Double Parton Distributions from Euclidean Lattice

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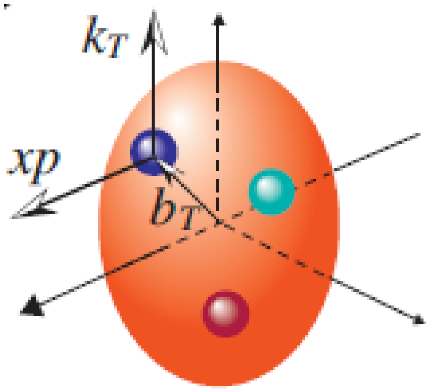
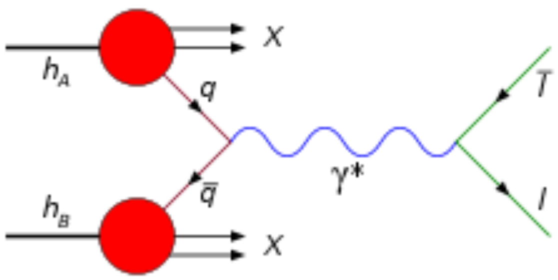
Introduction

- Parton physics plays an important role in mapping out the 3D structure of hadrons and interpreting the experimental data at hadron colliders





Example: Drell-Yan Process

- **Factorization**

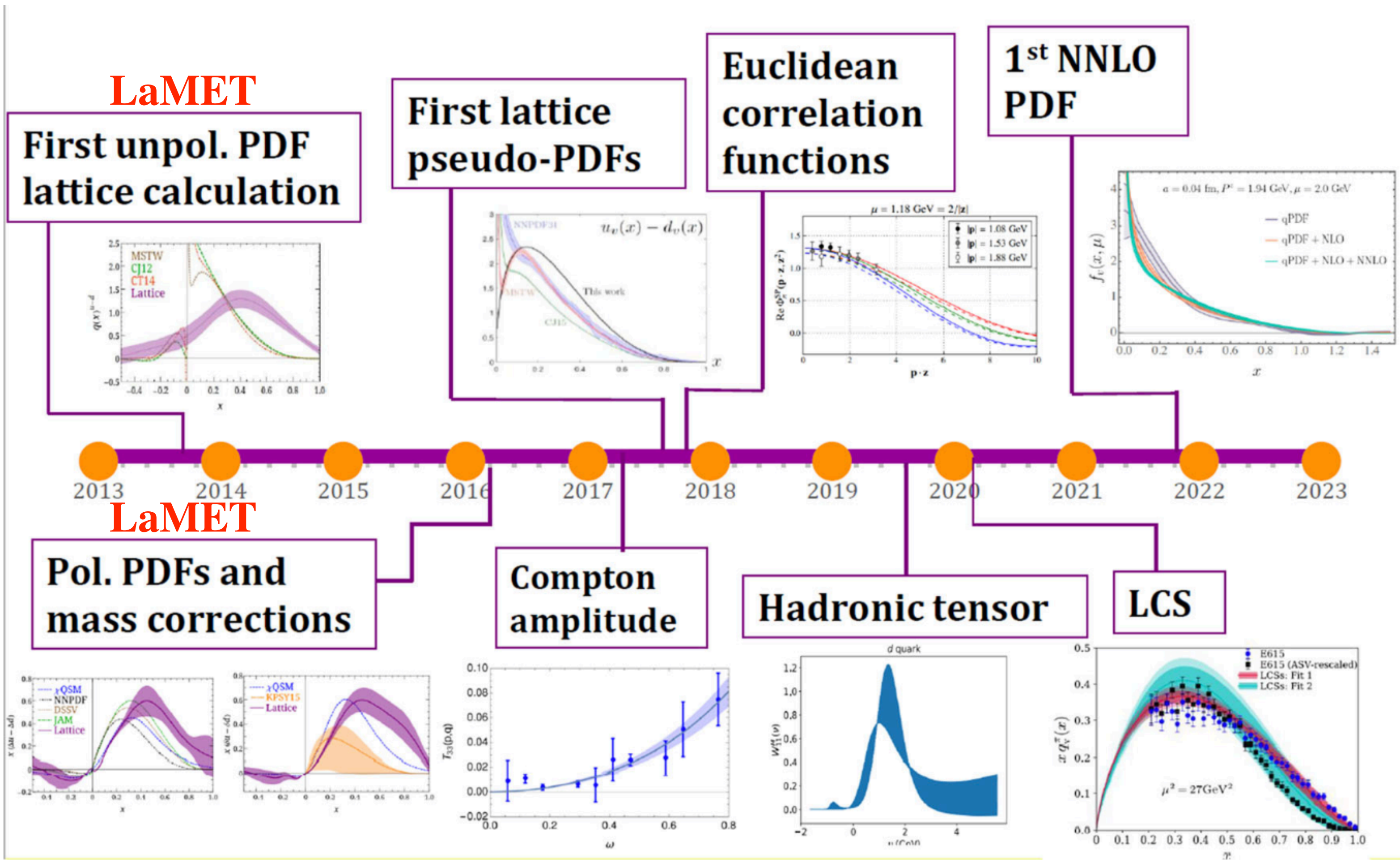
$$\frac{d\sigma}{dQ^2} = \sum_{i,j} \int_0^1 d\xi_a d\xi_b\, f_{i/P_a}(\xi_a) f_{j/P_b}(\xi_b) \frac{d\hat{\sigma}_{ij}(\xi_a, \xi_b)}{dQ^2} \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}\right)\right] \qquad Q = \sqrt{q^2}$$

$q_T \ll Q$:

$$\frac{d\sigma}{dQ^2 d^2\mathbf{q_T}} = \sum_{i,j} H_{ij}(Q) \int_0^1 d\xi_a d\xi_b \int d^2\mathbf{b_T}\, e^{i\mathbf{b_T}\cdot\mathbf{q_T}} \times f_{i/P}(\xi_a, \mathbf{b_T}) f_{j/P}(\xi_b, \mathbf{b_T}) \times \left[1 + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}, \frac{q_T}{Q}\right)\right]$$

Introduction

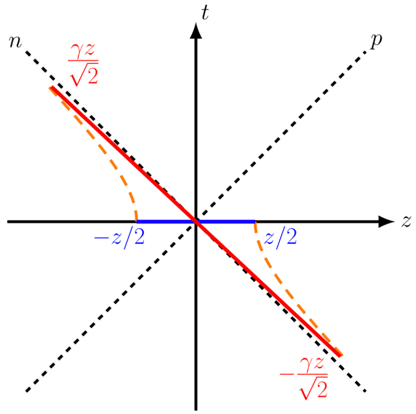
- Tremendous progress has been achieved on calculating the **x-dependent partonic structure** of hadrons from Euclidean lattice



H-W Lin, FBS 23’

Introduction

- **A popular approach: Large-momentum effective theory (LaMET)**
  Ji, PRL 13' & SCPMA 14', Ji, Liu, Liu, JHZ, Zhao, RMP 21'



$$q(x,\mu) = \int \frac{d\lambda}{4\pi} e^{ix\lambda} \langle P|\bar{\psi}(0) n\cdot\gamma L(0,\lambda n)\psi(\lambda n)|P\rangle, \quad n^2 = 0$$

$$\tilde{q}(y,P^z) = N \int \frac{dz}{4\pi} e^{-iyzP^z} \langle P|\bar{\psi}(0)\gamma^0 L(0,z)\psi(z)|P\rangle$$

$$\tilde{q}(y,P^z) = C\left(\frac{y}{x}, \frac{\mu}{xP^z}\right) \otimes q(x,\mu) + \mathcal{O}\left(\frac{\Lambda^2_{QCD}}{(yP^z)^2}, \frac{\Lambda^2_{QCD}}{((1-y)P^z)^2}\right)$$

- Theory studies and lattice calculations available for
  - Collinear PDFs, distribution amplitudes
  - GPDs, TMDPDFs/wave functions
  - Higher-twist distributions

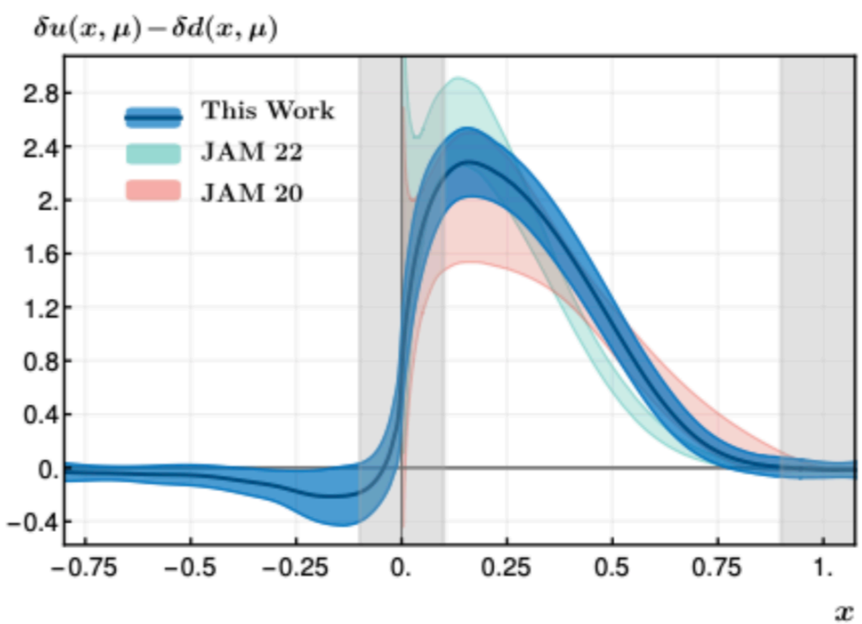
A huge number of references…

Lattice results

- **A popular approach: Large-momentum effective theory (LaMET)**
  Ji, PRL 13' & SCPMA 14', Ji, Liu, Liu, JHZ, Zhao, RMP 21'

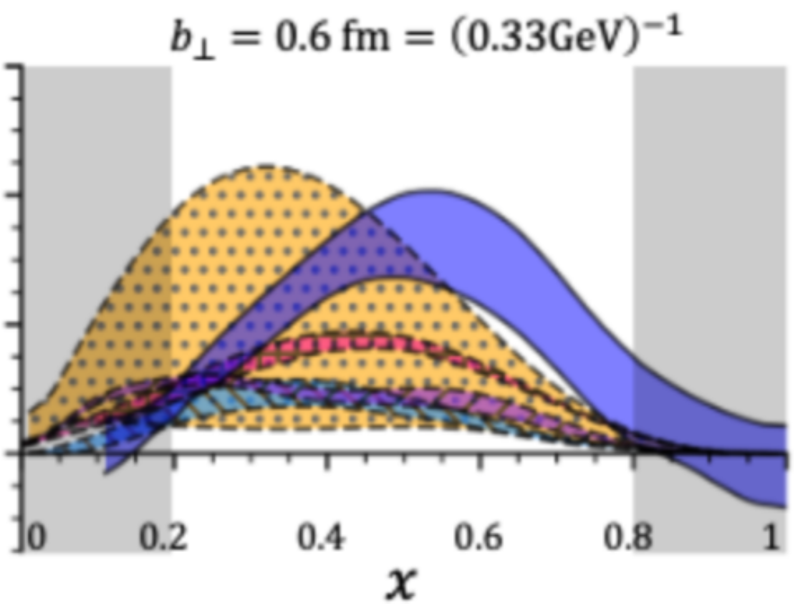
- Examples of the state-of-the-art:

Nucleon quark transversity



Yao, JHZ et al (LPC) 22'

Unpolarized quark TMD in the nucleon



He, JHZ et al (LPC) 22'

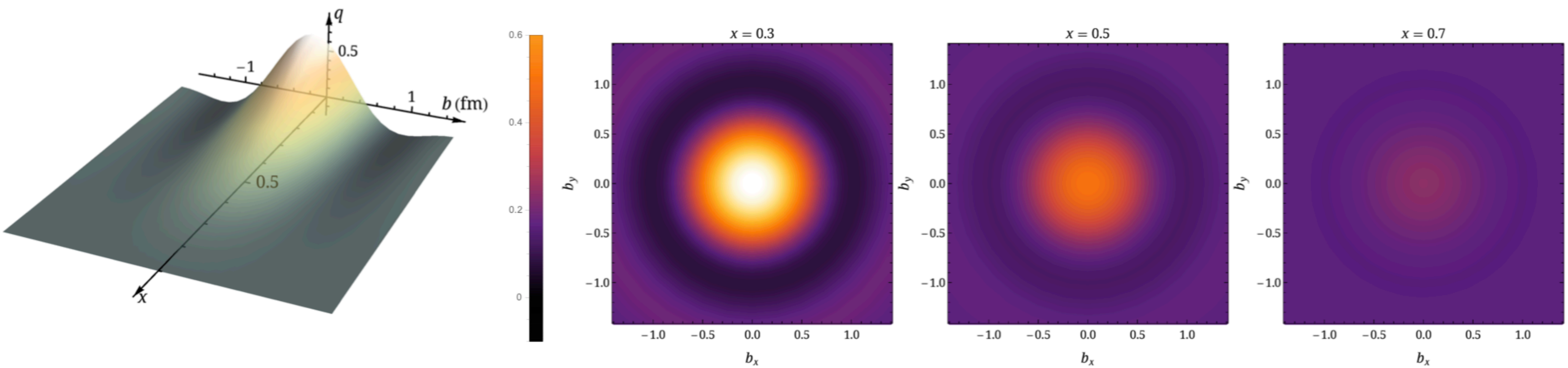
Lattice results

- **A popular approach: Large-momentum effective theory (LaMET)**
  Ji, PRL 13' & SCPMA 14', Ji, Liu, Liu, JHZ, Zhao, RMP 21'

- Examples of the state-of-the-art:

GPD: Impact parameter distribution

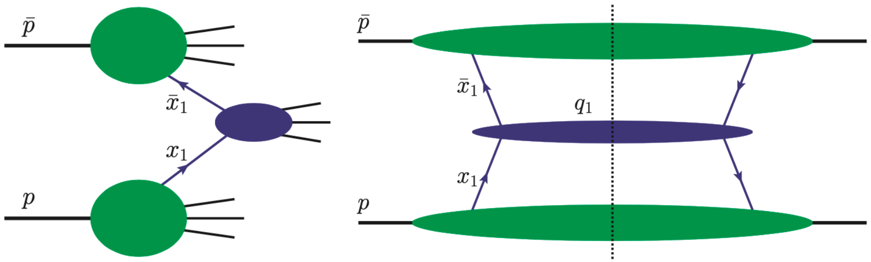
$$\mathsf{q}(x,b) = \int \frac{d\mathbf{q}}{(2\pi)^2} H(x,\xi=0,t=-\mathbf{q}^2)e^{i\mathbf{q}\cdot\mathbf{b}}$$



Lin, PRL 21'

From single- to multi-parton distributions

- The computational effort so far has been mainly focused on **single parton distributions** (relevant for single parton scattering)



$$\sigma = \sum_{ij} \int dx_1 \int d\bar{x}_1 f_i(x_1) f_j(\bar{x}_1) \hat{\sigma}_{ij}.$$

Single parton distributions

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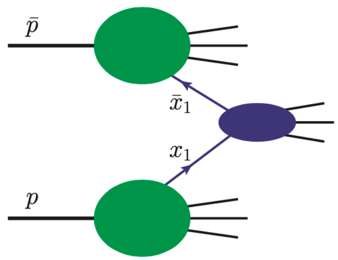
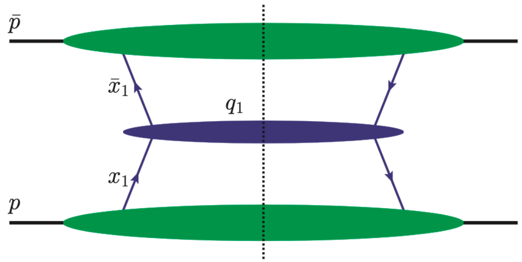
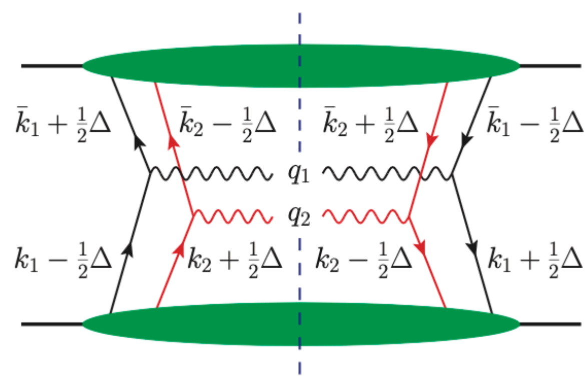




$$\sigma = \sum_{ij} \int dx_1 \int d\bar{x}_1 f_i(x_1) f_j(\bar{x}_1) \hat{\sigma}_{ij}.$$

Single parton distributions

- With the increasing energy of hadron colliders, **multiparton scattering** (e.g., double parton scattering) processes become increasingly important



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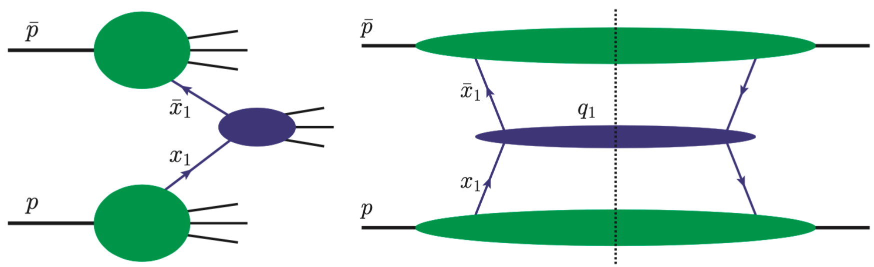

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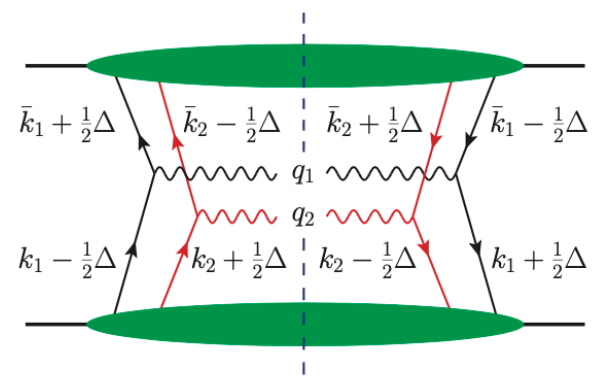
Single parton distributions

- With the increasing energy of hadron colliders, **multiparton scattering** (e.g., double parton scattering) processes become increasingly important

It can compete with single parton scattering in certain situations

Longitudinal parton momenta fixed by final state kinematics

Transverse parton momenta can differ by Δ, conjugate to transverse separation of partons



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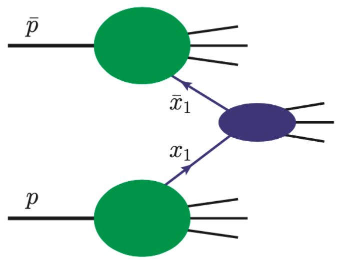

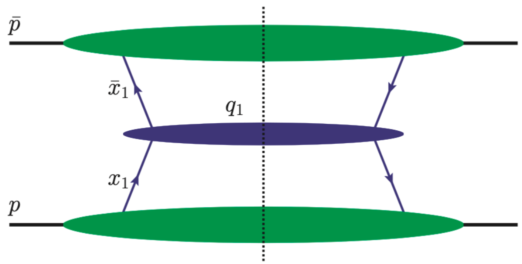

$$\sigma = \sum_{ij} \int dx_1 \int d\bar{x}_1 f_i(x_1) f_j(\bar{x}_1) \hat{\sigma}_{ij}.$$

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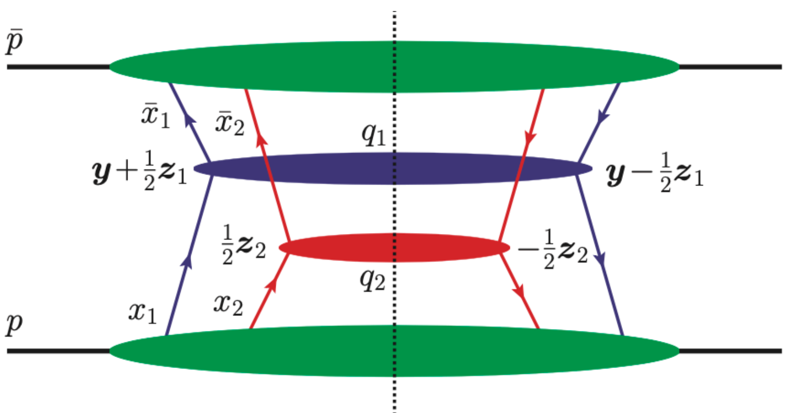


$$\frac{d\sigma}{\prod_{i=1}^{2} dx_i \, d\bar{x}_i \, d^2\boldsymbol{q}_i} =$$

$$= \frac{1}{C} \sum_{a_1 a_2 a_3 a_4} \int \frac{d^2\boldsymbol{z}_1}{(2\pi)^2} \frac{d^2\boldsymbol{z}_2}{(2\pi)^2} \, e^{-i\boldsymbol{z}_1\boldsymbol{q}_1 - i\boldsymbol{z}_2\boldsymbol{q}_2} \int d^2\boldsymbol{y}$$

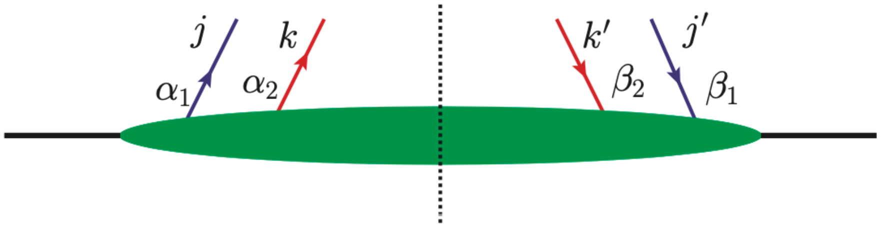
$$\times \Big\{ d\hat{\sigma}_{a_1\bar{a}_3} \, d\hat{\sigma}_{a_2\bar{a}_4} \left[{}^1F_{a_1a_2} \, {}^1\bar{F}_{\bar{a}_3\bar{a}_4} + c_8 \, {}^8F_{a_1a_2} \, {}^8\bar{F}_{\bar{a}_3\bar{a}_4} \right]$$

$$+ \, d\hat{\sigma}^I_{a_1\bar{a}_3} \, d\hat{\sigma}^I_{a_2\bar{a}_4} \left[{}^1F^I_{a_1a_2} \, {}^1\bar{F}^I_{\bar{a}_3\bar{a}_4} + c_8 \, {}^8F^I_{a_1a_2} \, {}^8\bar{F}^I_{\bar{a}_3\bar{a}_4} \right] \Big\}$$

Double parton distributions

From single- to multi-parton distributions

- Double parton distributions (DPDs) **Diehl, Ostermeier, Schaefer, JHEP 12', Diehl, Gaunt, 17'**



- Two-quark correlation

$$\Phi_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}(k_1,k_2,r) = \int \frac{d^4 z_1}{(2\pi)^4} e^{iz_1k_1} \frac{d^4 z_2}{(2\pi)^4} e^{iz_2k_2} \frac{d^4 y}{(2\pi)^4} e^{-iyr}$$
$$\times \langle p| \bar{T}\left[\bar{\psi}_{\Sigma_1'}(y-\tfrac{1}{2}z_1)\bar{\psi}_{\Sigma_2'}(-\tfrac{1}{2}z_2)\right] T\left[\psi_{\Sigma_2}(\tfrac{1}{2}z_2)\psi_{\Sigma_1}(y+\tfrac{1}{2}z_1)\right]|p\rangle$$

- Fourier transform to transverse position space

$$F_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}(x_1,x_2,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{y}) = 2p^+ \int \frac{dz_1^-}{2\pi}\frac{dz_2^-}{2\pi}\, dy^- \; e^{ix_1 z_1^- p^+ + ix_2 z_2^- p^+}$$
$$\times \langle p| \bar{\psi}_{\Sigma_2'}(-\tfrac{1}{2}z_2)\psi_{\Sigma_2}(\tfrac{1}{2}z_2)\bar{\psi}_{\Sigma_1'}(y-\tfrac{1}{2}z_1)\psi_{\Sigma_1}(y+\tfrac{1}{2}z_1)\,|p\rangle \Big|_{z_i^+=y^+=0}.$$

- Σ_i denotes collectively the spin, color and flavor of the corresponding quark, gauge links need to be inserted to ensure gauge invariance

DPDs in phenomenology

- Simplified modeling often ignores the correlation between partons in analyzing double parton scattering

- DPDs in terms of single parton impact parameter distributions

$$F_{a_1 a_2}(x_1, x_2, \boldsymbol{y}) \stackrel{?}{=} \int d^2\boldsymbol{b}\, f_{a_1}(x_1, \boldsymbol{b}+\boldsymbol{y})\, f_{a_2}(x_2, \boldsymbol{b})$$

- Such a factorization has been investigated on the lattice for the pion, significant differences have been observed between l.h.s. and r.h.s.

- DPDs in a completely factorized form

$$F_{a_1 a_2}(x_1, x_2, \boldsymbol{y}) \stackrel{?}{=} f_{a_1}(x_1)\, f_{a_2}(x_2)\, G(\boldsymbol{y})$$

- Double parton scattering X-sec is given by

$$\sigma_{\mathrm{DPS},ij} = \frac{1}{C} \frac{\sigma_{\mathrm{SPS},i}\ \sigma_{\mathrm{SPS},j}}{\sigma_{\mathrm{eff}}}$$

DPDs from lattice

- DPDs involve fields at lightlike distances and thus are not well-suited for lattice calculations
- What can be studied on the lattice are two-current correlations at space like separations, e.g., **Bali et al, JHEP 21'**

$$M^{\mu_1\cdots\mu_2\cdots}_{q_1q_2,i_1i_2}(p,y) = \langle h(p)|\, J^{\mu_1\cdots}_{q_1,i_1}(y)\, J^{\mu_2\cdots}_{q_2,i_2}(0)\,|h(p)\rangle$$

- Take the color singlet quark DPDs as an example:
- The correlations with currents

$$J^{\mu}_{q,V}(y) = \bar{q}(y)\gamma^{\mu}q(y)\,, \qquad J^{\mu}_{q,A}(y) = \bar{q}(y)\gamma^{\mu}\gamma_5\, q(y)\,, \qquad J^{\mu\nu}_{q,T}(y) = \bar{q}(y)\sigma^{\mu\nu}q(y)$$

are related to the lowest double Mellin moment of the DPDs, e.g.,

$$\int_{-\infty}^{\infty} dy^-\, M^{++}_{q_1q_2,VV}(p,y)\Big|_{y^+=0,\,\boldsymbol{p}=\boldsymbol{0}} = 2p^+\, I_{q_1q_2}(y^2)$$

$$I_{a_1a_2}(y^2) = \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2\, f_{a_1a_2}(x_1,x_2,y^2)$$

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- Take the color singlet quark DPDs as an example:
- On the other hand

$$\frac{1}{2}[M^{\mu\nu}_{q_1q_2,VV}(p,y) + M^{\nu\mu}_{q_1q_2,VV}(p,y)] = t^{\mu\nu}_{VV,A}\, A_{q_1q_2} + t^{\mu\nu}_{VV,B}\, m^2 B_{q_1q_2} + t^{\mu\nu}_{VV,C}\, m^4 C_{q_1q_2} + t^{\mu\nu}_{VV,D}\, m^2 D_{q_1q_2}$$

$$t^{\mu\nu}_{VV,A} = 2p^\mu p^\nu - \frac{1}{2} g^{\mu\nu} p^2\,, \qquad t^{\mu\nu}_{VV,C} = 2y^\mu y^\nu - \frac{1}{2} g^{\mu\nu} y^2\,,$$

$$t^{\mu\nu}_{VV,B} = p^\mu y^\nu + p^\nu y^\mu - \frac{1}{2} g^{\mu\nu} py\,, \qquad t^{\mu\nu}_{VV,D} = g^{\mu\nu}\,,$$

$$\Downarrow$$

$$A_{q_1q_2} = \frac{1}{8N^2}\left\{3(y^2)^2\, t^{\mu\nu}_{VV,A} - 6y^2 py\, t^{\mu\nu}_{VV,B} + \left[p^2y^2 + 2(py)^2\right] t^{\mu\nu}_{VV,C}\right\}\left[M_{q_1q_2,VV}\right]_{\mu\nu}$$

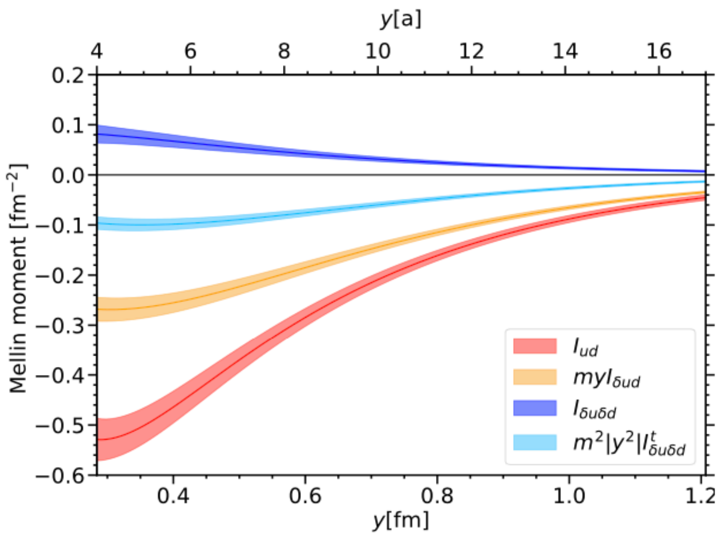
$$I_{a_1a_2}(y^2) = \int_{-\infty}^{\infty} d(py)\, A_{a_1a_2}(py, y^2)$$

DPDs from lattice

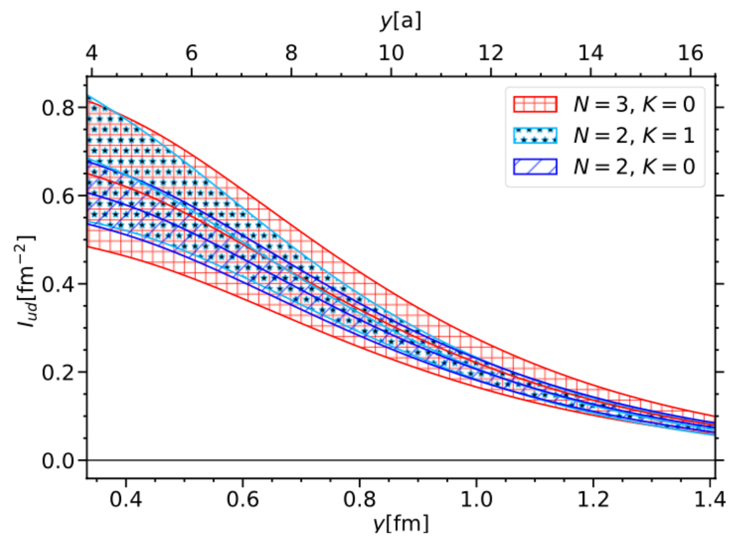
- DPDs involve fields at lightlike distances and thus are not well-suited for lattice calculations
- What can be studied on the lattice are two-current correlations at space like separations, e.g., **Bali et al, JHEP 21'**

$$M^{\mu_1\cdots\mu_2\cdots}_{q_1 q_2, i_1 i_2}(p, y) = \langle h(p)|\, J^{\mu_1\cdots}_{q_1, i_1}(y)\, J^{\mu_2\cdots}_{q_2, i_2}(0)\, |h(p)\rangle$$

- Take the color singlet quark DPDs as an example:



(a) Mellin moments $I(y^2, \zeta = 0)$



(a) fit comparison $I_{ud}(\zeta = 0, y^2)$

- Can we access the full DPD rather than the lowest double Mellin moments from lattice?

DPDs from lattice

- Start from the simplest DPD: unpolarized color singlet case **JHZ, 23'**

$$f_{q_1q_2}(x_1,x_2,y^2) = 2P^+\int dy^-\int \frac{dz_1^-}{2\pi}\frac{dz_2^-}{2\pi}e^{i(x_1z_1^-+x_2z_2^-)P^+}h_0(y,z_1,z_2,P)$$
$$= 2\int d\lambda\int\frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi}e^{i(x_1\lambda_1+x_2\lambda_2)}h(\lambda,\lambda_1,\lambda_2,y^2),$$

with

$$h_0(y,z_1,z_2,P) = \langle P|O_{q_1}(y,z_1)O_{q_2}(0,z_2)|P\rangle,$$
$$h(\lambda,\lambda_1,\lambda_2,y^2) = \frac{1}{(P^+)^2}h_0(y,z_1,z_2,P),$$
$$O_q(y,z) = \bar\psi_q\big(y-\frac{z}{2}\big)\frac{\gamma^+}{2}W\big(y-\frac{z}{2};y+\frac{z}{2}\big)\psi_q\big(y+\frac{z}{2}\big),$$
$$\lambda = P\cdot y,\quad \lambda_1 = P\cdot z_1,\quad \lambda_2 = P\cdot z_2,$$

- Double Mellin moments are given by

$$M_{q_1q_2}^{n_1n_2}(y^2) = \int_{-1}^{1}dx_1dx_2\,x_1^{n_1-1}x_2^{n_2-1}f_{q_1q_2}(x_1,x_2,y^2)$$
$$= \frac{(P^+)^{1-n_1-n_2}}{2}\int dy^-\langle P|\mathcal{O}_{q_1}^{+\cdots+}(y)\mathcal{O}_{q_2}^{+\cdots+}(0)|P\rangle$$

$$\mathcal{O}_q^{\mu_1\cdots\mu_n}(y) = \bar\psi_q(y)\gamma^{\{\mu_1}i\overleftrightarrow{D}^{\mu_2}(y)\cdots i\overleftrightarrow{D}^{\mu_n\}}(y)\psi_q(y),$$

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$$= 2\int d\lambda \int \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi} e^{i(x_1\lambda_1 + x_2\lambda_2)} h(\lambda,\lambda_1,\lambda_2,y^2),$$

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$$\lambda = P\cdot y, \quad \lambda_1 = P\cdot z_1, \quad \lambda_2 = P\cdot z_2,$$

- Double Mellin moments can be turned into a manifestly covariant form **Diehl, Ostermeier, Schaefer, JHEP 12'**

$$\langle P|\mathcal{O}_{q_1}^{\mu_1\cdots\mu_{n_1}}(y)\mathcal{O}_{q_2}^{\nu_1\cdots\nu_{n_2}}(0)|P\rangle =$$
$$2P^{\mu_1}\cdots P^{\mu_{n_1}}P^{\nu_1}\cdots P^{\nu_{n_2}}\langle \mathcal{O}_{q_1}^{n_1}\mathcal{O}_{q_2}^{n_2}\rangle(\lambda,y^2) + \cdots\,,$$

$$M_{q_1q_2}^{n_1n_2}(y^2) = \int d\lambda\, \langle \mathcal{O}_{q_1}^{n_1}\mathcal{O}_{q_2}^{n_2}\rangle(\lambda,y^2).$$

DPDs from lattice

- Consider the correlation of equal-time nonlocal operators following the spirit of LaMET **Ji, PRL 13' & SCPMA 14', Ji, Liu, Liu, JHZ, Zhao, RMP 21'**

$$\tilde{h}(z_1, z_2, y, P) = \frac{1}{N}\langle P|O_{q_1}(y, z_1)O_{q_2}(0, z_2)|P\rangle,$$

$$y^\mu = (0, \overrightarrow{y}_\perp, y^z), \quad z_i^\mu = (0, \overrightarrow{0}_\perp, z_i)$$

- From OPE **Izubuchi et al, PRD 18'**

$$\tilde{h}(z_i, \mu_i, y, P) = \frac{1}{4N}\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\frac{(-iz_1)^{n_1-1}}{(n_1-1)!}\frac{(-iz_2)^{n_2-1}}{(n_2-1)!}$$
$$\times C_{q_1}^{(n_1-1)}(\mu_1^2 z_1^2)C_{q_2}^{(n_2-1)}(\mu_2^2 z_2^2)\tilde{\mathcal{M}}_{q_1 q_2}^{n_1 n_2}(\mu_i, y, P) + \cdots,$$

$$\tilde{\mathcal{M}}_{q_1 q_2}^{n_1 n_2}(\mu_i, y, P) = n_{\mu_1}\cdots n_{\mu_{n_1}} n_{\nu_1}\cdots n_{\nu_{n_2}}$$
$$\times \langle P|\mathcal{O}_{q_1}^{\mu_1\cdots\mu_{n_1}}(y, \mu_1)\mathcal{O}_{q_2}^{\nu_1\cdots\nu_{n_2}}(0, \mu_2)|P\rangle$$
$$= 2(n\cdot P)^{n_1+n_2}\langle\mathcal{O}_{q_1}^{n_1}\mathcal{O}_{q_2}^{n_2}\rangle(\mu_i, \lambda, y^2) + \cdots,$$

- The same Lorentz invariant reduced matrix element appears both in correlations of lightcone and Euclidean nonlocal operators

DPDs from lattice

- Factorization

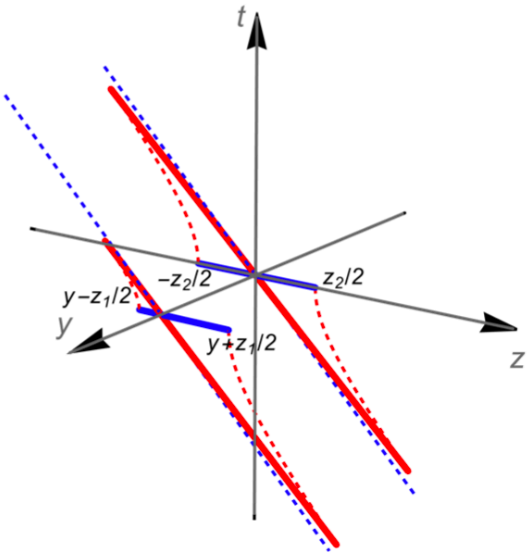
$$\tilde{H}(\lambda_i, \mu_i, z_i^2, y^2) = \int du_1 du_2\, \mathcal{C}_{q_1}(u_1, \mu_1^2 z_1^2)\mathcal{C}_{q_2}(u_2, \mu_2^2 z_2^2) H(u_i\lambda_i, \mu_i, y^2) + \cdots$$

$$\tilde{H}(\lambda_i, \mu_i, z_i^2, y^2) = \int d\lambda\, \tilde{h}(\lambda, \lambda_i, \mu_i, z_i^2, y^2),$$

$$H(\lambda_i, \mu_i, y^2) = \int d\lambda\, h(\lambda, \lambda_i, \mu_i, y^2).$$

- FT w.r.t. z_i with P fixed

$$\tilde{f}(x_1, x_2, \mu_i, y^2) = 2\int \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi} e^{i(x_1\lambda_1 + x_2\lambda_2)} \times \tilde{H}\Big(\lambda_i, \mu_i, -\frac{\lambda_i^2}{(P^z)^2}, y^2\Big)$$
$$= \int \frac{dx_1'}{|x_1'|}\frac{dx_2'}{|x_2'|} C_{q_1}\Big(\frac{x_1}{x_1'}, \frac{\mu_1^2}{(x_1' P^z)^2}\Big) C_{q_2}\Big(\frac{x_2}{x_2'}, \frac{\mu_2^2}{(x_2' P^z)^2}\Big) \times f(x_i', \mu_i^2, y^2) + \cdots ,$$



JHZ, 23'

DPDs from lattice

- Factorization

$$\tilde{H}(\lambda_i, \mu_i, z_i^2, y^2) = \int du_1 du_2\, \mathcal{C}_{q_1}(u_1, \mu_1^2 z_1^2)\mathcal{C}_{q_2}(u_2, \mu_2^2 z_2^2) H(u_i\lambda_i, \mu_i, y^2) + \cdots$$

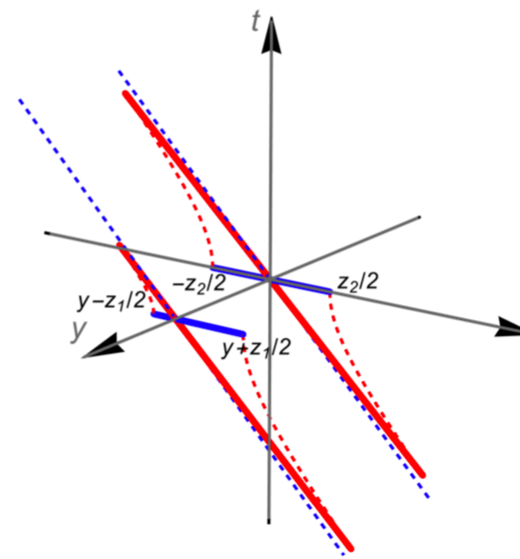
$$\tilde{H}(\lambda_i, \mu_i, z_i^2, y^2) = \int d\lambda\, \tilde{h}(\lambda, \lambda_i, \mu_i, z_i^2, y^2),$$

$$H(\lambda_i, \mu_i, y^2) = \int d\lambda\, h(\lambda, \lambda_i, \mu_i, y^2).$$

- FT w.r.t. λ_i with z_i^2 fixed

$$\mathcal{D}(x_i, \mu_i, z_i^2, y^2) = 2\int \frac{d\lambda_1}{2\pi}\frac{d\lambda_2}{2\pi} e^{i(x_1\lambda_1 + x_2\lambda_2)} \tilde{H}(\lambda_i, \mu_i, z_i^2, y^2)$$

$$= \int \frac{dx_1'}{|x_1'|}\frac{dx_2'}{|x_2'|} \mathcal{C}_{q_1}(\frac{x_1}{x_1'}, \mu_1^2 z_1^2)\mathcal{C}_{q_2}(\frac{x_2}{x_2'}, \mu_2^2 z_2^2) f(x_i', \mu_i, y^2) + \cdots,$$



JHZ, 23'

DPDs from lattice

- Color-correlated DPD **Jaarsma, Rahn, Waalewijn, 23', JHZ, in preparation**

$$
{}^{R_1R_2}\tilde{F}^{\mathrm{NS}}_{q_1q_2}(x_1,x_2,b_\perp,\mu,\tilde{\zeta}_p,\tilde{P}^z)
$$
$$
= \sum_{R_1',R_2'}\sum_{q_1',q_2'}\int_0^1 \frac{\mathrm{d}x_1'}{x_1'}\frac{\mathrm{d}x_2'}{x_2'}\,{}^{R_1R_1'}C_{q_1q_1'}\left(\frac{x_1}{x_1'},x_1'\tilde{P}^z,\mu\right){}^{R_2R_2'}C_{q_2q_2'}\left(\frac{x_2}{x_2'},x_2'\tilde{P}^z,\mu\right)
$$
$$
\times\exp\left[\frac{1}{2}{}^{R_{1/2}}J(b_\perp,\mu)\ln\left(\frac{\tilde{\zeta}_p}{\zeta_p}\right)\right]{}^{R_1'R_2'}F^{\mathrm{NS}}_{q_1'q_2'}(x_1',x_2',b_\perp,\mu,\zeta_p)\,.
$$

- Rapidity divergences show up in the collinear DPDs, and introduce rapidity scale dependence

- Checked by an explicit one-loop calculation, consistent with RG and rapidity evolution of DPDs

Summary and outlook

- Lattice calculations of single parton distributions have reached a stage of precision control

- Multiparton distributions are important both for collider phenomenology and for understanding the correlated partonic structure of hadrons

- Very limited knowledge even on the simplest case (DPDs)

- The full DPD can be accessed following the spirit of LaMET, lattice calculations can provide important inputs for phenomenological analyses

- The same development in single parton distributions (PDFs, TMDs, GPDs…) can be extended to DPDs, and generalized to multiparton distributions, a lot more to be explored…