Quantum Tomography of helicity states for general scattering processes.

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Motivation

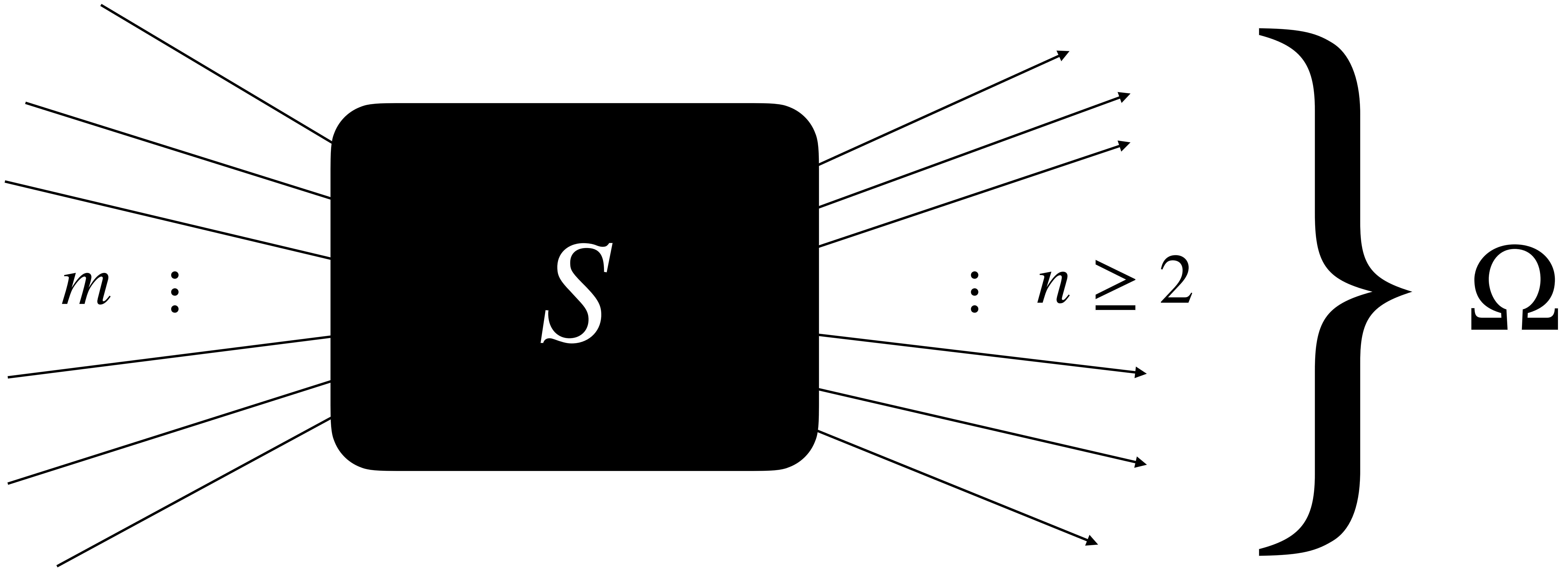
From knowing the helicity density matrix of a quantum state we have access to all the spin information of the system, in particular:

- Spin polarizations
- Spin correlations
- Entanglement
- Possible violation of Bell inequalities
- Etc

Simple and experimentally practical Quantum Tomography is highly important!

Main goal

Determine the initial helicity state ρ of a general scattering process from the angular distribution data of the final particles



Steps to follow:

- Generalize the definition of the production/decay matrix Γ
- Find the kinematic dependence of both Γ and the normalized differential cross section
- Expand ρ in terms of $\{T^L_M\}$ (Irreducible tensor operators) and compute the coefficients of the expansion from the previous results

Extra: Re-derivation from Quantum Information perspective (Weyl-Wigner-Moyal formalism)

Basic concepts

- A general quantum system in a Hilbert Space $\mathscr{H}$ of finite dimension d is described by a $d \times d$ density matrix ρ:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1 \iff Tr\{\rho\} = 1$$

- For N-partite systems $|\psi_i\rangle \in \mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_N$ with $\dim \mathscr{H}_i = d_i$

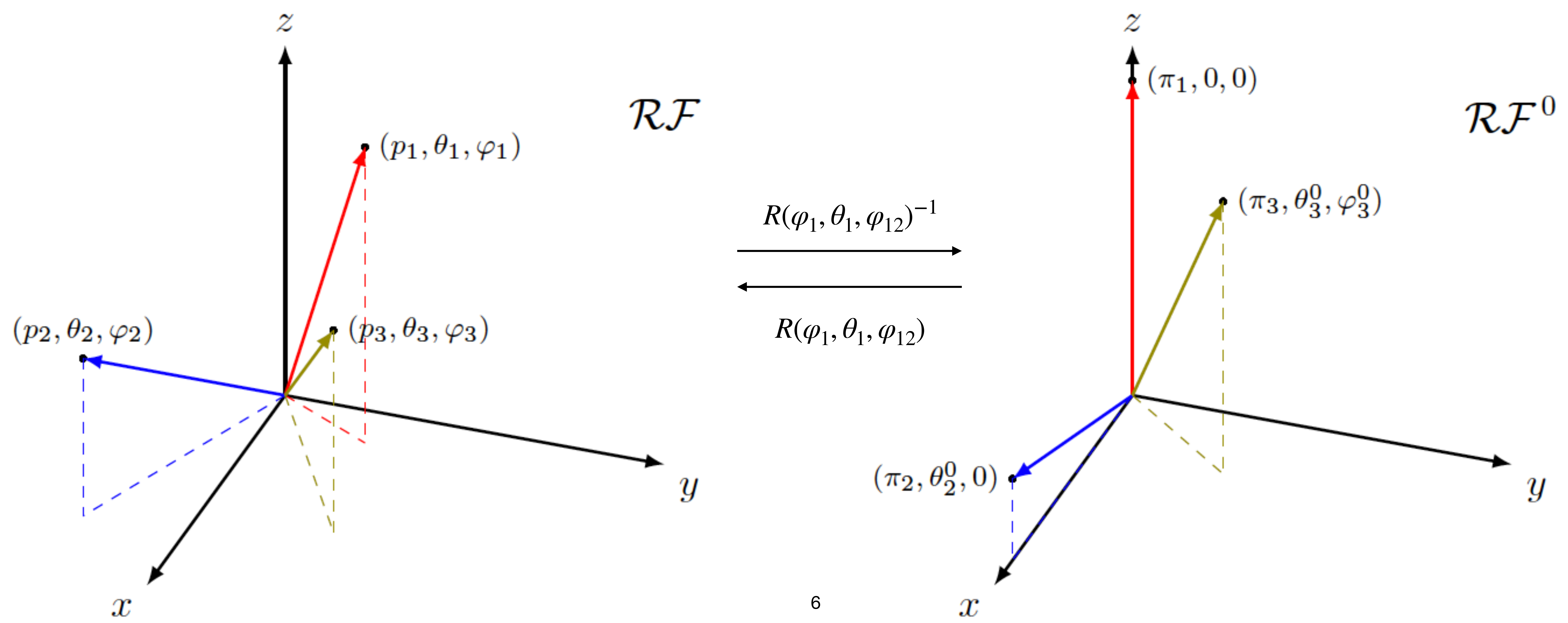
- Expectation values of operators are computed by

$$\langle \mathcal{O} \rangle_\rho = Tr\{\mathcal{O}\rho\}$$

State representation of relativistic many-particle systems

We consider a n-particle system with fixed λ_i and $\vec{p}_i$ such that $\vec{\chi} = \sum \vec{p}_i = \vec{0}$.

We distinguish 2 reference frames relevant for the work:



Following this setup, the quantum state of the n-particle system is given by

$$\prod_{i=1}^{n} |\vec{p}_i\lambda_i\rangle = \hat{D}(R)\prod_{i=1}^{n} |\vec{\pi}_i\lambda_i\rangle = |R\pi\lambda\rangle$$

Here $\hat{D}(R)$ is the unitary representation of R, λ are the particle's helicities and π are the $3n-3$ spherical coordinates in $\mathscr{R}\mathscr{F}^0$.

A more convenient representation is

$$|RE\vec{\chi}\kappa\lambda\rangle = |R\kappa\lambda\rangle$$

$\uparrow$

E and $\vec{\chi}$ fixed

Now, κ is a set of $3n-7$ parameters to be chosen depending on the case

We are interested in the relation between these states and the ones with definite angular momentum (J, M):

$$|R\,\kappa\,\lambda\rangle = \sum_{J}\sum_{M\,\Lambda}\frac{(J+1/2)^{1/2}}{2\pi}D^{J}_{M\,\Lambda}(R)\,|\,J\,M\,\Lambda\,\kappa\,\lambda\rangle,$$

where $D^{J}_{M\,\Lambda}(R)\,\delta_{J,\,J'} = \langle J\,M\,|\,\hat{D}(R)\,|\,J'\,M'\rangle$ is the Wigner D-matrix associated with R and Λ is the projection of the total angular momentum of the system over $\hat{p}_1$.

With these tools we can move forward to define the production matrix Γ and to compute its elements and kinematic dependence

Generalized production matrix Γ

We define the production matrix as:

$$\Gamma_{\bar{\lambda}\,\bar{\lambda}'} \propto \sum_{\lambda} \mathscr{M}_{\lambda\,\bar{\lambda}} \mathscr{M}^{*}_{\lambda\,\bar{\lambda}'}, \qquad \mathscr{M}_{\lambda\,\bar{\lambda}} = \langle R\,\kappa\,\lambda\,|\,S\,|\,\bar{R}\,\bar{\kappa}\,\bar{\lambda}\rangle$$

with $\mathscr{M}_{\lambda\,\bar{\lambda}}$ the so-called helicity amplitudes given in terms of the scattering matrix S and the previously introduced quantum states.

We are particularly interested in the transposed matrix Γ^T, whose simplified expression after some algebra is given by

$$\Gamma^{T}_{\bar{\lambda}\,\bar{\lambda}'} \propto \langle \bar{1}\,\bar{\kappa}\,\bar{\lambda}\,|\,\hat{D}(\bar{R}^{-1}R)\left[\sum_{\lambda} (S^{\dagger}\,|\,1\,\kappa\,\lambda\rangle\langle 1\,\kappa\,\lambda\,|\,S)\right]\hat{D}(\bar{R}^{-1}R)^{-1}\,|\,\bar{1}\,\bar{\kappa}\,\bar{\lambda}'\rangle$$

For instance, this implies

$$\Gamma^T(\bar{R}, R) = \Gamma^T(\bar{R}^{-1}R) = \hat{D}(\bar{R}^{-1}R)\,\Gamma^T(1)\,\hat{D}(\bar{R}^{-1}R)^{-1}$$

We can set the initial configuration as the one defining $\mathscr{R}\mathscr{F}^{\,0}$, hence $\bar{R} = \bar{1}$

$$\Gamma^T(\bar{R}^{-1}R) = \Gamma^T(R) = \hat{D}(R)\,\Gamma^T(1)\,\hat{D}(R)^{-1}, \quad R = R(\varphi_1, \theta_1, \varphi_{12}) = R(\Omega)$$

One only needs to compute the elements of $\Gamma^T(1)$ and then rotate the matrix accordingly. In general, in the canonical basis

$$\Gamma^T(R) = \frac{1}{a_{\sigma\,\sigma}} \sum_{\sigma\,\sigma'} a_{\sigma\,\sigma'}\,\hat{D}(R)\,e_{\sigma\,\sigma'}\,\hat{D}(R)^{-1}, \quad \boldsymbol{\sigma}^{(\prime)} = (\sigma_1^{(\prime)}, \ldots, \sigma_m^{(\prime)})$$

$$a_{\sigma\,\sigma'} = \sum_{\lambda} \langle \bar{1}\,\bar{\kappa}\,\sigma\,|\,S^\dagger\,|\,1\,\kappa\,\lambda\rangle\langle 1\,\kappa\,\lambda\,|\,S\,|\,\bar{1}\,\bar{\kappa}\,\sigma'\rangle, \quad a_{\sigma\,\sigma} = \sum_{\lambda} |\,\langle 1\,\kappa\,\lambda\,|\,S\,|\,\bar{1}\,\bar{\kappa}\,\sigma\rangle\,|^2$$

Red. helicity amplitudes
(Specific expressions in def. J rep.)

Reconstruction of density matrix ρ

In order to develop the quantum tomography, we will make use of the relation between ρ, Γ and the normalized differential cross section:

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa} = \frac{d}{8\pi^2 \bar{K} K} Tr\{\rho\, \Gamma^T(R)\}, \quad \bar{K} = \int d\bar{\kappa} \text{ and } K = \int d\kappa$$

Actually, we first perform an expansion of both Γ and ρ over a convenient basis. Due to its transformation property under rotations, the optimal one is composed by the irreducible tensor operators $\{T^L_M\}$.

As we deal with n particles with spin s_i each, the dimension of the helicity Hilbert space is $d = \prod_i d_i = \prod_i (2s_i + 1)$.

In particular this fixes the dimensionality of the basis $\{T^L_M\}$: $L \in \{0,1,\ldots,(d-1)\}$ and $M \in \{-L,\ldots,L\}$.

The elements of each operator are given by

$$\left[T^L_M\right]_{\sigma_T\,\sigma'_T} = (2L+1)^{1/2}\, C^{s_T\,\sigma_T}_{s_T\,\sigma'_T\,L\,M}$$

where s_T is an "effective" spin of the whole system: $d = 2s_T + 1$. Another important property is

$$Tr\{T^L_M\,(T^{L'}_{M'})^\dagger\} = Tr\{T^L_M\,(T^{L'}_{M'})^T\} = d\,\delta_{L\,L'}\delta_{M\,M'}\,.$$

Using the orthogonality condition, we get for any operator

$$A = \frac{1}{d} \sum_{LM} A_{LM} T^L_M, \text{ with } A_{LM} = Tr\{A\,(T^L_M)^\dagger\}$$

Applying this result to $e_{\sigma\,\sigma'}$

$$Tr\{e_{\sigma\,\sigma'}(T^L_M)^\dagger\} = Tr\{e_{\sigma\,\sigma'}(T^L_M)^T\} = \left[T^L_M\right]_{\sigma_T\,\sigma'_T} = (2L+1)^{1/2} C^{s_T\,\sigma_T}_{s_T\,\sigma'_T\,L\,M}$$

and plugging this expression in $\Gamma^T(1)$ leads to

$$\Gamma^T(1) = \frac{1}{d} \sum_{L\,\sigma_T^-} \tilde{B}_{L\,\sigma_T^-} T^L_{\sigma_T^-}, \quad \tilde{B}_{L\,\sigma_T^-} \equiv \frac{(2L+1)^{1/2}}{a_+} \sum_{\substack{\sigma\,\sigma' \\ (\sigma-\sigma')\cdot d^{(v)} = \sigma_T^-}} a_{\sigma\,\sigma'}\, C^{s_T\,\sigma_T}_{s_T\,\sigma'_T\,L\,\sigma_T^-}$$

Using the transformation of $\{T^L_M\}$ under rotations

$$\hat{D}(R)T^L_{M'}\hat{D}(R)^{-1} = \sum_M D^L_{MM'}(R)T^L_M \implies \Gamma^T(R) = \frac{1}{d}\sum_{LM}\left[\sum_{M'}\tilde{B}_{LM'}D^L_{MM'}(R)\right]T^L_M$$

In this way, we have given the expansion of $\Gamma^T(R)$ and we have factorized the kinematic dependence as

$$\tilde{B}_{LM'} = \tilde{B}_{LM'}(\bar{\kappa}, \kappa), \quad D^L_{MM'}(R) = D^L_{MM'}(\Omega)$$

In the same fashion,

$$\rho = \frac{1}{d}\sum_{LM} A_{LM}\, T^L_M \implies \frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa} = \frac{1}{8\pi^2\bar{K}K}\sum_{LM} A_{LM} Tr\{T^L_M\,\Gamma^T(R)\} \implies$$

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa} = \frac{1}{8\pi^2 K\bar{K}}\sum_{LM} A_{LM}\sum_{M'}\tilde{B}^*_{LM'}D^L_{MM'}(R)^*$$

Finally, from the orthogonality conditions for the Wigner D-matrices we get

$$\boxed{\int d\Omega\ \left[\frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa}\right]\left(\frac{2L+1}{4\pi}\right)^{1/2} D^{L}_{M\,M'}(\Omega) = \frac{B_{L\,M'}(\bar{\kappa},\kappa)^*}{4\pi}A_{L\,M}(\bar{\kappa})}$$

with

$$B_{L\,M'}(\bar{\kappa},\kappa) \equiv \left(\frac{4\pi}{2L+1}\right)^{1/2}\frac{\tilde{B}_{L\,M'}(\kappa,\bar{\kappa})}{\bar{K}K}$$

We have accomplished the **Quantum Tomography**, since from the angular data we can obtain $A_{L\,M}$ knowing $B_{L\,M'}$ (theoretically computable)

For $M' = 0$,

$$\int d\Omega\ \left[\frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa}\right] Y_L^{M\,*}(\Omega) = \frac{B_{L\,M'}(\bar{\kappa},\kappa)^*}{4\pi}A_{L\,M}(\bar{\kappa})$$

Factorizable case

Let us consider a scattering process of the form

$$(\bar{A}_1\,\bar{B}_1\,\bar{C}_1\ldots)\,(\bar{A}_2\,\bar{B}_2\,\bar{C}_2\ldots)\ldots\,(\bar{A}_N\,\bar{B}_N\,\bar{C}_N\ldots) \to (A_1\,B_1\,C_1\ldots)\,(A_2\,B_2\,C_2\ldots)\ldots\,(A_N\,B_N\,C_N\ldots)$$

The production matrix Γ and the diff. cross section are in this case

$$\Gamma = \bigotimes_{j=1}^{N} \Gamma_j(R_j) \implies \frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa} = \mathcal{N}\, Tr\left\{\rho\left(\bigotimes_{j=1}^{N}\Gamma_j^T(R_j)\right)\right\}$$

In this context instead of using $\{T^L_M\}$, it is convenient to use the factorized one:

$$\left\{\bigotimes_{j=1}^{N} T^{L_j}_{M_j}\right\}_{L_j,\,M_j} \implies \rho = \frac{1}{d}\sum_{L_1\,L_2\ldots L_N}\ \sum_{M_1\,M_2\ldots M_N} A_{L_1 M_1,\,L_2 M_2,\ldots,\,L_N M_N}\ \bigotimes_{j=1}^{N} T^{L_j}_{M_j}$$

Applying a similar reasoning than for the general case

$$\int d\Omega \left[\frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa}\right] \left[\prod_{j=1}^{N}\left(\frac{2L_j+1}{4\pi}\right)^{1/2} D^{L_j}_{M_j\, M'_j}(\Omega_j)\right] = \frac{\prod_{j=1}^{N} B_{L_j\, M'_j}(\bar{\kappa},\kappa)^*}{4\pi} A_{L_1\, M_1,\, L_2\, M_2, \ldots,\, L_N\, M_N}(\bar{\kappa})$$

Furthermore, when all the processes are decays ($N = m$)

$$\bar{A}_1\, \bar{A}_2 \ldots \bar{A}_m \to (A_1\, B_1\, C_1 \ldots)\,(A_2\, B_2\, C_2 \ldots) \ldots\ (A_m\, B_m\, C_m \ldots),$$

$L_j = L_{j_0}\delta_{j\, j_0} \longrightarrow A_{L_1\, M_1,\, L_2\, M_2, \ldots,\, L_m\, M_m}$ spin polarization vector of particle j_0

$L_j = 0$ except for $L_{j_1}, L_{j_2} \longrightarrow A_{L_1\, M_1,\, L_2\, M_2, \ldots,\, L_m\, M_m}$ spin correlation matrix of particles j_1 and j_2

$\vdots$

$L_j \neq 0 \quad \forall j \longrightarrow A_{L_1\, M_1,\, L_2\, M_2, \ldots,\, L_m\, M_m}$ spin correlation tensor of the whole system

Weyl-Wigner-Moyal formalism

We define the generalized Wigner Q symbol of an operator A with respect to F by

$$\Phi_A^Q \equiv Tr\{A\,F\}$$

Here F is a Positive Operator-Valued Measure, i.e. an element of a set of positive semi-definite hermitian operators $\{F_l = \mathscr{K}_l^\dagger \mathscr{K}_l\}_l$, where $\sum_l \mathscr{K}_l^\dagger \mathscr{K}_l = \sum_l F_l = 1$

We can identify $\mathscr{K}_{\lambda\bar{\lambda}} \propto \mathscr{M}_{\lambda\bar{\lambda}} \implies F_{\bar{\lambda}\bar{\lambda}'} = \Gamma^T_{\bar{\lambda}\bar{\lambda}'} \implies \Phi_A^Q = Tr\{A\,\Gamma^T\}$

We also define the generalized Wigner P symbol of an operator A as the function Φ_A^P (not unique) such that for any other operator B (or for a basis of operators)

$$Tr\{A\,B\} = \frac{d}{8\pi^2 \bar{K}K} \int d\Omega\, \Phi_B^Q\, \Phi_A^P$$

Applying the definition of the Q symbol for the operator T^L_M

$$\Phi^Q_{LM}(\Omega, \bar{\kappa}, \kappa) = Tr\{T^L_M \Gamma^T(\Omega, \bar{\kappa}, \kappa)\} = \sum_{M'} \tilde{B}_{LM'}(\bar{\kappa}, \kappa)^* D^L_{MM'}(\Omega)^*$$

Regarding the P symbol for $(T^L_M)^\dagger$, it is easy to check that a suitable family is given by

$$\Phi^P_{\hat{L}\hat{M},\hat{M}'}(\Omega, \bar{\kappa}, \kappa)^\dagger = \frac{4\pi}{B_{\hat{L}\hat{M}'}(\bar{\kappa}, \kappa)^*} \left(\frac{2\hat{L}+1}{4\pi} \right)^{1/2} D^{\hat{L}}_{\hat{M}\hat{M}'}(\Omega)$$

By definition of P symbol we have

$$\frac{d}{8\pi^2 \bar{K} K} \int d\Omega\, \Phi^Q_\rho \left(\Phi^P_{\hat{L}\hat{M},\hat{M}'} \right)^\dagger = Tr\left\{ \rho \left(T^{\hat{L}}_{\hat{M}} \right)^\dagger \right\} = A_{\hat{L}\hat{M}}$$

Finally, from the diff. cross section we get

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa} = \frac{d}{8\pi^2\bar{K}K} Tr\{\rho\,\Gamma^T\} = \frac{d}{8\pi^2\bar{K}K}\Phi^Q_\rho \implies \Phi^Q_\rho = \frac{8\pi^2\bar{K}K}{d}\frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa}$$

Using this expression as well as the explicit formula for $\Phi^P_{\hat{L}\,\hat{M},\,\hat{M}'}(\Omega,\bar{\kappa},\kappa)^\dagger$, we deduce after some simplifications that

$$\int d\Omega\, \left[\frac{1}{\sigma}\frac{d\sigma}{d\Omega\, d\bar{\kappa}\, d\kappa}\right]\left(\frac{2L+1}{4\pi}\right)^{1/2} D^L_{M\,M'}(\Omega) = \frac{B_{L\,M'}(\bar{\kappa},\kappa)^*}{4\pi} A_{L\,M}(\bar{\kappa}),$$

recovering the exact same result as with the other formalism.

Conclusions

- We have developed a practical way of performing the Quantum Tomography of the initial helicity state ρ in general scattering processes.

- The method is based on computing the coefficients of the expansion over $\{T^L_M\}$ by averaging the angular distribution of the final particles under Wigner D-matrices kernels.

- We have further given explicit formulas for the angular dependence of both a generalization of the production/decay matrix Γ and of the diff. cross section, elaborating on the factorizable case.

- We have re-derived everything using the Weyl-Wigner-Moyal formalism.

Thank you for your attention!