Universal approximation and error bounds for quantum neural networks

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joint work with Antoine Jacquier

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Quantum neural networks

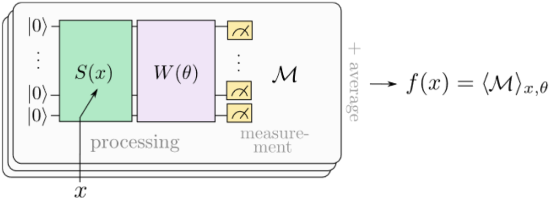
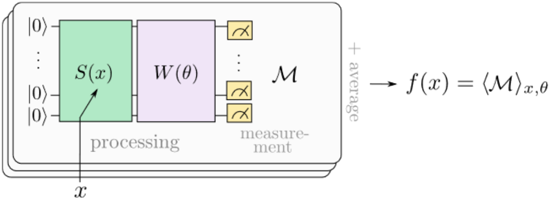


Figure: Variational quantum circuit *(Schuld & Petruccione, 2021)*

- Learning using quantum computers: variational quantum circuits have become a central focus of research
- Training facilitated by software (e.g. PennyLane, TensorFlow Quantum, ...)
- First universal approximation results available *(A. Pérez-Salinas et al., 2020), ...*
  - Approximation error? (approximation accuracy $\leftrightarrow$ circuit size)
  - Dimensionality? Quantum advantage?

Quantum neural networks



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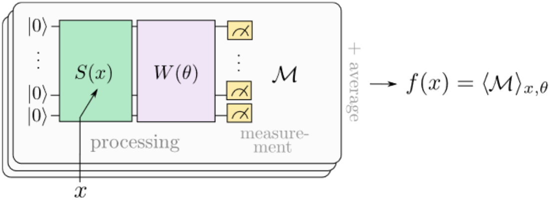

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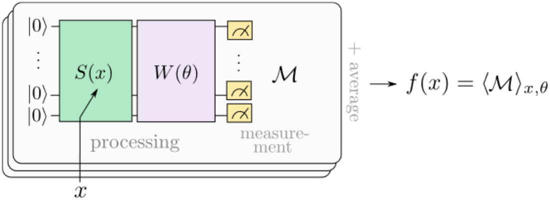

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Quantum neural network approximation bound

Fix a probability measure μ on $\mathbb{R}^d$ to measure the error.

Theorem (G. & Jacquier (2023))

For any $n \in \mathbb{N}$ we can construct a parametrized quantum circuit operating on $\log_2(4n) + 1$ qubits with the following property: for any integrable and continuous $f : \mathbb{R}^d \to \mathbb{R}$ with $\int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{\xi})| \mathrm{d}\boldsymbol{\xi} < \infty$ there exist circuit parameters $\boldsymbol{\theta}$ such that the circuit output $f_{n,\boldsymbol{\theta}}$ satisfies

$$\left(\int_{\mathbb{R}^d} |f(\boldsymbol{x}) - f_{n,\boldsymbol{\theta}}(\boldsymbol{x})|^2 \mu(\mathrm{d}\boldsymbol{x}) \right)^{1/2} \leq \frac{\int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{\xi})| \mathrm{d}\boldsymbol{\xi}}{\sqrt{n}}.$$

- No curse of dimensionality occurs.
- The number of qubits is logarithmic in n.
- Analogous results for $\sup_{\boldsymbol{x} \in [-R,R]^d}$-error.

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Circuit construction: Define the following gates acting on a single qubit

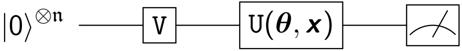
$$\mathtt{U}_1^{(i)} := \mathtt{H}\,\mathtt{R_z}\left(-b^i\right)\mathtt{R_z}\left(-a_d^i x_d\right)\cdots\mathtt{R_z}\left(-a_1^i x_1\right)\mathtt{H}, \quad \mathtt{R_z}(\alpha) := \begin{pmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{pmatrix}$$

$$\mathtt{U}_2^{(i)} := \begin{pmatrix} \cos\left(\frac{\gamma^i}{2}\right) & -\sin\left(\frac{\gamma^i}{2}\right) \\ \sin\left(\frac{\gamma^i}{2}\right) & \cos\left(\frac{\gamma^i}{2}\right) \end{pmatrix}$$

with $\mathtt{H}$ the Hadamard gate. Now let $n_0 \in \mathbb{N}_0$ such that $\log_2(4n + n_0) \in \mathbb{N}$, write $\boldsymbol{\theta} = (\boldsymbol{a}^{(i)}, b^{(i)}, \gamma^{(i)})_{i=1,\dots,n} \in \boldsymbol{\Theta} := (\mathbb{R}^d \times \mathbb{R} \times [0, 2\pi])^n$ and define

$$\mathbf{U} := \mathbf{U}(\boldsymbol{\theta}, \boldsymbol{x}) := \begin{bmatrix} \mathtt{U}_1^{(1)} \otimes \mathtt{U}_2^{(1)} & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times 4} & \cdots & \mathbf{0}_{4\times 4} & \mathbf{0}_{4\times n_0} \\ \mathbf{0}_{4\times 4} & \mathtt{U}_1^{(2)} \otimes \mathtt{U}_2^{(2)} & \mathbf{0}_{4\times 4} & \cdots & \mathbf{0}_{4\times 4} & \vdots \\ \vdots & & \ddots & & \vdots & \vdots \\ \mathbf{0}_{4\times 4} & \cdots & \mathbf{0}_{4\times 4} & \mathtt{U}_1^{(n-1)} \otimes \mathtt{U}_2^{(n-1)} & \mathbf{0}_{4\times 4} & \vdots \\ \mathbf{0}_{4\times 4} & \cdots & \cdots & \mathbf{0}_{4\times 4} & \mathtt{U}_1^{(n)} \otimes \mathtt{U}_2^{(n)} & \mathbf{0}_{4\times n_0} \\ \mathbf{0}_{n_0\times 4} & \cdots & \cdots & \cdots & \mathbf{0}_{n_0\times 4} & \mathbf{1}_{n_0\times n_0} \end{bmatrix},$$

Let $N = 4n + n_0$ and $\mathtt{V} \in \mathbb{C}^{N\times N}$ a unitary matrix that maps $|0\rangle^{\otimes \mathfrak{n}}$ to the state $|\psi\rangle = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}|4i\rangle$. Then we consider



Thank you!