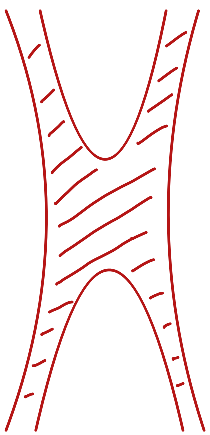
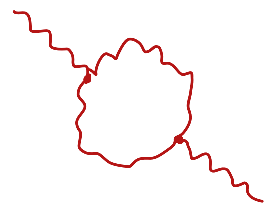
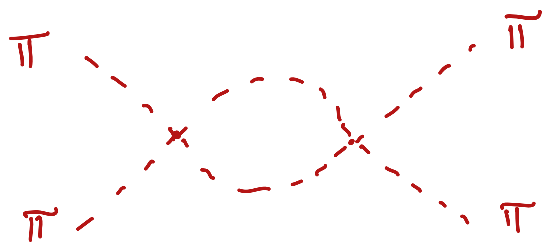
Real World Amplitudes from Curves on Surfaces



Carolina Figueiredo



Strings 2024
CERN
June 2024



w/ N. Arkani-Hamed
Q. Cao, J. Dong
S. He

Outline...

I. Surface Integral Formalism vs. Standard Worldsheet

[Built around manifesting singularities].

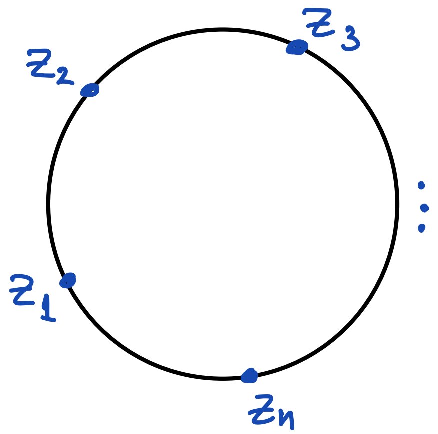
II. Revealing Qualitatively New Features

* Hidden Factorizations away from Poles + Zeros.

* Kinematic connection between different ths.: $\mathrm{Tr}\,\varphi^3 \leftrightarrow \mathrm{NLSM} \leftrightarrow \mathrm{YM}$

* "Perfect" Integrands for real world amplitudes.

Bosonic String Theory $\xrightarrow{\alpha' p^2 \ll 1}$ Field Theory: $\mathrm{Tr}\,\varphi^3$



$z_1 < z_2 < z_3 < \dots < z_n$

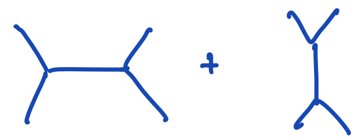
$$A_n = \int \frac{d^n z}{SL(2,\mathbb{R})} \langle e^{iP_1 X(z_1)} \dots e^{iP_n X(z_n)} \rangle$$

$$= \int \frac{d^n z}{SL(2,\mathbb{R})} \frac{1}{z_{12} z_{23} \cdots z_{n,1}} \prod_{i<j} z_{ij}^{2\alpha' p_i \cdot p_j}$$

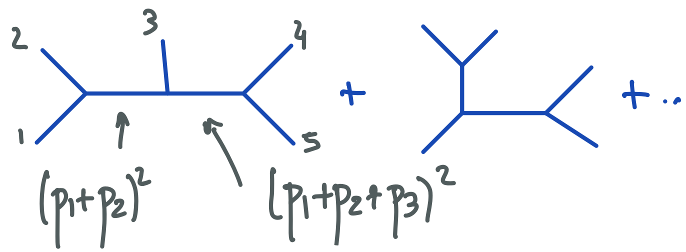
$$\mathcal{L}_{\mathrm{Tr}\varphi^3} = \frac{1}{2} \mathrm{Tr}(\partial\varphi)^2 + \frac{g}{3} \mathrm{Tr}(\varphi^3)$$

$$A^{\mathrm{Tr}\varphi^3} = \sum_{\mathcal{D},\ \text{diagrams}} \left(\prod_{P \in \mathcal{D}} \frac{1}{P^2} \right)$$

$A_4 =$



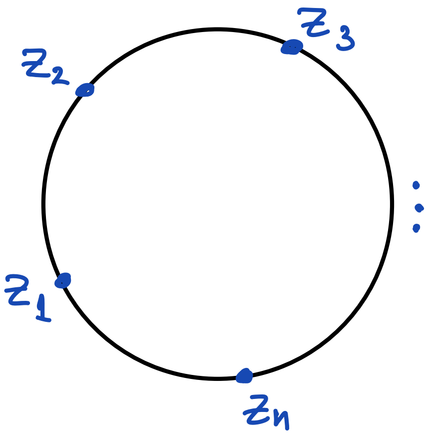
$A_5 =$



Singularities $\equiv$

$$(p_i + p_{i+1} + \dots + p_{j-1})^2 = 0$$

Bosonic String Theory $\xrightarrow{\alpha' p^2 \ll 1}$ Field Theory: Tr φ^3



$z_1 < z_2 < z_3 < \dots < z_n$

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Kin. dependence poles NOT manifest

z_{ij} $\longleftrightarrow$

Need Blow Ups

$$\mathcal{L}_{\mathrm{Tr}\varphi^3} = \frac{1}{2} \mathrm{Tr}(\partial\varphi)^2 + \frac{g}{3} \mathrm{Tr}(\varphi^3)$$

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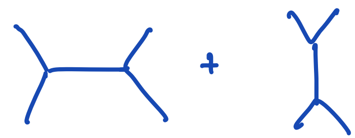
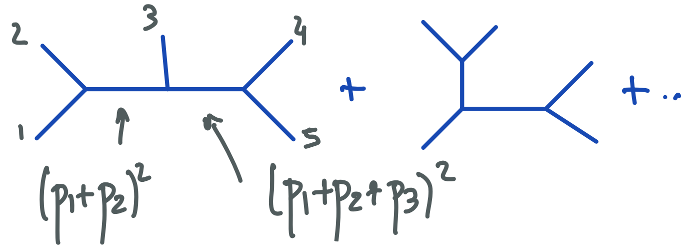
$A_4 =$ 

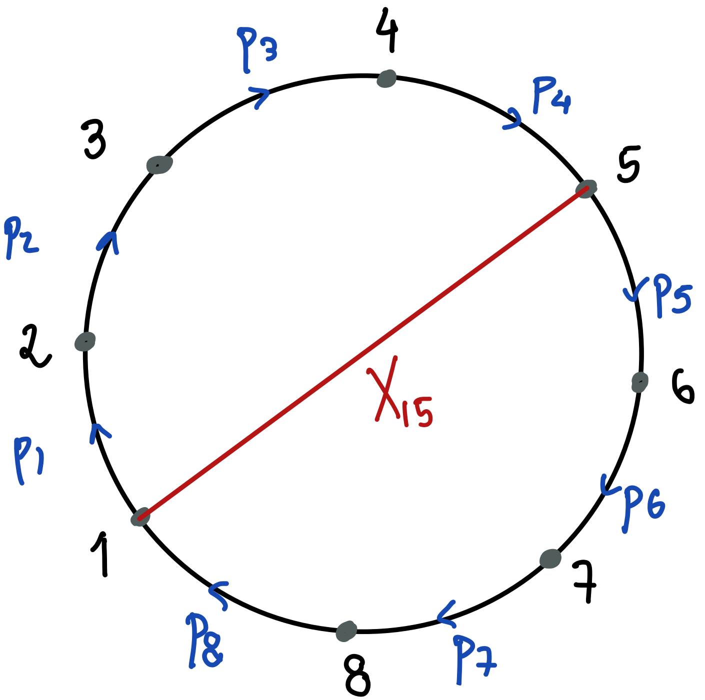
$A_5 =$ 

Singularities $\equiv$

$$(p_i + p_{i+1} + \dots + p_{j-1})^2 = 0$$

[Kinematics and curves on Surfaces]

Color-ordered amplitude: Singularities $\equiv (p_i + p_{i+1} + \cdots + p_{j-1})^2 = 0$



$$\Vert\!\!\!\Vert$$

Curves, $\mathcal{C}$, on the Surface. S

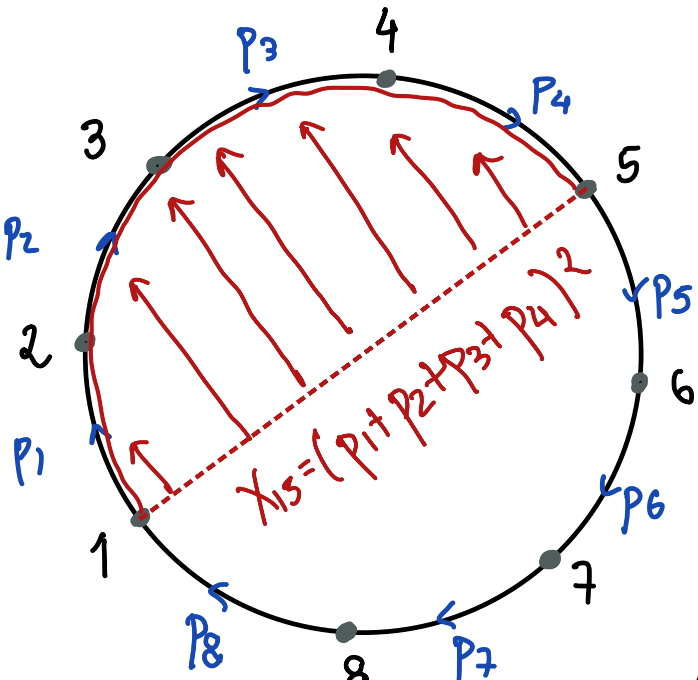
$X_{\mathcal{C}}$ = Kinematics associated to the curve read off by Homology!

$$\Vert$$

$$X_{i,j} = (p_i + p_{i+1} + \cdots + p_{j-1})^2$$

[Momentum $\leftrightarrow$ Homology]

Color-ordered amplitude: Singularities $\equiv (p_i + p_{i+1} + \cdots + p_{j-1})^2 = 0$



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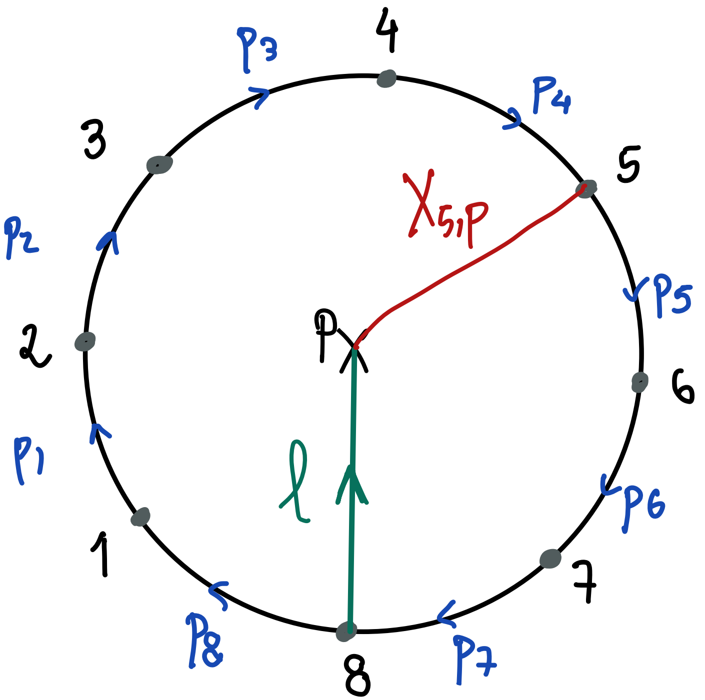
Curves, $\mathcal{C}$, on the Surface. $\mathcal{S}$

$X_{\mathcal{C}}$ = Kinematics associated to the curve read off by Homology!

$A^{\mathrm{Tr}\,\phi^3}\left[X_{\mathcal{C}} \equiv X_{i,j}\right]$ manifest singularities. ✓

[Momentum ↔ Homology]

Loop-level : S punctures + more interesting topology.

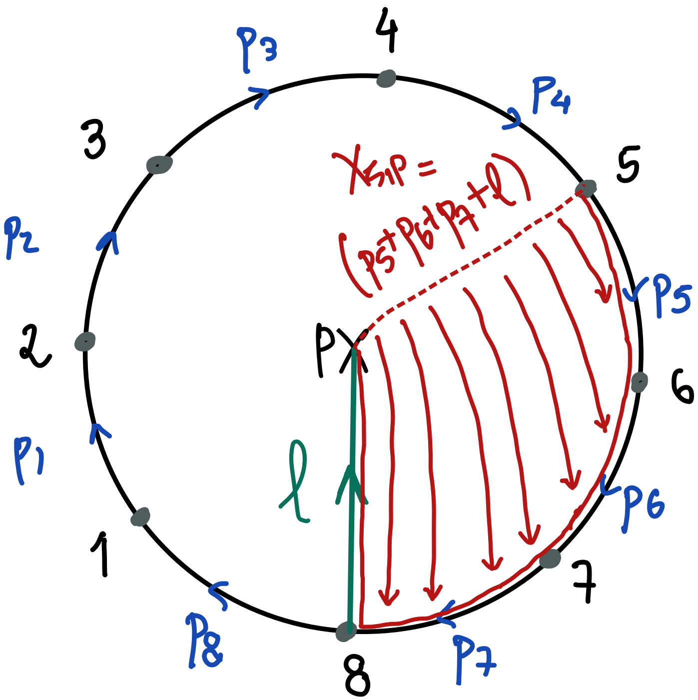


To each curve $\mathcal{C}$ on S associate:

$$X_{\mathcal{C}} = \text{Homology}!$$

[Momentum ↔ Homology]

Loop-level : $\mathcal{S}$ punctures + more interesting topology.



To each curve $\mathcal{C}$ on $\mathcal{S}$ associate:

$$X_{\mathcal{C}} = \text{Homology!}$$

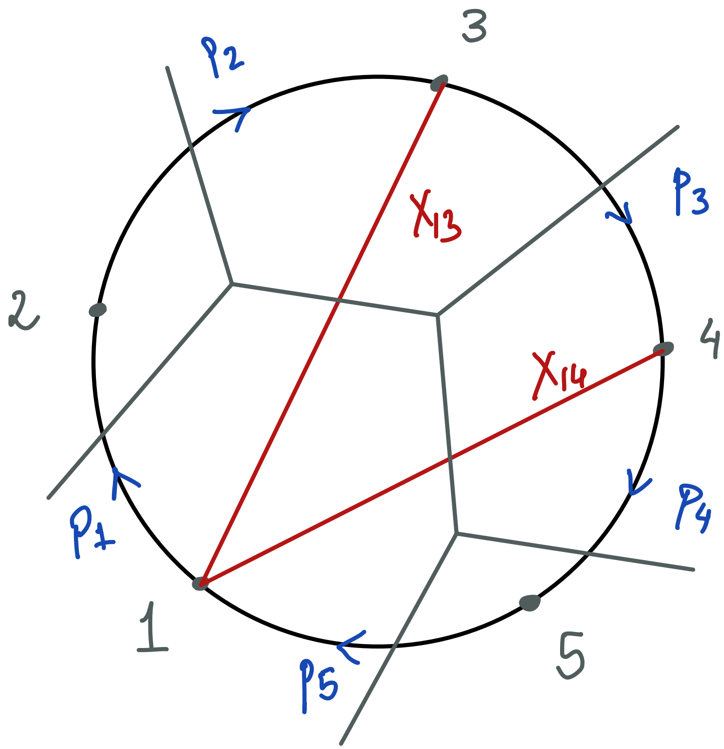
$$|||$$

Propagators in Feynman Diagrams.

$A^{\mathrm{Tr}\,\varphi^3}\left[X_{\mathcal{C}} \equiv X_{i,j}\right]$ manifest singularities. ✓

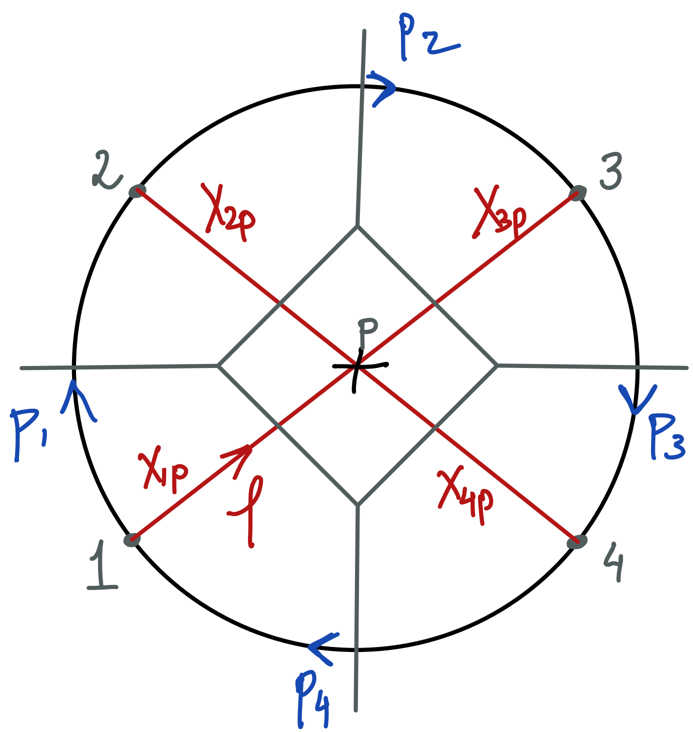
[Feynman Diagrams as Triangulations of S]

5-point Tree



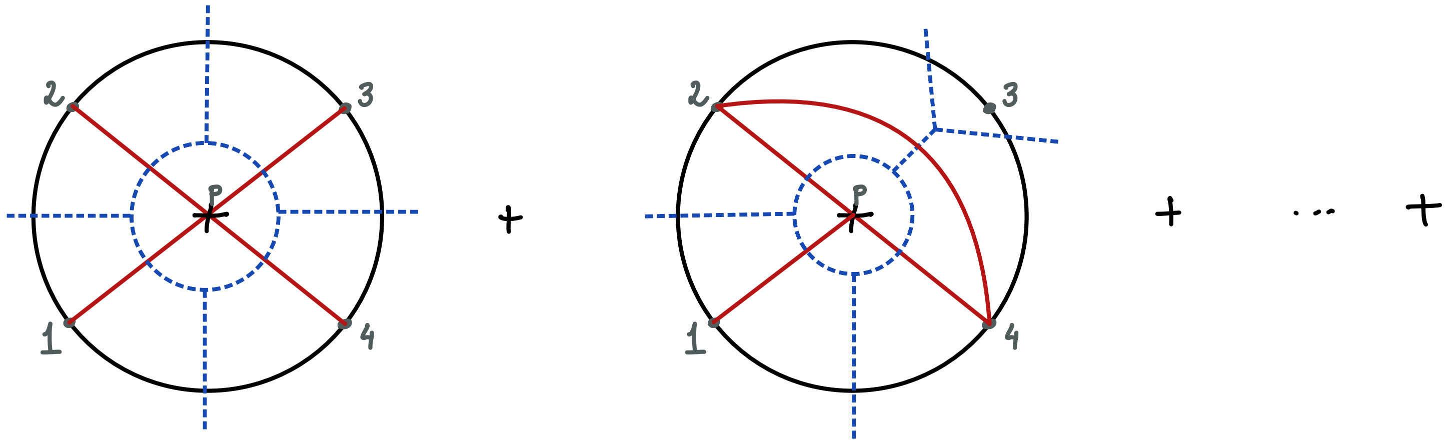
$$\frac{1}{X_{1,3}\,X_{1,4}}$$

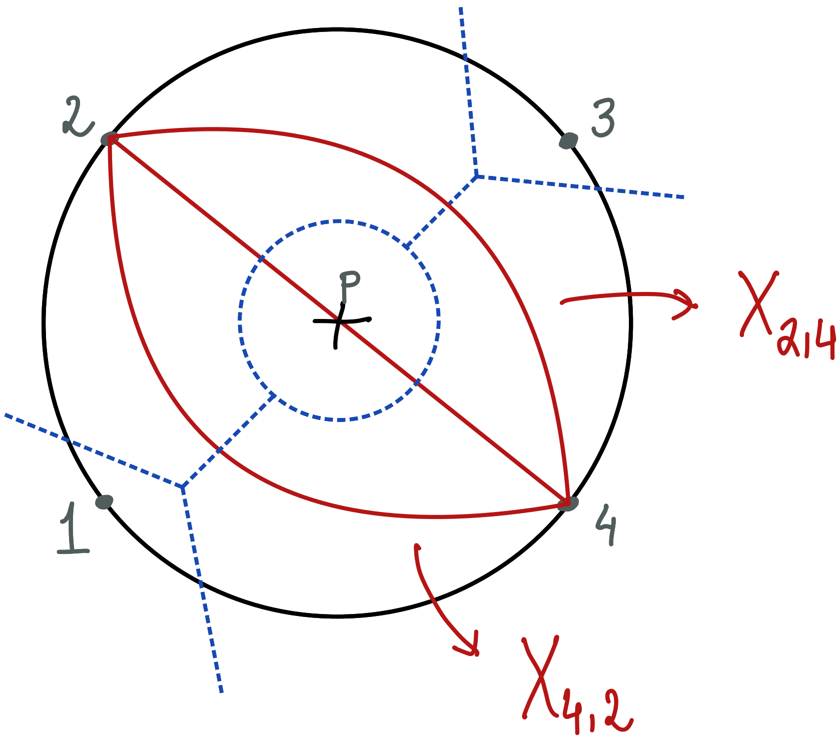
4-point 1-loop



$$\frac{1}{X_{1,p}\,X_{2,p}\,X_{3,p}\,X_{4,p}}$$

[Amplitude $\leftrightarrow$ $\sum$ Triangulations of S]





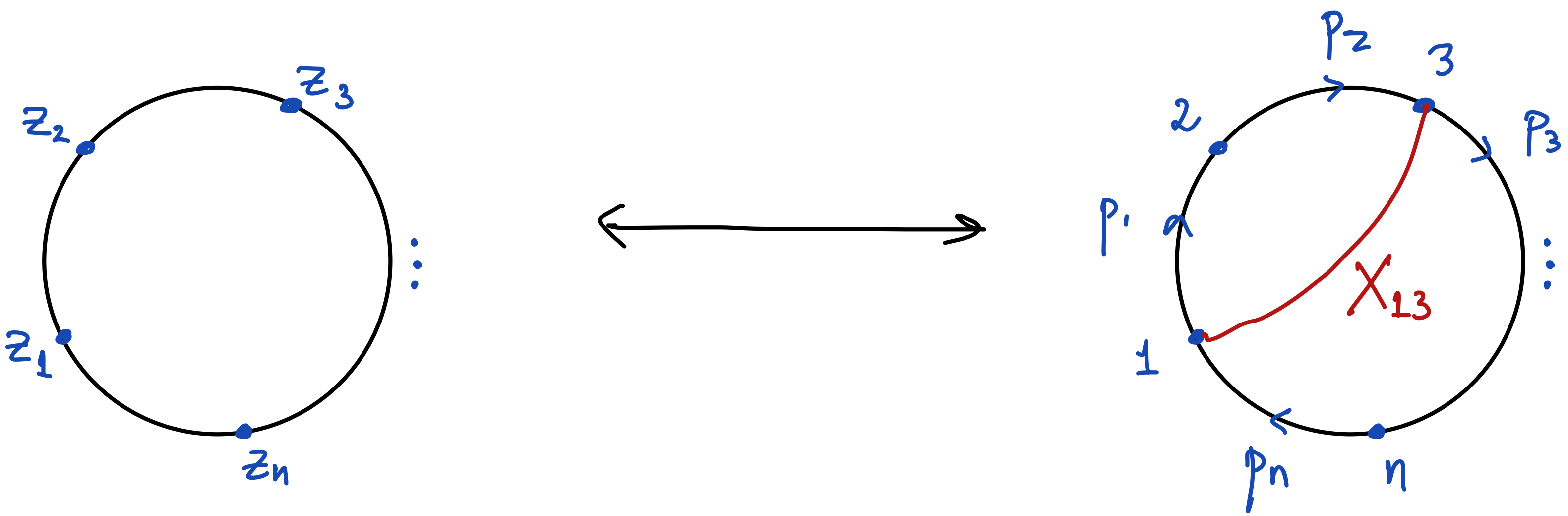
Keep Curves up to **Homotopy!**

$X_{2,4} \neq X_{4,2}$ as CURVES

Generalizing away from Momenta $\leftrightarrow$ Homology.

$(X_{24} = X_{42})$.

Q: Can we formulate string amplitudes in a way that:



Points in Worldsheet

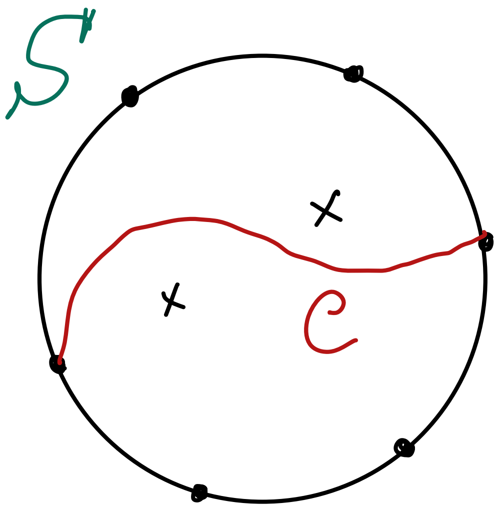
$z_i \Leftrightarrow p_i^{\mu}$

Curves on S

$\Updownarrow$

Manifest location of poles.

Surface Integral Formalism



u-variables:

Positive coordinates y

$$u_C[y] = \frac{F^{1,2}(y)\, F^{2,1}(y)}{F^{1,1}(y)\, F^{2,2}(y)}$$

Counting Problem associated to C on S.

$$A = \int_0^{+\infty} \underbrace{\left(\prod \frac{dy}{y}\right)}_{\text{dlog form}} \times \prod_{C \in S} u_C[y]^{\alpha' X_C}$$

Product over all curves on S

[N. Arkani-Hamed, H. Frost, G. Salvatori, P-G. Plamondon, H. Thomas 23']

Surface Integral Formalism: Tree-level.

$$A_n^{\text{tree}} = \int \underbrace{\frac{d^n z}{SL(2,\mathbb{R})} \times \frac{1}{z_{12}\, z_{23} \cdots z_{n1}}}_{= \prod_{i=1}^{n-3} \frac{dy_i}{y_i}} \times \underbrace{\prod_{i<j} z_{ij}^{-2\alpha' p_i \cdot p_j}}_{\prod_{i<j} u_{ij}^{\alpha' X_{ij}}} \qquad \text{[Worldsheet]}$$

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$$u_{ij} = \frac{z_{i-1,j}\, z_{i,j-1}}{z_{i,j}\, z_{i-1,j-1}}$$

Manifest all poles
$[u_{ij} \to 0]$

Surface Integral Formalism: Tree-level.

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$$= \underbrace{\prod_{i=1}^{n-3} \frac{dy_i}{y_i}}_{\text{Blows up ALL Singularities}} \times \underbrace{\prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\text{Manifest all poles } [u_{ij} \to 0]}$$

$$u_{ij} = \frac{z_{i-1,j}\, z_{i,j-1}}{z_{i,j}\, z_{i-1,j-1}}$$

Surface Integral Formalism: Tree-level.

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Blows up ALL Singularities

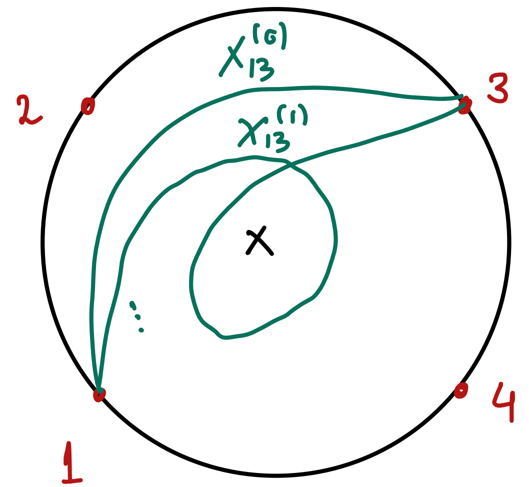
Manifest all poles $[u_{ij} \to 0]$

* No Gauge Redundancy/Fixing
* Trivial to extract Field Theory Limit.

Surface Integral Formalism: Loop-level.

$$A_n = \int_0^{+\infty} \underbrace{\prod \frac{dy_p}{y_p}}_{\text{dlog form}} \times \underbrace{\prod_{c\in S} u_c[y_p]^{\alpha' X_c}}$$

$\longrightarrow$ Now ∞ many Curves!



∞ self intersection

$$\prod_{c^{(0)}\in S} u_c^{\alpha' X_{c^{(0)}}} \times \underbrace{\prod_{c^{(q)}\in S} u_c^{\alpha' X_{c^{(q)}}}}_{\infty\text{ product}}$$

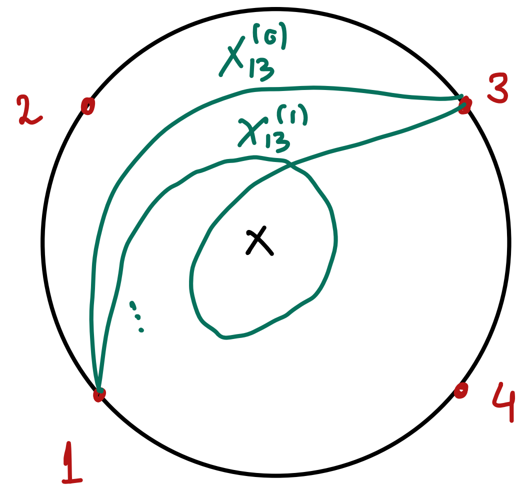
$$|||$$

1-loop Bosonic String.

Surface Integral Formalism: Loop-level.

$$A_n = \int_0^{+\infty} \underbrace{\prod \frac{dy_p}{y_p}}_{\text{dlog form}} \times \underbrace{\prod_{C \in S'} u_C[y_p]^{\alpha' X_C}}$$

Now ∞ many Curves!



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$$\prod_{C^{(0)} \in S} u_C^{\alpha' X_{C^{(0)}}} \times \cancel{\underbrace{\prod_{C^{(q)} \in S'} u_C^{\alpha' X_{C^{(q)}}}}_{\infty \text{ product}}}$$

@ low-energies:

$(\mathrm{Tr}\,\varphi^3)$

Surface Integral Formalism: Loop-level.

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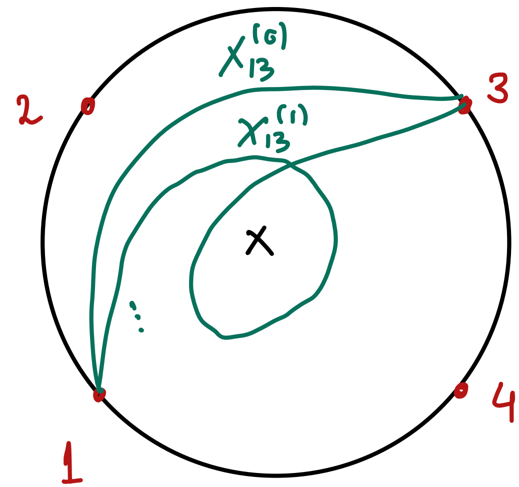
Now ∞ many Curves!



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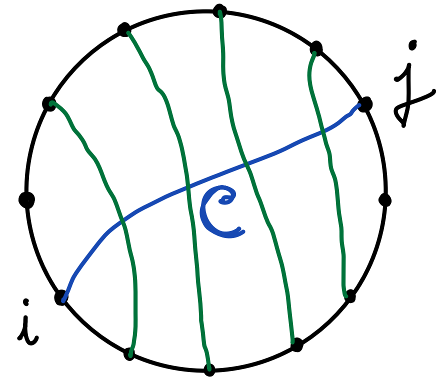
@ low-energies: $(\mathrm{Tr}\,\varphi^3)$

$$\int_0^{+\infty} \prod \frac{dy_p}{y_p} \prod_{C^{(0)}\in S} u_C^{\alpha' X_{C^{(0)}}}$$

* Much Simpler!
* "Stringy" UV regularization

Factorization $\longleftrightarrow$ U-variables

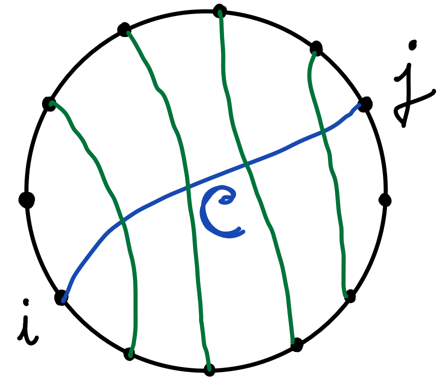


$$\boxed{u_e + \prod_{e' \in S} u_{e'}^{\mathrm{Int}(e';e)} = 1}$$

U- equations

$u_e \geqslant 0 \Rightarrow u_e \in [0,1]$

Factorization $\leftrightarrow$ U-variables



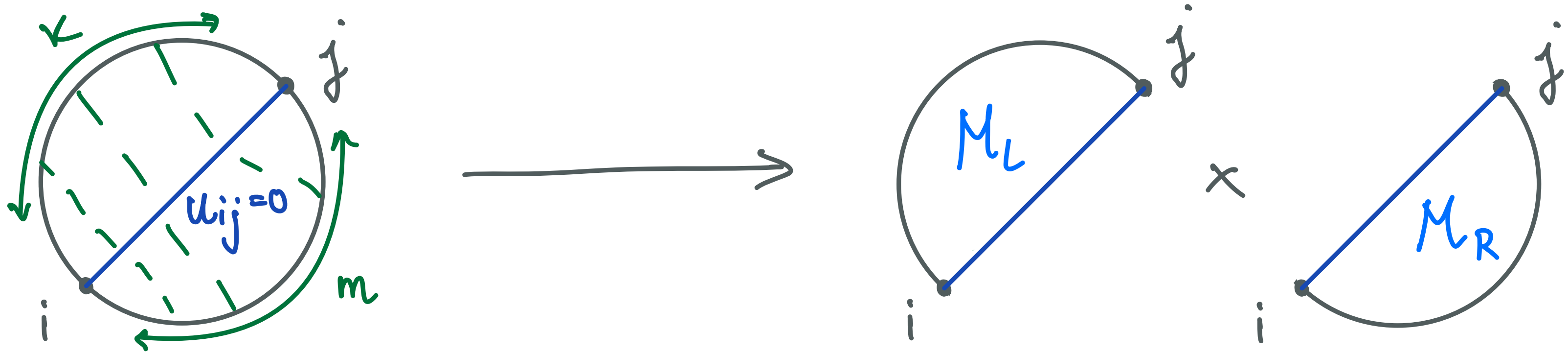
$$\boxed{u_e + \prod_{e' \in S} u_{e'}^{\mathrm{Int}(e';e)} = 1}$$

U-equations $\quad u_e \geqslant 0 \Rightarrow u_e \in [0,1]$

* **Factorization:** $[X_{ij} = -n \Rightarrow \text{singularity } u_{ij}=0]$

$$u_{ij} \to 0 \Rightarrow \forall_{(km)\text{ incomp. }(ij)} \; u_{km} \to 1 \qquad [\text{Binary}]$$

$$\prod_c u_c^{\alpha' X_c}$$



Revealing New Features.

* Hidden Factorizations away from poles

  \+ Zeros of string/particle amplitudes

[2312.16282, 2405.09608 w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

New Kind of Factorizations ($\Rightarrow$ Zeros)

$$\mathcal{A}_S\left[\{p_i \cdots p_j \to 0\}\right] \longrightarrow \mathcal{A}_{S_1} \times \mathcal{A}_{S_2}$$

w/ i, j not-adjacent

(no poles)

New Kind of Factorizations ($\Rightarrow$ Zeros)

$$\mathcal{A}_S\left[\{p_i \cdot p_j \to 0\}\right] \longrightarrow \mathcal{A}_{S_1} \times \boxed{\mathcal{A}_{S_2}}$$

w/ i, j not-adjacent

(no poles)

$\longrightarrow 0$

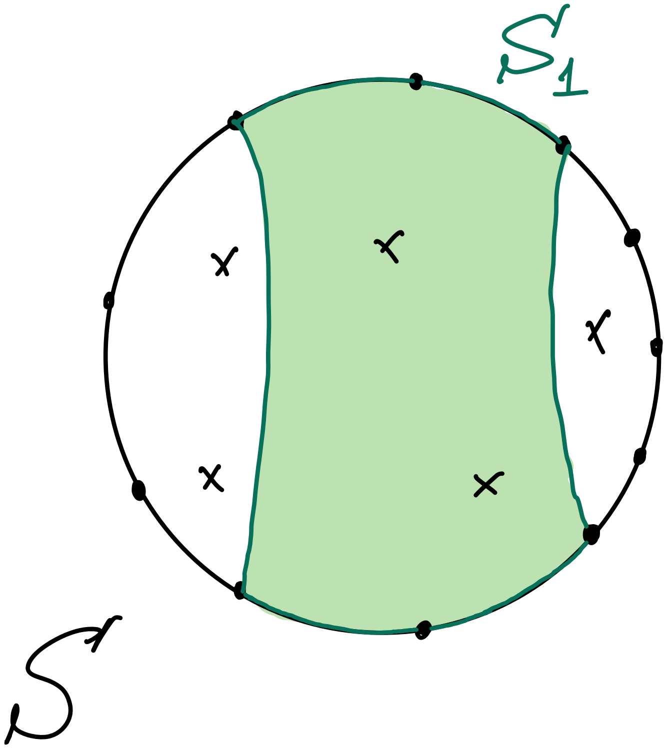
$p'_i \cdot p'_j \to 0$

SPLITs: AWAY FROM Poles (Near Zeros)

U-variables for Subsurfaces

Simple fact about u-variables!

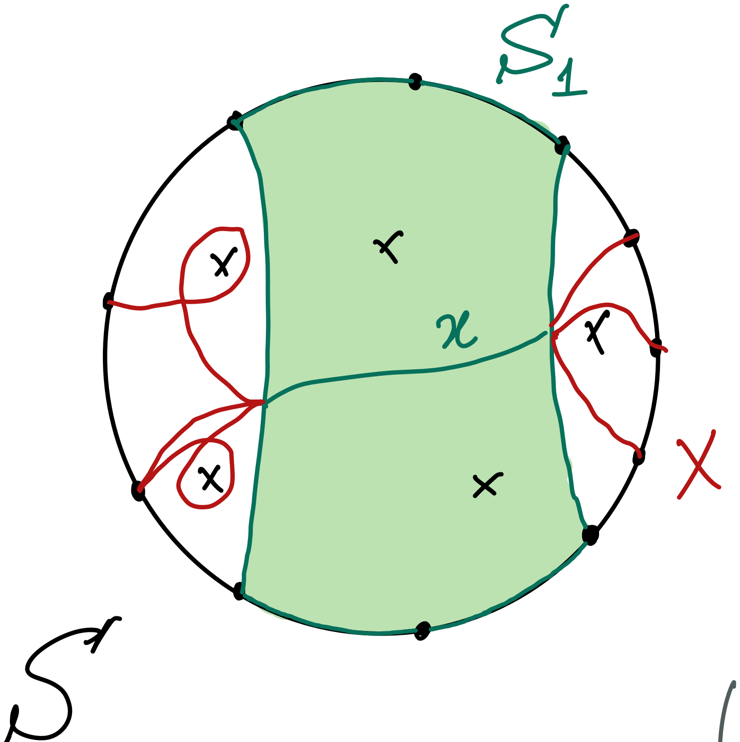
u-variables for subsurface $S_1' \subset S'$ can be written in terms of u-variables of S'



U-variables for Subsurfaces

Simple fact about u-variables !

u-variables for subsurface $S_1 \subset S$ can be written in terms of u-variables of S

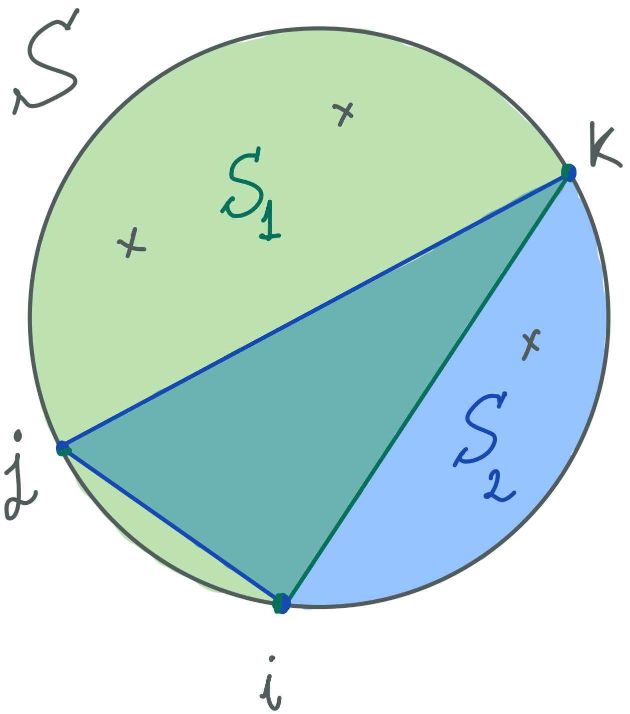


$$u_x = \prod U_X^{\#(x \subset X)}$$

Extension Formula

(all ways of extending x into a curve in S)

Split Factorization: Choose $\mathcal{S}_1$ and $\mathcal{S}_2$ such that all curves X in $\mathcal{S}$ belong to at least one of subsurfaces.

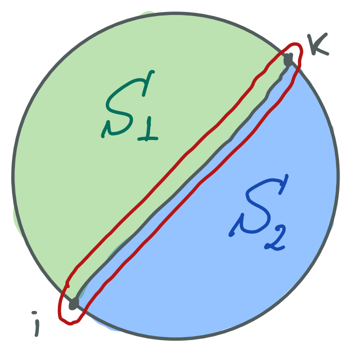


$$A_{\mathcal{S}_1} = \int \prod \frac{dy}{y} \times \prod_{c_1 \in \mathcal{S}_1} u_{c_1}^{\alpha' x_{c_1}} \to \text{Kinematics of } \mathcal{S}_1$$

$$A_{\mathcal{S}_2} = \int \prod \frac{dy}{y} \times \prod_{c_2 \in \mathcal{S}_2} u_{c_2}^{\alpha' y_{c_2}} \to \text{Kinematics of } \mathcal{S}_2$$

OVERLAP ON TRIANGLE!

[



Factorization.

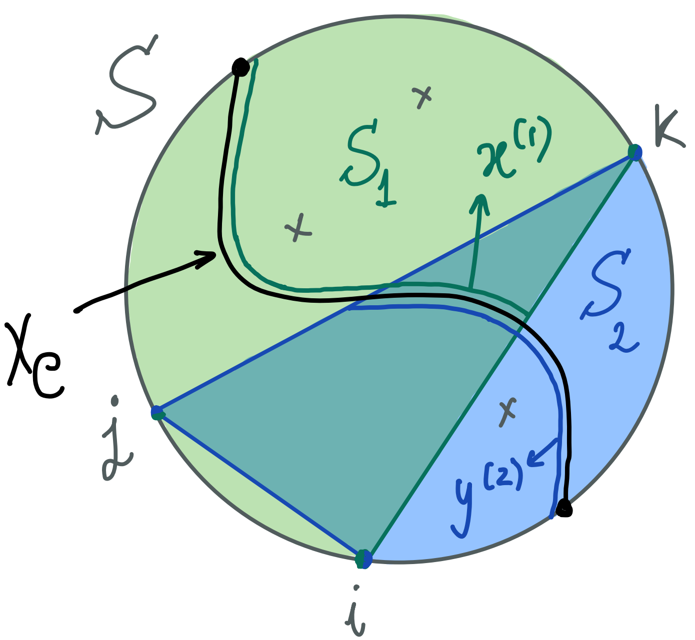
$X_{i,k}$ NOT in $\mathcal{S}_1 / \mathcal{S}_2$.

]

Extension Formula ⇒ Split Kinematics

$$\mathcal{A}_{S_1}\times\mathcal{A}_{S_2}=\left(\int\prod\frac{dy}{y}\prod_{c_1\in S_1}u_{c_1}^{\alpha' x_{c_1}}\right)\times\left(\int\prod\frac{dy}{y}\prod_{c_2\in S_2}u_{c_2}^{\alpha' y_{c_2}}\right)=\int\prod\frac{dy}{y}\prod_{c\in S}u_c^{\alpha'\tilde{X}_c}$$

$$\text{w/}\quad \tilde{X}_c=\sum_{c_1\,c_2}[\#c_1\subset c]\,x_{c_1}+[\#c_2\subset c]\,y_{c_2}$$



$$\tilde{X}_c\to x^{(1)}+y^{(2)}$$

$$\left(\Rightarrow\ \tilde{X}_c\neq 0\ \text{NO POLES}\right)$$

$$\mathcal{A}_S(\tilde{X}_c)\to\mathcal{A}_{S_1}(x_{c_1})\times\mathcal{A}_{S_2}(y_{c_2})$$

From Splits to Zeros:

Consider a split in which one subsurface is a 4 point:

$$A_4(s,t) \times A_{S_2} \quad \text{or} \quad A_4(s,t) \times A_{S_1} \times A_{S_2} \times \cdots$$

$$\left(\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \right)$$

which vanishes for $s+t=0$ (or $s+t=-n$)

$\Rightarrow$ Broad class of Zeros for particle / string amplitudes

* Standard Physical Interpretation

Zeros? Factorization near Zeros?

Split Factorizations & Zeros

* Surface Integrals [u-variables] $\xrightarrow{\alpha' X \ll 1}$ Field Theory
  (string amplitudes) $\mathrm{Tr}\,\phi^3$ amplitudes.

Q: Are they also present for other [coloured] theories ?

* Non-linear Sigma model (NLSM) ✓✓ (Zeros $\Downarrow$ Adler Zero)

* Non-SUSY Yang-Mills Theory ✓✓

Revealing New Features.

* Kinematic Connection between different theories: $\mathrm{Tr}\,\varphi^3 \leftrightarrow \mathrm{NLSM} \leftrightarrow \mathrm{YM}$

[2401.00041, 2401.05483, 2403.04826]
w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

Hint: NLSM, YM share all the Zeros (and Splits) found for $\mathrm{Tr}\varphi^3$.

Build simplest deformation of $\mathrm{Tr}\varphi^3$

$$\mathcal{A}^{\alpha'}_{\mathrm{Tr}\varphi^3}\left[\,X_{ij} \to X_{i,j} + \delta_{i,j}\,\right]$$

preserving ALL ZEROS.

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$$A^{\alpha'}_{\mathrm{Tr}\varphi^3}\left[X_{ij} \to X_{i,j} + \delta_{i,j}\right]$$

preserving ALL ZEROS.

UNIQUE SOLUTION!

$$\delta_{i,j} = \begin{cases} +\delta & (i,j) \text{ even} \\ -\delta & (i,j) \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Hint: NLSM, YM share all the Zeros (and Splits) found for $\mathrm{Tr}\varphi^3$.

Build simplest deformation of $\mathrm{Tr}\varphi^3$

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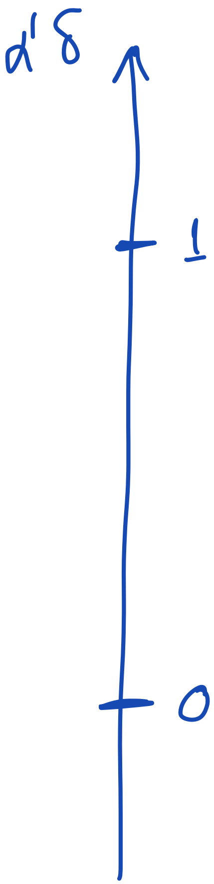
$$A^{\alpha'}_{\mathrm{Tr}\varphi^3}\left[X_{i,j} + \delta_{i,j}\right] = \int_0^{+\infty} \prod \frac{dy}{y} \prod u_{i,j}^{\alpha' X_{i,j}} \times \left(\frac{\prod u_{e,e}}{\prod u_{o,o}}\right)^{\alpha'\delta}$$

Only Changing Measure!

$$= \int_0^{+\infty} \prod \frac{dy}{y} \prod u_{i,j}^{\alpha' X_{i,j}} \times \left(\frac{1}{\prod y}\right)^{\alpha'\delta}$$

δ- Shifted $\mathrm{Tr}\,\varphi^3$

$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\mathrm{Tr}\,\varphi^3} \times \underbrace{\left(\frac{\prod u_{\text{even},\text{even}}}{\prod u_{\text{odd},\text{odd}}} \right)^{\alpha'\delta}}_{\delta\text{-shift}}$$



@ low energies

$\to \alpha'\delta = 0$ $\mathrm{Tr}\,\varphi^3$ theory

δ- Shifted $\text{Tr}\,\varphi^3$

$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\text{Tr}\,\varphi^3} \times \underbrace{\left(\frac{\prod u_{\text{even},\text{even}}}{\prod u_{\text{odd},\text{odd}}} \right)^{\alpha'\delta}}_{\delta\text{-shift}}$$

$\alpha'\delta$

1

$\alpha'\delta \in \mathbb{R}^+ \setminus \mathbb{Z}$ NLSM @ low energies

$0 \rightarrow \alpha'\delta = 0$ $\text{Tr}\,\varphi^3$ theory

δ-Shifted $\mathrm{Tr}\,\phi^3$

$$\mathcal{A}_{2n}[\alpha' X_{ij}] \propto \underbrace{\int \prod \frac{dy}{y} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}}_{\mathrm{Tr}\,\phi^3} \times \underbrace{\left(\frac{\prod u_{\text{even,even}}}{\prod u_{\text{odd,odd}}} \right)^{\alpha'\delta}}_{\delta\text{-shift}}$$

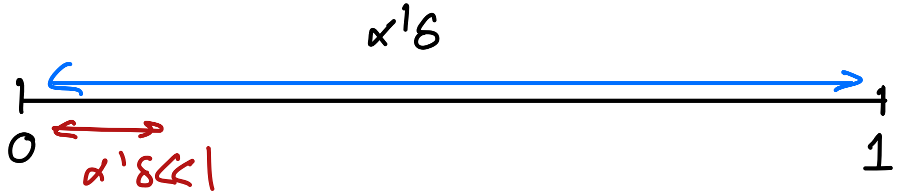
$\alpha'\delta$

$1 \to \alpha'\delta = 1$ YM theory

$\alpha'\delta \in \mathbb{R}^+ \setminus \mathbb{Z}$ NLSM @ low energies

$0 \to \alpha'\delta = 0$ $\mathrm{Tr}\,\phi^3$ theory

Claim:



$\Longrightarrow$ NLSM

$\alpha'\delta \ll 1$, @ low energies: $A_\delta^{\mathrm{tr}\varphi^3}\left[X_{ij} \to X_{ij} + \delta_{ij}\right]$

Ex. 4 pts

$$A_4^\delta = \frac{1}{X_{13}-\delta} + \frac{1}{X_{24}+\delta} \xrightarrow[X \ll \delta]{} 1/\delta^2 \underbrace{\left(X_{13}+X_{24}\right)}_{A_4^{NLSM}} + O\left(\delta^{-3}\right)$$

Claim:

$\alpha'\delta$

0 ——————————————— 1

$\alpha'\delta \ll 1$

$\Longrightarrow$ NLSM

$\alpha'\delta \ll 1$, @ low energies: $A_\delta^{\mathrm{tr}\phi^3}\left[X_{ij} \to X_{ij} + \delta_{ij}\right]$

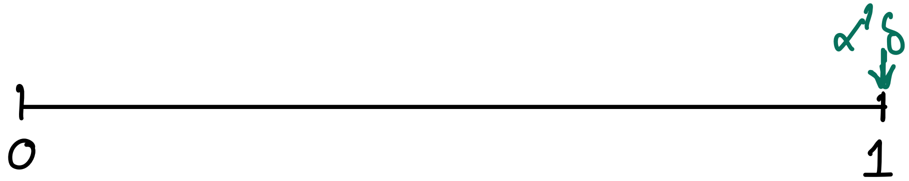
Ex. 4 pts

$$A_4^\delta = \frac{1}{X_{13}-\delta} + \frac{1}{X_{24}+\delta} \xrightarrow{X \ll \delta} 1/\delta^2 \underbrace{\left(X_{13}+X_{24}\right)}_{A_4^{NLSM}} + \mathcal{O}\left(\delta^{-3}\right)$$

Why NLSM?

* Factorization ✓
* Split Zeros $\Longrightarrow$ Adler Zero
* Lagrangian derivation of $\mathrm{Tr}\phi^3 \Leftrightarrow$ NLSM.
* Useful (Tree + Loop-level)!

Claim:



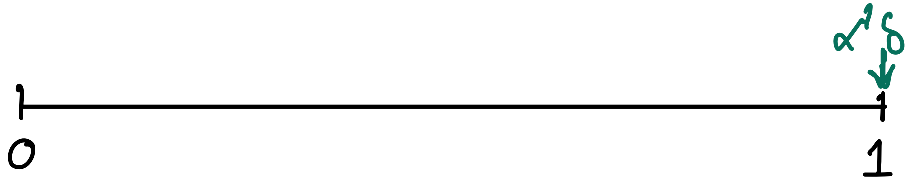
$\Rightarrow$ YM

$$\alpha'\delta = 1$$

$$A_{2n}[\alpha' X_{ij}] \propto \int \prod \frac{dy}{y^2} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}$$

different measure

Claim:



$\Longrightarrow$ YM

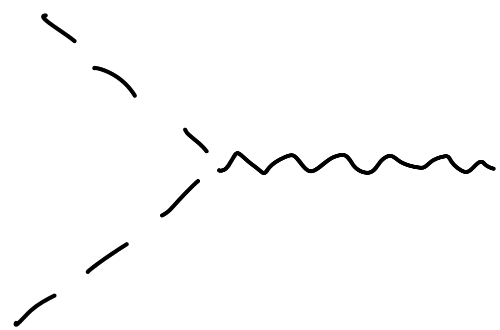
$$\alpha'\delta = 1$$

$$A_{2n}[\alpha' X_{ij}] \propto \int \prod \frac{dy}{y^2} \times \prod_{i<j} u_{ij}^{\alpha' X_{ij}}$$

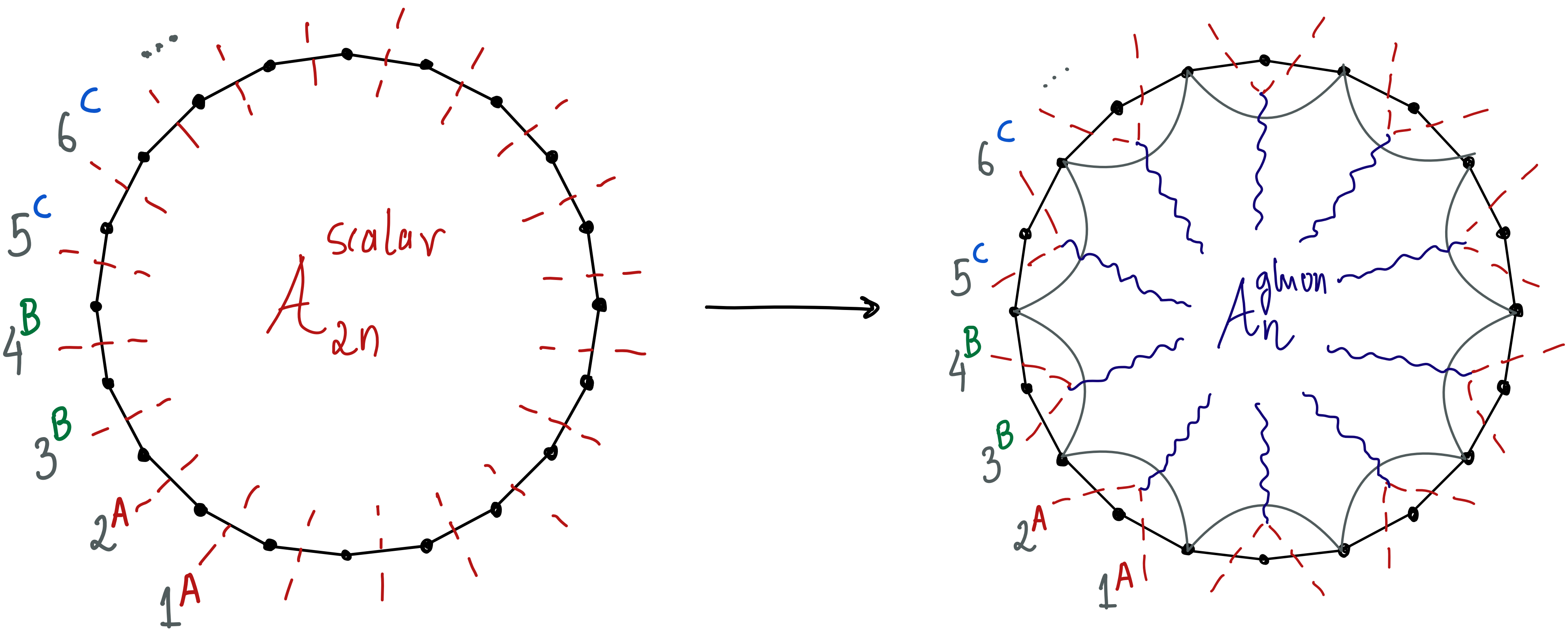
different measure

Q: Where are the polarizations?

$A_{2n}[X] \equiv$ scattering of scalars that couple to gluons



General 2n-scattering scalars $\longrightarrow$ n-gluons



$$A_{2n} = \int \prod \frac{dy}{y_{2}} \prod u_{ij}^{\alpha' X_{ij}} \longrightarrow A_n^{\text{gluon}} = \underset{\text{Scaffolding } X=0}{\mathrm{Res}} \left(A_{2n} \right) [X_{ij}].$$

Getting $A_{2n} = \int \prod \frac{dy}{y^2} \prod u_{ij}^{\alpha' X_{ij}}$ from Bosonic String (Tree-level)

$$A_{2n}^{\text{tree}}(1,2,\ldots,2n) = \int \frac{dz_1 \cdots dz_{2n}}{SL(2,\mathbb{R})} \times \prod_{i<j} z_{ij}^{2\alpha' p_i \cdot p_j} \exp\left\{-\sum_{i\neq j}\left(\frac{\sqrt{\alpha'}\,\varepsilon_i \cdot p_j}{z_{ij}} - 2\,\frac{\varepsilon_i \cdot \varepsilon_j}{z_{ij}^2}\right)\right\}\Bigg|_{\text{multilinear in } \varepsilon_i}$$

$$\Big\downarrow \quad \varepsilon_1 \cdot \varepsilon_2 = \varepsilon_3 \cdot \varepsilon_4 = \ldots = \varepsilon_{2n-1} \cdot \varepsilon_{2n} = 1 \; ; \quad p_i \cdot \varepsilon_j = 0, \quad \varepsilon_i \cdot \varepsilon_j = 0$$

(Gluons → Scalars)

$$A_{2n \text{ scalars}}^{\text{tree}}(1,2,\ldots,2n) \propto \int \frac{dz_1 \cdots dz_{2n}}{SL(2,\mathbb{R})} \times \prod_{i<j} z_{ij}^{2\alpha' p_i \cdot p_j} \times \frac{1}{z_{12}^2\, z_{34}^2 \cdots z_{2n-1,2n}^2}$$

$$\Big\downarrow \quad z_{ij} \to u[y]$$

$$A_{2n}(1,2,\ldots,2n) \propto \int \prod \frac{dy}{y} \prod u_{ij}^{\alpha' X_{ij}} \times \left(\frac{\prod u_{ee}}{\prod u_{oo}}\right)$$

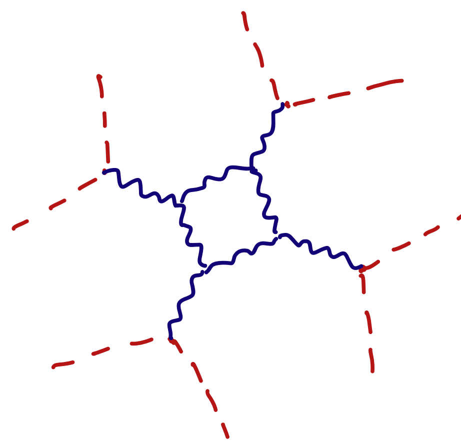
Gluons @ Loop-level.

$$A^{\mathrm{Tr}\varphi^3}_{\mathrm{Loop}} = \int \prod \frac{dy}{y} \prod_{C \in S} u_C^{\alpha' X_C} \longrightarrow \underbrace{\int \prod \frac{dy}{y^{2}} \prod_{C \in S} u_C^{\alpha' X_C}}_{\text{Surface Gluon Integrands.}} \equiv A^{\mathrm{Gluons}}_{\mathrm{Loop}}.$$

* General Dimensions

* Leading Singularities ✓ (2 loops)

$$\begin{aligned}
&-X14\,X15\,X27\,X36 + X15^2\,X27\,X36 + X14\,X15\,X28\,X36 - X15^2\,X28\,X36 + X14\,X15\,X27\,X37 - X15^2\,X27\,X37 - X14\,X16\,X27\,X37 + X15\,X16\,X27\,X37 - X14\,X15\,X28\,X37 + X15^2\,X28\,X37 + \\
&X14\,X16\,X28\,X37 - X15\,X16\,X28\,X37 + X14\,X15\,X36\,X37 - X15^2\,X36\,X37 - X15\,X24\,X36\,X37 + X15\,X25\,X36\,X37 - X14\,X15\,X37^2 + X15^2\,X37^2 + X14\,X16\,X37^2 - X15\,X16\,X37^2 + X15\,X24\,X37^2 - \\
&X16\,X24\,X37^2 - X15\,X25\,X37^2 + X16\,X25\,X37^2 - X14\,X15\,X36\,X38 + X15^2\,X36\,X38 + X15\,X24\,X36\,X38 - X15\,X25\,X36\,X38 + X14\,X15\,X37\,X38 - X15^2\,X37\,X38 - X14\,X16\,X37\,X38 + \\
&X15\,X16\,X37\,X38 - X15\,X24\,X37\,X38 + X16\,X24\,X37\,X38 + X15\,X25\,X37\,X38 - X16\,X25\,X37\,X38 - X15^2\,X27\,X46 + X15^2\,X28\,X46 + X15^2\,X37\,X46 - X15\,X25\,X37\,X46 - X15^2\,X38\,X46 + \\
&X15\,X25\,X38\,X46 + X15^2\,X27\,X47 - X15\,X16\,X27\,X47 - X15^2\,X28\,X47 + X15\,X16\,X28\,X47 - X15^2\,X37\,X47 + X15\,X16\,X37\,X47 + X15\,X25\,X37\,X47 - X16\,X25\,X37\,X47 + X15^2\,X38\,X47 - \\
&X15\,X16\,X38\,X47 - X15\,X25\,X38\,X47 + X16\,X25\,X38\,X47 + X14\,X27\,X36\,X58 - X15\,X27\,X36\,X58 - X14\,X27\,X37\,X58 + X15\,X27\,X37\,X58 - X14\,X36\,X37\,X58 + X15\,X36\,X37\,X58 + X24\,X36\,X37\,X58 - \\
&X25\,X36\,X37\,X58 + X14\,X37^2\,X58 - X15\,X37^2\,X58 - X24\,X37^2\,X58 + X25\,X37^2\,X58 + X15\,X27\,X46\,X58 - X15\,X37\,X46\,X58 + X25\,X37\,X46\,X58 - X15\,X27\,X47\,X58 + X15\,X37\,X47\,X58 - \\
&X25\,X37\,X47\,X58 + X14\,X27\,X37\,X68 - X15\,X27\,X37\,X68 - X14\,X37^2\,X68 + X15\,X37^2\,X68 + X24\,X37^2\,X68 - X25\,X37^2\,X68 + X15\,X27\,X47\,X68 - X15\,X37\,X47\,X68 + X25\,X37\,X47\,X68 + \\
&X15\,X37\,X46\,Y2 - X15\,X38\,X46\,Y2 - X15\,X36\,X47\,Y2 + X15\,X38\,X47\,Y2 - X16\,X38\,X47\,Y2 + X15\,X36\,X48\,Y2 - X15\,X37\,X48\,Y2 + X16\,X37\,X48\,Y2 - X14\,X36\,X58\,Y2 + X14\,X37\,X58\,Y2 - \\
&X16\,X37\,X58\,Y2 - X37\,X46\,X58\,Y2 + X16\,X47\,X58\,Y2 + X36\,X47\,X58\,Y2 - X14\,X37\,X68\,Y2 + X15\,X37\,X68\,Y2 - X15\,X47\,X68\,Y2 + X15\,X27\,X36\,Y4 - X15\,X28\,X36\,Y4 + X16\,X25\,X37\,Y4 - \\
&X15\,X26\,X37\,Y4 + X15\,X28\,X37\,Y4 - X16\,X28\,X37\,Y4 - X16\,X25\,X38\,Y4 + X15\,X26\,X38\,Y4 - X15\,X27\,X38\,Y4 + X16\,X27\,X38\,Y4 + X16\,X27\,X58\,Y4 - X27\,X36\,X58\,Y4 - X16\,X37\,X58\,Y4 + \\
&X26\,X37\,X58\,Y4 - X15\,X27\,X68\,Y4 + X15\,X37\,X68\,Y4 - X25\,X37\,X68\,Y4 - 2\,X16\,X58\,Y2\,Y4 + 2\,X15\,X68\,Y2\,Y4 + X15\,X24\,X37\,Y6 - X14\,X25\,X37\,Y6 - X14\,X28\,X37\,Y6 + X15\,X28\,X37\,Y6 - \\
&X15\,X24\,X38\,Y6 + X14\,X25\,X38\,Y6 + X14\,X27\,X38\,Y6 - X15\,X27\,X38\,Y6 - X15\,X28\,X47\,Y6 + X15\,X38\,X47\,Y6 - X25\,X38\,X47\,Y6 + X15\,X27\,X48\,Y6 - X15\,X37\,X48\,Y6 + X25\,X37\,X48\,Y6 - \\
&X14\,X27\,X58\,Y6 + X14\,X37\,X58\,Y6 - X24\,X37\,X58\,Y6 + 2\,X14\,X37\,Y2\,Y6 - 2\,X15\,X37\,Y2\,Y6 - 2\,X14\,X38\,Y2\,Y6 + 2\,X15\,X38\,Y2\,Y6 + 2\,X15\,X47\,Y2\,Y6 + 2\,X38\,X47\,Y2\,Y6 - 2\,X15\,X48\,Y2\,Y6 - \\
&2\,X37\,X48\,Y2\,Y6 + 2\,X14\,X58\,Y2\,Y6 + 2\,X37\,X58\,Y2\,Y6 - 2\,X47\,X58\,Y2\,Y6 + 2\,X28\,X37\,Y4\,Y6 - 2\,X27\,X38\,Y4\,Y6 - X15\,X24\,X36\,Y8 + X14\,X25\,X36\,Y8 - X14\,X27\,X36\,Y8 + X15\,X27\,X36\,Y8 + \\
&X15\,X24\,X37\,Y8 - X16\,X24\,X37\,Y8 - X14\,X25\,X37\,Y8 + X16\,X25\,X37\,Y8 + X14\,X26\,X37\,Y8 - X15\,X26\,X37\,Y8 - X15\,X27\,X46\,Y8 + X15\,X37\,X46\,Y8 - X25\,X37\,X46\,Y8 - X16\,X25\,X47\,Y8 + \\
&X15\,X26\,X47\,Y8 - X15\,X36\,X47\,Y8 + X25\,X36\,X47\,Y8 + 2\,X37\,X46\,Y2\,Y8 - 2\,X36\,X47\,Y2\,Y8 + 2\,X16\,X25\,Y4\,Y8 - 2\,X15\,X26\,Y4\,Y8 + 2\,X15\,X27\,Y4\,Y8 - 2\,X16\,X27\,Y4\,Y8 + 2\,X15\,X36\,Y4\,Y8 - \\
&2\,X25\,X36\,Y4\,Y8 + 2\,X27\,X36\,Y4\,Y8 - 2\,X15\,X37\,Y4\,Y8 + 2\,X16\,X37\,Y4\,Y8 + 2\,X25\,X37\,Y4\,Y8 - 2\,X26\,X37\,Y4\,Y8 + 2\,X15\,X24\,Y6\,Y8 - 2\,X14\,X25\,Y6\,Y8 - Y2\,Y4\,Y6\,Y8 - Y2\,Y4\,Y6\,Y8\,\triangle
\end{aligned}$$



A_4^{Gluon}

Revealing New Features.

* Surface Kinematics → Determining "Perfect" Integrands for real world amplitudes.

[2403.048261, 2401.00041, 2401.05483, to appear

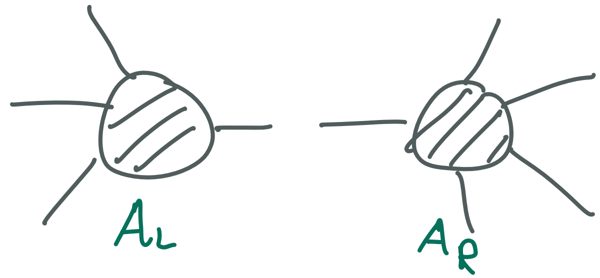
w/ N. Arkani-Hamed, Q. Cao, J. Dong, S. He]

CHALLENGE: Determine real world amplitudes from

SINGLE CUTS ≡ single poles ?

Tree-level ✓

Rational Function + On poles:

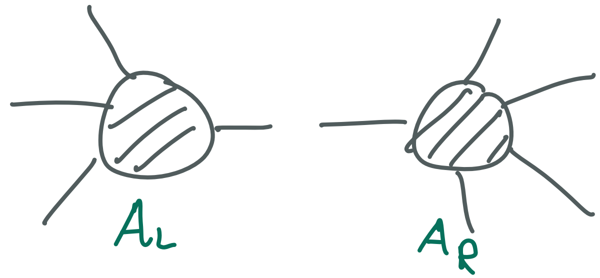


⇒ Can be used to recursively construct A. [BCFW].

CHALLENGE: Determine real world amplitudes from SINGLE CUTS $\equiv$ single poles ?

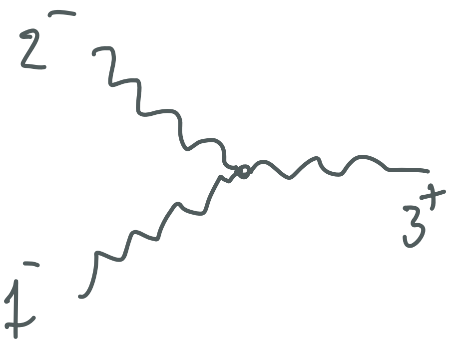
Tree-level ✓

Rational Function + On poles:



$\Rightarrow$ Can be used to recursively construct A. [BCFW].

BUT, Needs extension of momenta $\mathbb{R} \to \mathbb{C}$.



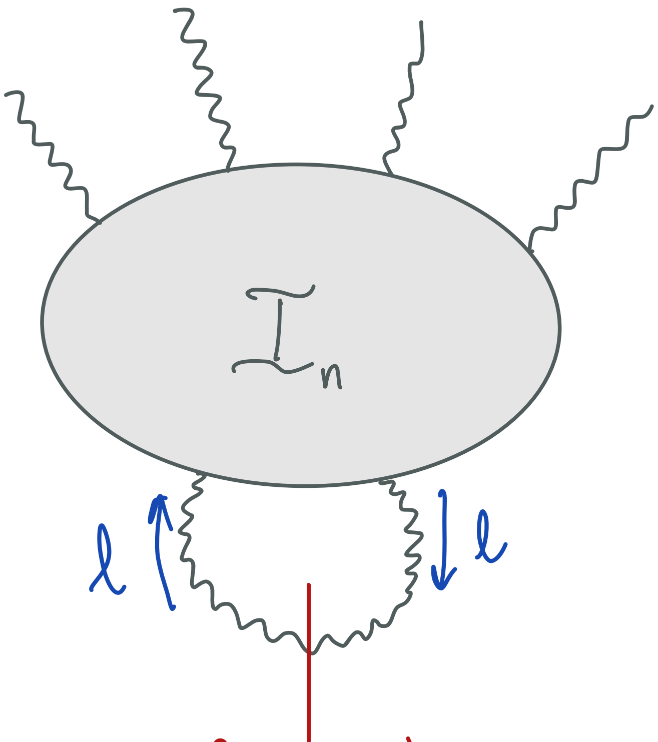
3 point amplitude $= 0$ Lorentzian

$\neq 0$ Complex Kinematics!

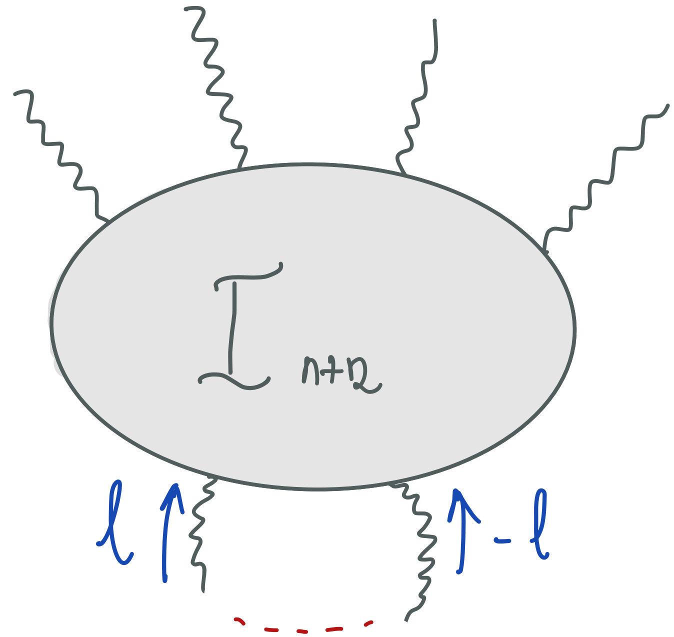
CHALLENGE: Determine real world amplitudes from

SINGLE CUTS ≡ single poles ?

Loop-level? No!



$\stackrel{?}{\equiv}$



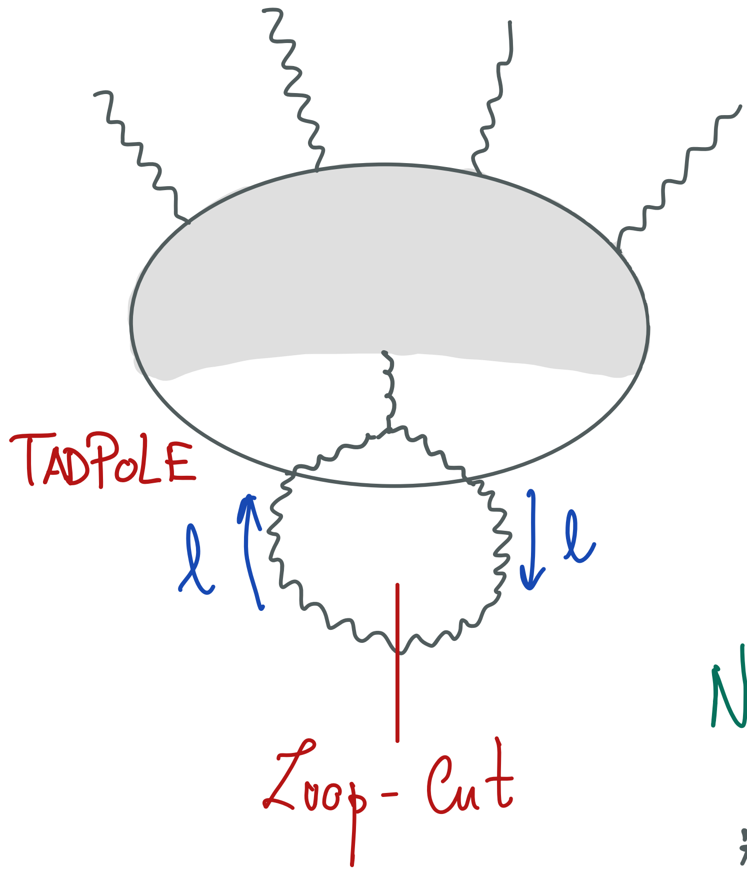
Loop-Cut

Tree-Glue

CHALLENGE: Determine real world amplitudes from

SINGLE CUTS ≡ single poles ?

Loop-level? No!



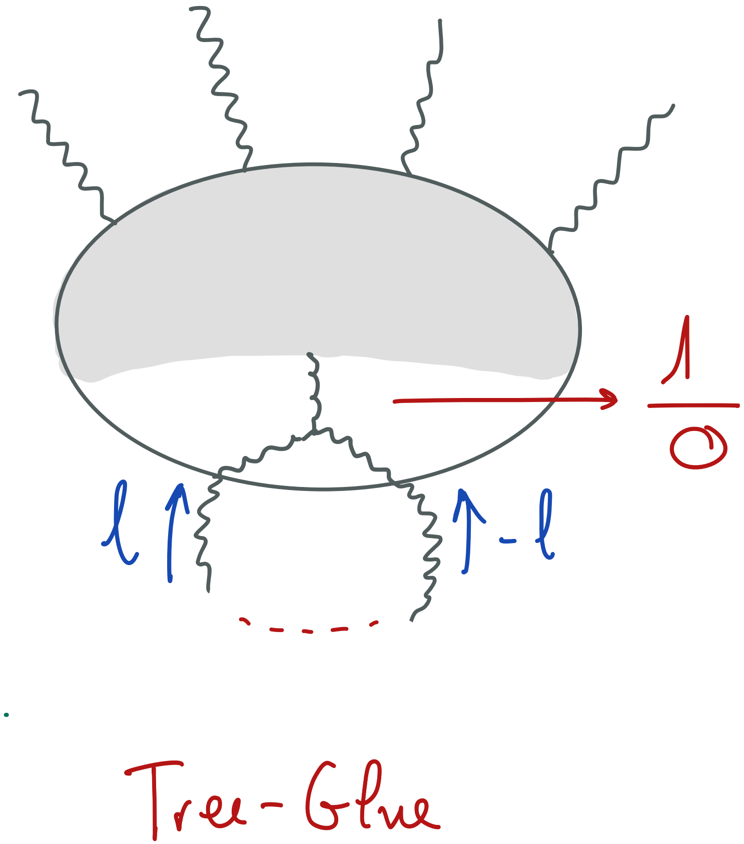
$\overset{?}{\equiv}$

DIVERGENT!

⇓

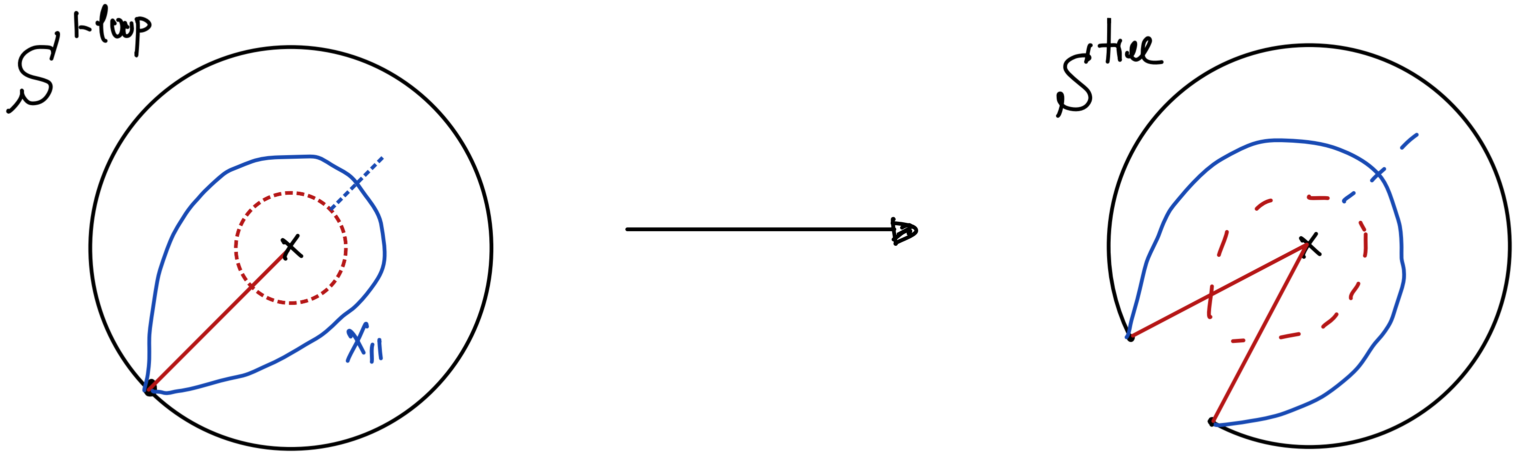
No "The" Integrand.

* Adler Zero ✗
* Gauge Invariant ✗



[cancels in maximal SUSY]

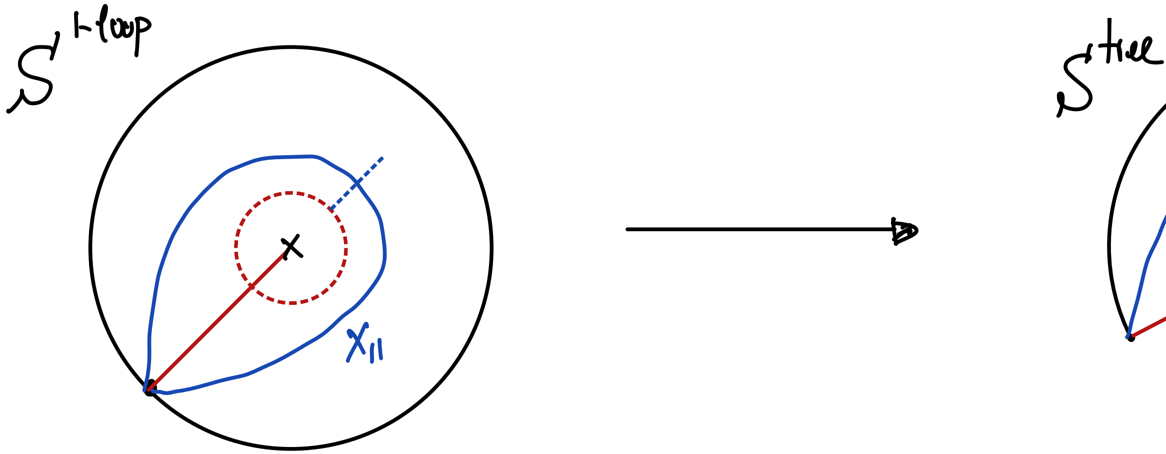
In Surface language: Loop-cut



$X_{11} \equiv 0$ in momentum space

$A^{tree}\big|_{X_{11}=0}$ blows up!

In Surface language: Loop-cut



$X_{11} \equiv 0$ in momentum space

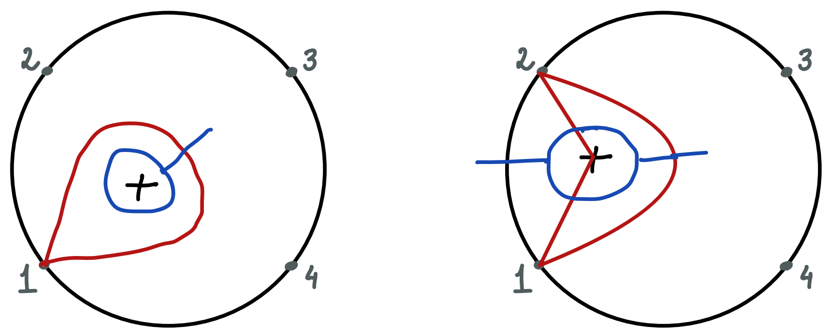
$A^{\text{tree}}\big|_{X_{11}=0}$ blows up!

BUT,

in Surface integral CAN keep curves w/o standard momentum

$$A^{\delta}_{2n} = \int \prod \frac{dy}{y^{1+\delta}} \prod_{C \in S} u_{ij}^{X_C}$$

HOMOLOGY → HOMOTOPY!



X_{11} ~ tadpoles

$X_{2,1}$ ~ Bubbles

Lorentzian $\xrightarrow{\text{Tree}}$ Complex $\xrightarrow{\text{Loop}}$ "Surface" Kinematics
(Curves on surfaces)

$$\underbrace{\hspace{20em}}$$

"The" (Surface) Integrand for Pions & Gluons

* Exists! ⇒ Can be obtained Recursively.

[Tree, One-loop, general D Gluons
(higher-loops in progress)
All-loop Pions]

$$A_{2n}^{\delta} = \int \prod \frac{dy}{y^{1+\delta}} \prod_{C \in S} u_{ij}^{X_C}$$

Lorentzian $\xrightarrow{\text{Tree}}$ Complex $\xrightarrow{\text{Loop}}$ "Surface" Kinematics

(Curves on surfaces)

"The" (Surface) Integrand for Pions & Gluons

* Exists! ⇒ Can be obtained Recursively. [Tree, One-loop, general D Gluons (higher-loops in progress) All-loop Pions]
* Adler Zero (Pions)
* Gauge Invariance (Gluons).
* Splits ~ Multi-soft limits. ⇒ All-order statements for pions and Gluons.

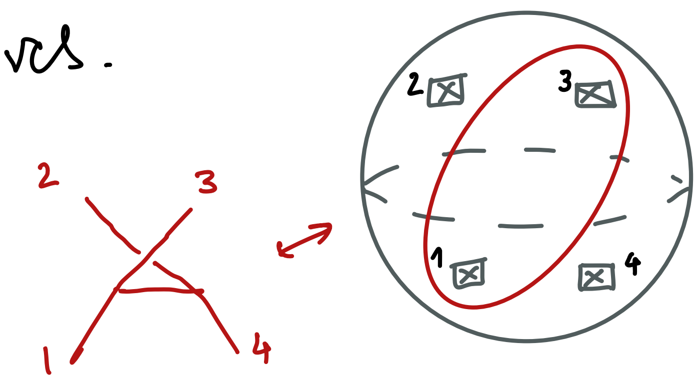
$$A_{2n}^{\delta} = \int \prod \frac{dy}{y^{1+\delta}} \prod_{C \in S} u_{ij}^{X_C}$$

Outlook...

* Fermions.
* Realistic particle content. (Beyond) SM

$$\phi^i_j \;\underset{}{\overset{u(N)}{\rightleftarrows}}\; \longrightarrow \;\overset{SU(3)\times SU(2)\times U(1)}{\begin{array}{l} Q\,(3,2,+1/6) \\ H\,(1,2,-1/2) \end{array}} \;\longrightarrow\; \text{Extension of SM?}$$

* Gravity from Closed Curves.



Thank You !