Remarks on Entanglement in Quantum Field Theory

Top Quark Physics at the Precision Frontier

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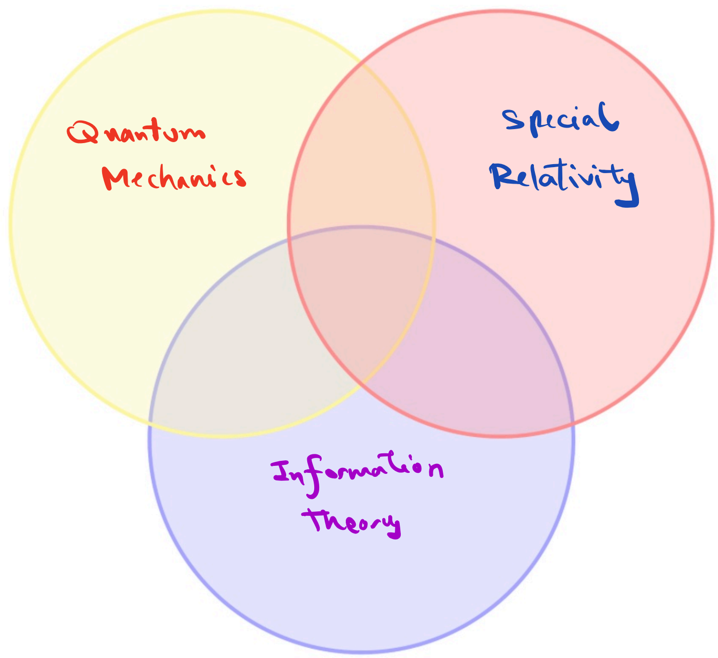


- Entanglement of relativistic particles
- Entanglement of fields

Quantum Information theory:

- **Information** is **physical**: needs a physical carrier.

QIT: How quantum info is **stored** (states) **processed** (dynamics) **transferred** (channels)

observables: self-adjoint operators

$\uparrow$ spin-$\frac{1}{2}$

$\mathbb{I}, \sigma_x, \sigma_y, \sigma_z$

$a_1 + a_2$ addition

$a_1 a_2$ multiplication

algebra of observables (noncommutative)

time-ordering

states $\psi : \mathcal{A} \longrightarrow \mathbb{C}$

$|\psi\rangle = \sum_i c_i |i\rangle$

positive $\psi(a^\dagger a) \geq 0$.

$\rho = \sum_i p_i |i\rangle\langle i|$

Pure state $\psi(a) = \langle \psi | a | \psi \rangle$

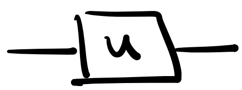
mixed state (density matrix) $\psi_\rho(a) = \mathrm{tr}(\rho a)$

Measurements: POVM (projection-operator valued measure)

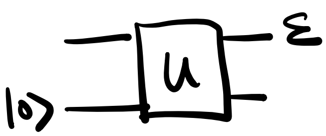
$\{ |i\rangle\langle i|, p_i \}$ $\quad p_i = |\langle i|\psi\rangle|^2$

$\{ F_i \geq 0, p_i, \sum_i F_i = \mathbb{I} \}$ $\quad p_i = \mathrm{tr}(\rho F_i)$

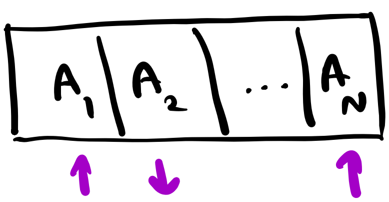
other operations:



Apply unitaries $\quad \rho \to U^\dagger \rho U$



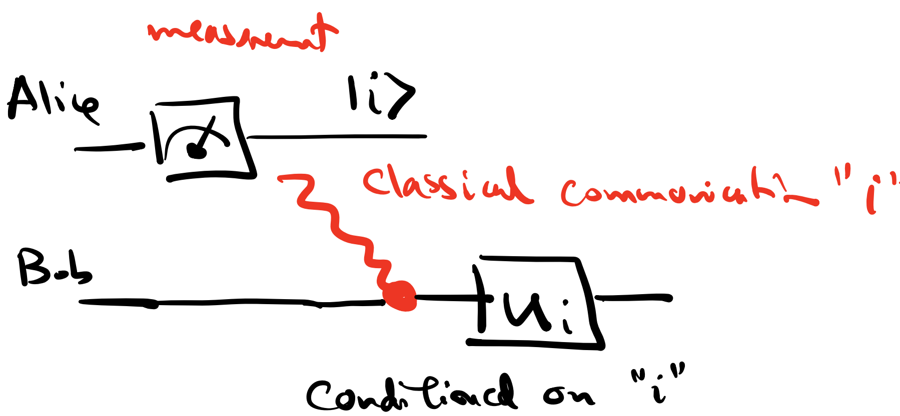
Apply a quantum channel $\quad \rho \to \mathcal{E}(\rho)$



- **Multipartite quantum systems**

$$|\Psi_{A_1\cdots A_N}\rangle = \sum_{i_1\cdots i_N} c_{i_1\cdots i_N} |i_1, \cdots, i_N\rangle$$

- **LOCC:** classical communication & local operators



Entanglement theory

Informal def:

Quantum Entanglement : non classical correlations between subsystems A_i & A_j.

But what are classical correlations?

LOCC can NOT create entanglement.

Entanglement is a resource.

↑ ↗

Entangled state : $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

- For bipartite pure state there's a complete entanglement theory

  von Neumann entropy (Entanglement entropy)

  $$S(\rho) = -\mathrm{tr}(\rho \log \rho)$$

  is the ultimate measure of entanglement

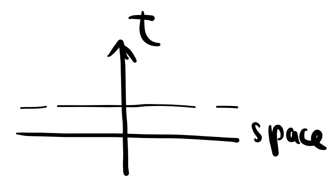
- More than bipartite: Entanglement theory is too complex. (Art)

Relativistic Quantum Info.

- Relativistic particles as subsystems

Relativity sets constraints on info. dynamics:

Speed of signal propagation $\leq c$

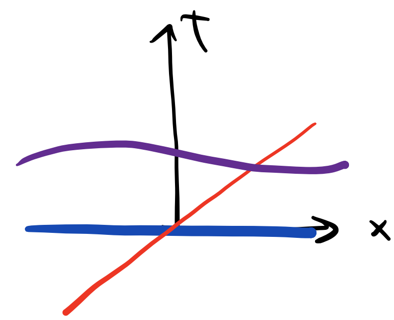


Non-relativistic physics

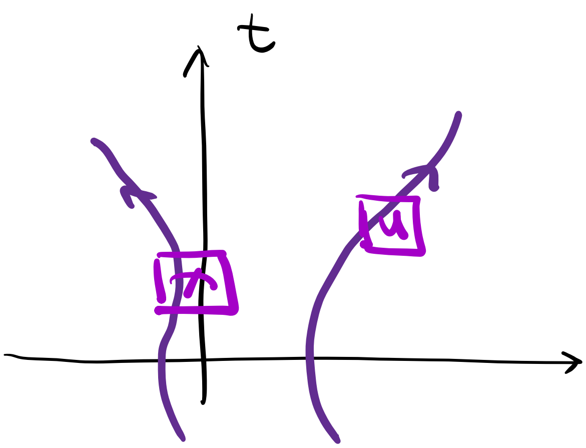
space × time

Relativistic physics

frames



States depend on the frame



Measurements & operations occur at points in spacetime

Spacelike separated operators commute.

Lorentz transf:

$$\begin{cases} \rho_0' = U \rho_0 U^\dagger \\ \rho_f' = V \rho_0 V^\dagger \end{cases}$$

Time evolution:

depends on Lorentz frame

$$\rho_f = \sum_n A_n \rho_0 A_n^\dagger$$

$$\rho_f' = \sum_n (V A_n U^\dagger) \rho_0' (U A_n V^\dagger)$$

Spin & momentum:

the "reduced state" on spin depends on the Lorentz frame.

massive spin ½ particle

$$\psi(\vec{p}) = \begin{pmatrix} \psi_1(\vec{p}) \\ \psi_2(\vec{p}) \end{pmatrix}$$

spin density matrix

$$\rho_{spin} = \int d\vec{p} \; \psi(\vec{p}) \psi^\dagger(\vec{p})$$

But ρ_{spin} depends on the Lorentz frame.

Entanglement entropy of ρ_{spin}

$$S(\rho_{spin}) = -tr(\rho_{spin} \log \rho_{spin})$$

depends on the frame.

- Orthogonal states in Alice's frame might not be orthogonal in Bob's frame.

Relativistic Entanglement:

For a pair of particles we can choose the center of mass frame. But this does not generalize to multiple particles.

Why fields?

Two paradigms from wave-particle duality:

1) Fundamental building blocks are particles

thermodynamic limit

many particles $\longrightarrow$ statistical many body physics

volume $\to \infty$

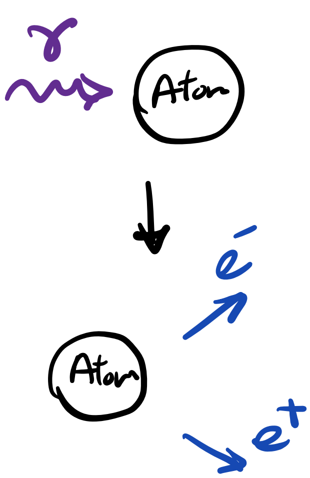
\# of particles $\to \infty$

density fixed

2) Fundamental building blocks are fields

("functions" in spacetime)

But fields are more Fundamental

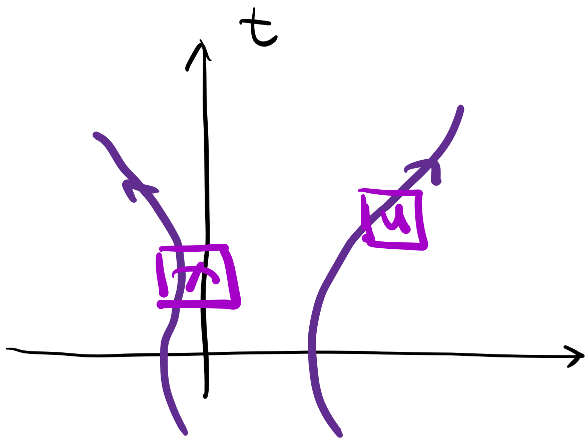


- Field theory is intrinsically many body.
- Field theory allows for pair creation

Quantum Mechanics + Special Relativity

$\downarrow$

particle number is not conserved.



Consequences for quantum info:

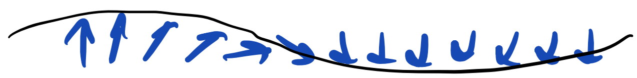
Particles can NOT be localized

Measurements " NOT " "

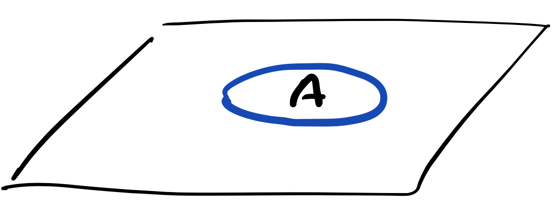
Operations " " " "

Quantum Info. in QFT:

Particles $\longrightarrow$ fields

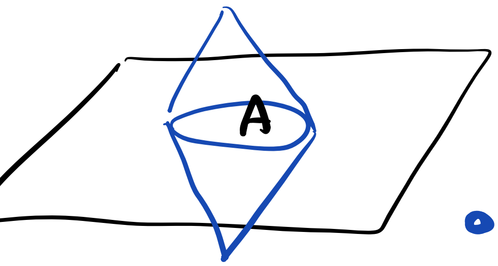


There might be particles, quasi-particle & sometimes at strong coupling just a quantum soup.



$$\langle \varphi(x) | \Psi \rangle = \Psi(\varphi(x))$$

$$\rho_A = \mathrm{tr}_{A^c} |\Psi\rangle\langle\Psi|$$



Real-space Entanglement:

Entanglement entropy of local regions in QFT is

- frame independent
- Infinite & non-renormalizable

Too fine-grained.

Momentum space Entanglement:

In free QFT the Hilbert space

$$\mathcal{H} = \bigotimes_k \mathcal{H}_k$$

we can define low momenta density matrix

$$\rho_{k<\Lambda} = \mathrm{tr}_{k>\Lambda} |\Psi\rangle\langle\Psi| \quad \&$$

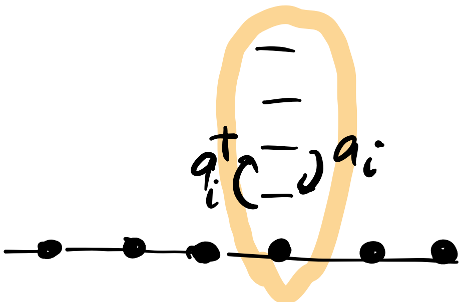
study entanglement in momentum space in perturbation theory.

Summary

- Relativity sets a constraint on the propagation of quantum info and entanglement.
- Defining the reduced states of spin d.o.f. or a few particles in QFT is tricky.
- Entanglement in QFT is often studied for all d.o.f. reduced to subregions of spacetime.

Entanglement entropy in free QFT

First approach:



$\phi_i\,,\ \pi_i$

$$\phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}$$

Bosons on a lattice

$$H = \frac{1}{2}\sum_i^N \dot\phi_i^2 + \frac{1}{2}\sum_{i,j=1}^N \phi_i K_{ij} \phi_j$$

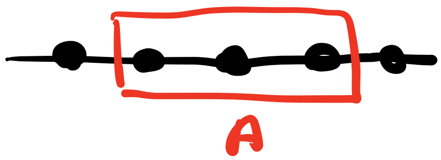
$$= \frac{1}{2}\sum_i^N \overbrace{\left(\dot\phi_i - i(\sqrt{K})_{ij}\phi_j\right)}^{\tilde a_i}\overbrace{\left(\dot\phi_i + i(\sqrt{K})_{ij}\phi_j\right)}^{\tilde a_i^\dagger} + \frac{1}{2}\operatorname{tr}\sqrt{K}$$

$$[\dot\phi_i, \phi_j] = i\delta_{ij}$$

ground-state wave-functional

$$\langle\phi|\Omega\rangle = \det\left(\frac{\sqrt{K}}{\pi}\right)^{1/4} e^{-\frac{1}{2}\phi^T\cdot(\sqrt{K})\cdot\phi}$$

$$\langle\phi|\Omega\rangle\langle\Omega|\phi'\rangle = \det\left(\frac{\sqrt{K}}{\pi}\right)^{1/2} e^{-\frac{1}{2}\left(\phi^T\cdot\sqrt{K}\cdot\phi + \phi'^T\cdot\sqrt{K}\cdot\phi'\right)}$$



split the sites into two groups

Inside $\phi^{(i)}$: if $i \in A$

outside $\phi^{(o)}$: if $i \notin A$

some def:

$$\sqrt{K} \equiv W$$

$$W = \begin{pmatrix} W_{(ii)} & W_{(io)} \\ W_{(oi)} & W_{(oo)} \end{pmatrix}$$

$$\tilde{\phi} = W_{(ii)}^{\frac{1}{2}} \cdot \phi^{(i)}$$

matrix multiplication

$$\Lambda = W_{(ii)}^{-\frac{1}{2}} \cdot W_{(io)} \cdot W_{(oo)}^{-1} \cdot W_{(oi)} \cdot W_{(ii)}^{-\frac{1}{2}}$$

we trace over outside modes

$$\rho_A(\tilde{\phi}, \tilde{\phi}') = \sqrt{\det \frac{(1-\Lambda)}{\pi}}\, e^{-\frac{1}{2}(\tilde{\phi}^T \cdot \tilde{\phi} + \tilde{\phi}'^T \cdot \tilde{\phi}')}$$

$$\times\, e^{-\frac{1}{4}(\tilde{\phi} + \tilde{\phi}')^T \cdot \Lambda \cdot (\tilde{\phi} + \tilde{\phi}')}$$

The entanglement entropy is $S(\rho_A) = -\mathrm{tr}(\rho_A \log \rho_A)$

to compute it we first need to diagonalize Λ:

$$\tilde{\phi}^T \cdot \tilde{\phi} + \tilde{\phi}'^T \cdot \phi' + \frac{1}{2}(\tilde{\phi} + \tilde{\phi}') \cdot \Lambda \cdot (\tilde{\phi} + \tilde{\phi}')$$

$$\Lambda = S^T \cdot \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} \cdot S$$

real symmetric matrix

$$S^T S = 1.$$

orthogonal matrix S