R^4 = \alpha'^2 g_{YM}^2 N = \alpha'^2 (4\pi g_s) N$.
•Then reducing on an S^5 of radius R, with $\Omega_5 = \pi^3$ gives

$$G_{N,10} = \frac{\pi^4}{2N^2} R^8 \Rightarrow G_{N,5} = \frac{G_{N,10}}{\Omega_5 R^5} = \frac{\pi}{2N^2} R^3 \Rightarrow s_{\lambda=\infty} = \frac{\pi^2}{2} N^2 T^3.$$

•This is entropy density at ∞ coupling. From $\sigma = \partial P / \partial T$ and $\epsilon = -P + Ts$, we find

$$P_{\lambda=\infty} = \frac{\pi^2}{8} N^2 T^4 \, , \quad \epsilon_{\lambda=\infty} = \frac{3\pi^2}{8} N^2 T^4 \, ,$$

•But at weak coupling ($\lambda = 0$), one free bosonic d.o.f has $s = 2\pi^2 T/45$, and one free fermionic d.o.f. has 7/8 of that. The for $\mathcal{N} = 4$ SYM (8 bosonic d.o.f and 8 fermionic d.o.f., all in adjoint of $SU(N)$), we have

$$s_{\lambda=0} = \left(8 + 8\frac{7}{8}\right)(N^2 - 1)\frac{2\pi^2 T^3}{45} \simeq \frac{2\pi^2}{3} N^2 T^3 \, ,$$

so we obtain the ratios (for pressure, we use the same thermod. relations)

$$\frac{s_{\lambda=\infty}}{s_{\lambda=0}} = \frac{3}{4} \, , \quad \frac{P_{\lambda=\infty}}{P_{\lambda=0}} = \frac{\epsilon_{\lambda=\infty}}{\epsilon_{\lambda=0}} = \frac{3}{4}.$$

•In lattice QCD, one finds about 80% reduction from $\lambda = 0$ (UV) to $\lambda = \infty$ (IR), instead of the above 75% for $\mathcal{N} = 4$ SYM, consistent with reduction of effective d.o.f.'s along the RG flow.

•But reversing the logic, we can say that

$$S_{\text{semi-classical}} = \frac{A_H}{4} \to S_{\text{strongly-quantum}} = \frac{4}{3} S_{\text{semi-classical}} = \text{,,} \frac{A_H}{3} \text{,,}$$

so that in quantum gravity, we have an increase of the entropy density (effective nr. of d.o.f.'s) from semiclassical to strongly quantum.

•There are many, many other observables that have been computed in AdS/CFT, though not many have an easy to understand definition in the strong quantum gravity regime.