The BFSS conjecture, a review

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The BFSS conjecture says that:

M-theory scattering amplitudes

=

the large N limit of a scattering problem in a matrix quantum mechanics.

Banks, Fischler, Shenker, Susskind 1996

(All references are hyperlinks).

We will review this conjecture and related issues.

It will not be historical,

but, hopefully,

it will be pedagogical.

Let us start with some comments on M-theory

M-theory is a theory of quantum gravity in eleven dimensions which reduces to eleven dimensional supergravity at low energies.

It is also supposed to reduce to IIA string theory when compactified on a very small spatial circle.

M-theory can be obtained from various limits:

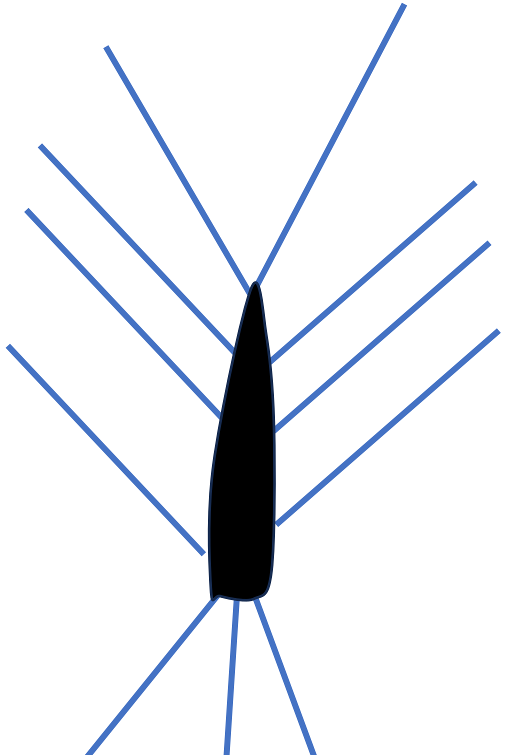
- Strong coupling limit of IIA string theory → M-theory on S^1

- $\boxed{\text{Via the BFSS conjecture}}$

- Large N limit of $AdS_4 x S^7$ via the dual field theory, ABJM (and AdS_7 x S^4)

We are interested in the theory with asymptotically flat space boundary conditions, $R^{1,10}$

The main observable is the S-matrix of massless particles, namely the ones in the graviton supermultiplet.



If there were other stable particles, we would also have to include them to get a unitary S-matrix.

(We do not expect other stable particles)

We are talking about the full non perturbative amplitudes.

$$\mathcal{A}(p_1, \cdots, p_n) = \delta^{11}\Big(\sum_i p_i^\nu\Big)\mathcal{M}(p_1, \cdots, p_n)$$

For now the statement is that we expect them to be well defined, even if we can’t predict them.

Light front, or light-cone, coordinates

$$ds^2 = -2dx^+dx^- + d\vec{y}^2 \ , \qquad \vec{y} \in R^9$$

$$e^{ip_\mu x^\mu} \ , \qquad p_\mu = -i\frac{\partial}{\partial x^\mu}$$

$$-p_- \geq 0 \quad \text{and} \quad -p_+ \geq 0$$

$$-p_+ = \frac{\vec{p}^2}{2(-p_-)}$$

For a massless particle.

View x^+ as time and $-p_+$ as the Hamiltonian →

looks like a non-relativistic particle of mass $M = -p_-$

Now we will do a tricky operation.

Compactify the x^- direction

$$x^- \sim x^- + 2\pi R \ , \qquad \rightarrow \qquad -p_- = \frac{N}{R}$$

$$N \geq 0$$

Note that

$$\mathcal{A}_{\text{compactified}}(p_1, \cdots, p_n) \neq \mathcal{A}_{\text{un-compactified}}(p_1, \cdots, p_n)\big|_{p_i^- \to -\frac{N_i}{R}}$$

The null compactification

Susskind, Sen, Seiberg

- A null circle has zero proper length.
- We can view it as as the zero size limit of a spacelike circle.
- Small spacelike circle → IIA string theory at weak coupling.
- Momentum → D0 branes

Carefully working out the limit one finds:

Amplitudes in compactified theory = low energy limit of D0 brane scattering → computed by a matrix model

Polchinski , Witten

$$\mathcal{A}_{\text{compactified}}(p_1, \cdots, p_n) = \mathcal{A}_{\text{Matrix Model}}(N_1, \vec{p}_1; \cdots ; N_n, \vec{p}_n)$$

(We will define the matrix model in more detail later.)

This follows from the M-theory/IIA relationship.

This is not yet the BFSS conjecture.

The BFSS conjecture is the following

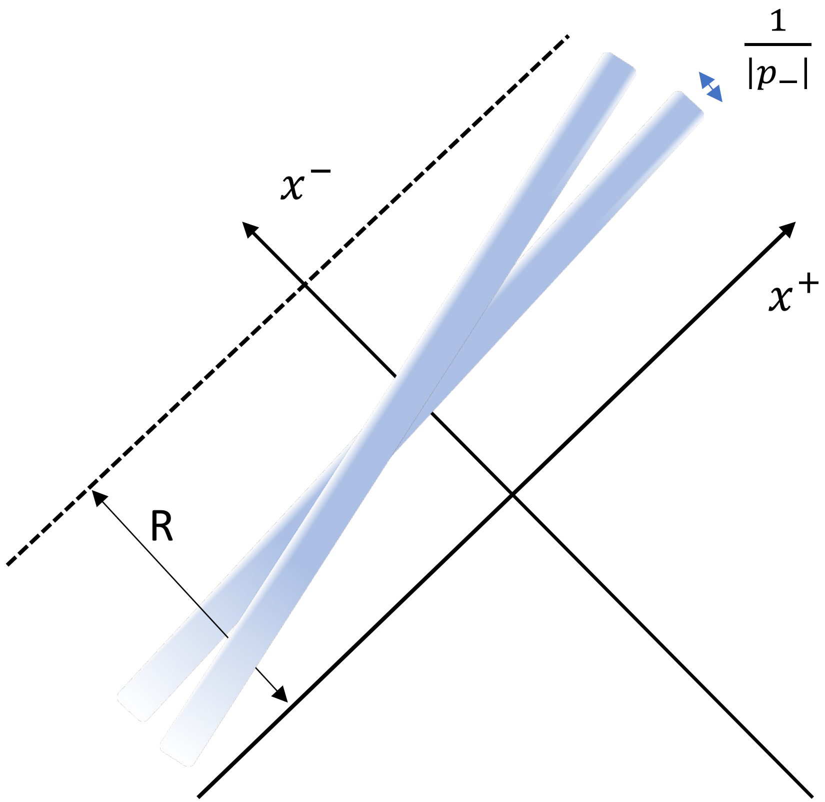
$$\lim_{N^i\to\infty,\ R\to\infty}\left[\mathcal{A}_{\text{Matrix Model}}(N_1,\vec{p}_1;\cdots;N_n,\vec{p}_n)\right]=(2\pi R)^{1-\frac{n}{2}}\mathcal{A}_{\text{un-compactified}}(p_1,\cdots,p_n)$$

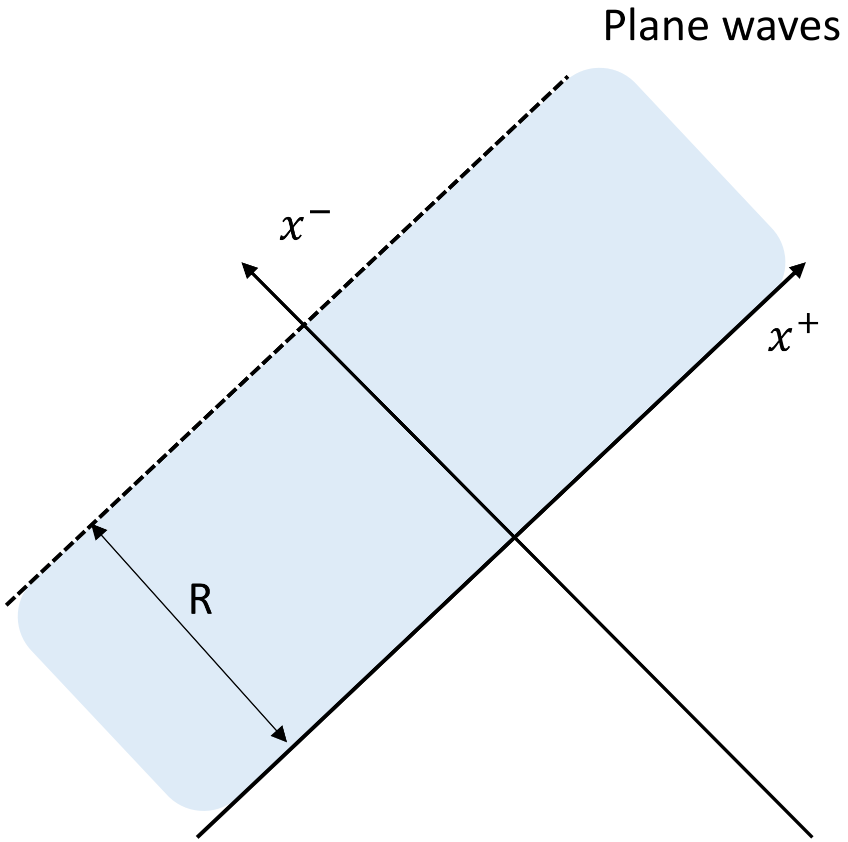
$$\text{with}\quad \frac{N^i}{R}=-p^i_-=\text{fixed}$$

Banks, Fischler, Shenker, Susskind 1996

Intuition

Banks, Fischler, Shenker, Susskind 1996





Let us now define more carefully the matrix model

The matrix model

$$S = \int dt Tr \left\{ \sum_I \frac{(D_t X^I)^2}{2R} + \frac{R}{4(2\pi)^2 l_p^6} \sum_{IJ} [X^I, X^J]^2 + \psi_\alpha D_t \psi_\alpha + i \frac{R}{(2\pi) l_p^3} \psi_\alpha \gamma^I_{\alpha\beta} [\psi_\beta, X^I] \psi_b \right\}$$

SO(9) symmetry + 16 supersymmetries.

$$G_{N,11} = 16\pi^7 l_p^9$$

$X^I, \ \psi_\alpha$ are NxN Hermitian matrices.

I,J = 1,…, 9

$\alpha, \beta = 1, \cdots, 16$, $\gamma^I_{\alpha,\beta}$ = nine real symmetric traceless gamma matrices

$$D_0 Y = \partial_t Y + [A_0, Y]$$

Gauged: A_0 → Imposes the U(N) singlet constraint.

The U(1) sector decouples

$$S = N \int dt \sum_I \frac{(\dot{X}^I)^2}{2R} + \psi_\alpha \dot{\psi}_\alpha$$

<u>If</u> $E_{SU(N)} = 0$,

Superparticle action in light cone gauge

$$S = N \int dt \sum_I \frac{(\dot{X}^I)^2}{2R} + \psi_\alpha \dot{\psi}_\alpha$$

Fill out the 256 = 2^8 states of the massless supergraviton multiplet

$$H = -p_+ = \vec{p}^{\,2} \frac{1}{2} \frac{R}{N}$$

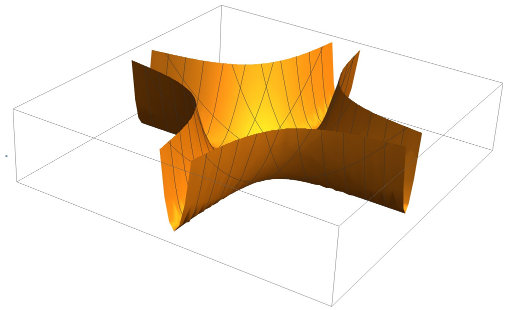
All the states of a massless particle in 11 dimensions

It is believed that the SU(N) problem has a single bound state at energy E=0.

Single zero energy bound state

The potential has flat directions, so the quantum mechanics will have a continuous spectrum (more on that later).

This is a truly normalizable zero energy state.



Evidence: Index arguments. (It is subtle because of the flat directions.)

Piljin Yi ; Sethi, Stern; Moore, Nekrasov, Shatashvili ; Konechny ; Porrati, Rozenberg ;Sethi, Stern ;

We also have results on how fast the wavefunction decays for large r: $\psi \sim r^{-9}$

Plefka, Waldron ;Froelich, Graf, Hasler, Hoppe, Yau ; Hoppe, Plefka ; Hasler, Hoppe ; Y.H. Lin, Xi Yin

Low energy states of the matrix model.

The potential vanishes when the matrices commute: $[X^I, X^J] = 0 \rightarrow$ diagonalize them

$$X^i = \begin{pmatrix} x_1 \mathbf{1}_{N_1} & & & \\ & x_2 \mathbf{1}_{N_2} & & \\ & & x_3 \mathbf{1}_{N_3} & \\ & & & x_4 \mathbf{1}_{N_4} \end{pmatrix}$$

Consider $|x_i - x_j| \gg$ large. And add a bound state wavefunction in each sub-block

→ we have 4 gravitons with momenta N_i.

Asymptotic states of the matrix model.

In general the N x N matrix will separate into n submatrices, where the center of mass coordinate of each will be very far away from the others.

With each group of size N_i forming a bound state.

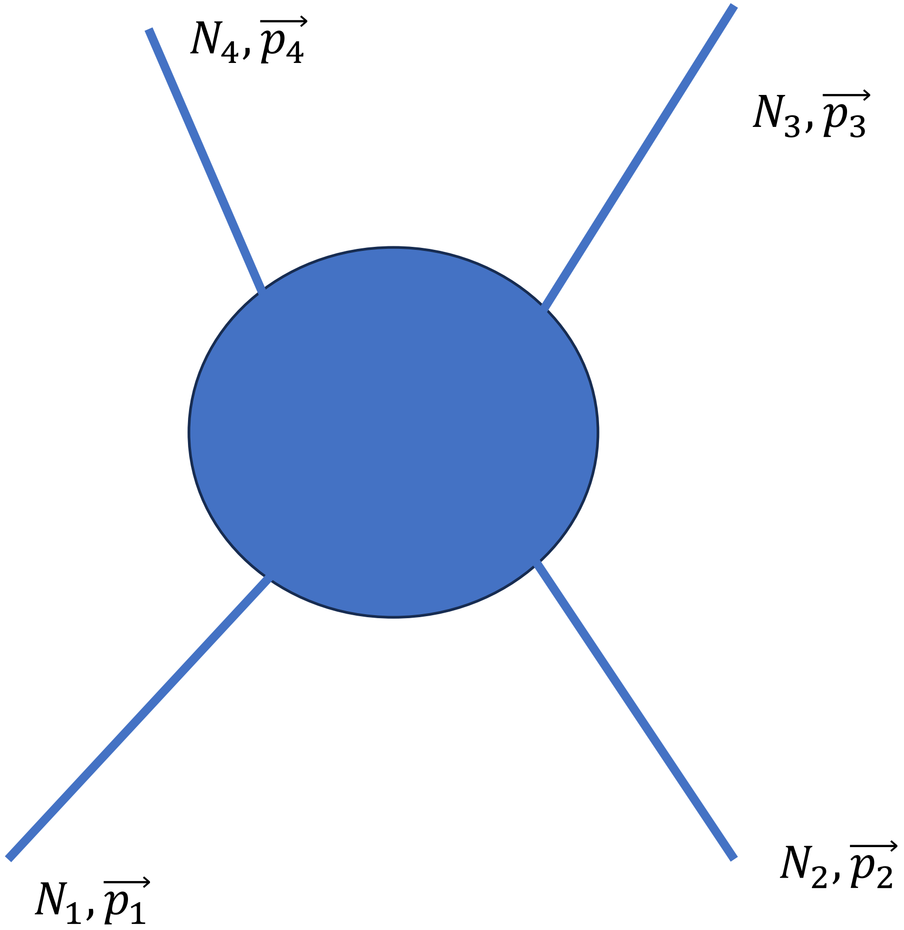
This gives an asymptotic n graviton state.

Scattering in the matrix model

$$\begin{pmatrix} x_3 \mathbf{1}_{N_3} & 0 \\ 0 & x_4 \mathbf{1}_{N_4} \end{pmatrix}$$

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$\begin{pmatrix} x_1 \mathbf{1}_{N_1} & 0 \\ 0 & x_2 \mathbf{1}_{N_2} \end{pmatrix}$$



$$N_1 + N_2 = N_3 + N_4$$

Now that we defined the scattering problem in the matrix model, we will restate the BFSS conjecture

$$\lim_{N^i\to\infty,\ R\to\infty} \left[\mathcal{A}_{\text{Matrix Model}}(N_1, \vec{p}_1; \cdots; N_n, \vec{p}_n)\right] = (2\pi R)^{1-\frac{n}{2}} \mathcal{A}_{\text{un-compactified}}(p_1, \cdots, p_n)$$

$$\text{with} \quad \frac{N^i}{R} = -p_-^i = \text{fixed}$$

Subconjectures:

- The limit exists.
- The limit defines a suitably analytic function of p_-^i
- The result is fully Lorentz invariant.
- The S-matrix is unitary in the Fock space. (e.g. the total probability that produce finite N^i gravitons goes to zero as the other N^j go to infinity).

Disputed in: Banks, Fischler

- We have all the properties we expect from M-theory: reduces to supergravity amplitudes, contains membranes, fivebranes, black holes, etc.

It contains membranes

- Configurations where the commutators of the matrices are non-zero.

de Wit, Hoppe, Nicolai 1988

$$S = \int dt Tr \left\{ \sum_I \frac{(D_t X^I)^2}{2R} + \underline{\frac{R}{4(2\pi)^2 l_p^6} \sum_{IJ} [X^I, X^J]^2} + \psi_\alpha D_t \psi_\alpha + i \frac{R}{(2\pi) l_p^3} \psi_\alpha \gamma^I_{\alpha\beta} [\psi_\beta, X^I] \psi_b \right\}$$

Example: we could consider a spherical membrane, using $X^I = \lambda(t)\, J^I$, with J^I SU(2) generators, $I = 1,2,3$ in a representation of dimension N.

$$T = \frac{1}{(2\pi)^2 l_p^3}$$

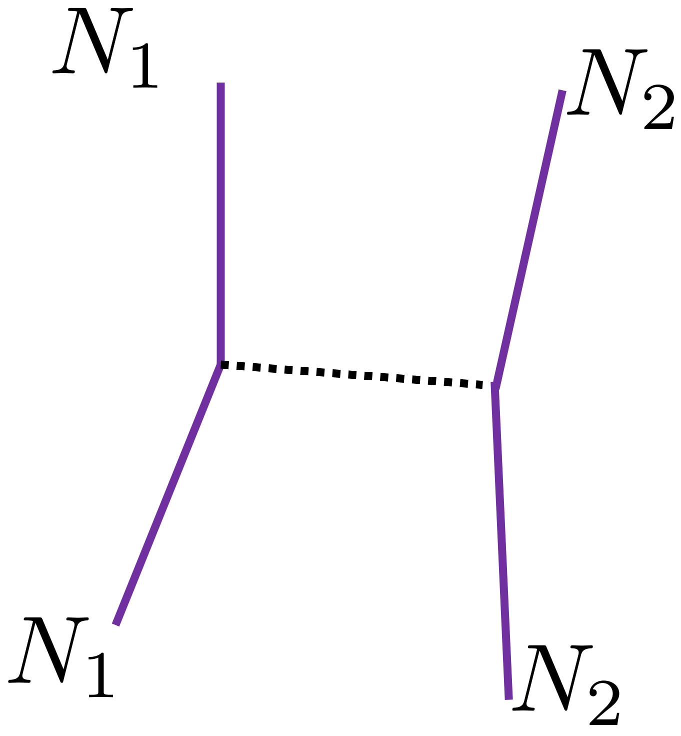
Some simple scattering problems

A scattering of D0 branes at low velocity and large distances.

Related to small momentum transfer limit of amplitudes.

Low velocity expansion

Douglas, Kabat, Pouliot, Shenker



$$S = S_{\text{free}} + \int dt \frac{N_1 N_2 l_p^9}{R^3}\frac{v^4}{r^7} = S_{free} + p_-^1 p_-^2 \frac{l_p^9}{R}\int dt \frac{v^4}{r^7}$$

This particular term arises after we integrate out the massive off diagonal terms at 1-loop order. At leading order in low velocity.

At higher looks we expect corrections with extra factors going like $\frac{N l_p^3}{r^3}$.

These diverge in the BFSS limit.

But they vanish by a non-renormalization theorem. Paban, Sethi, Stern

(Also, there is no correction to the v^2 terms).

This is a check of the conjecture

Note that computing the one loop term is not enough to check it.

We need the one loop term + non renormalization argument, so that we can take the BFSS large N limit.

In fact, all computations in the BFSS limit involve very strong coupling in the matrix model.

Stronger coupling than the 't Hooft limit.

In the above computation, the dimensionless 't Hooft coupling is $= \frac{N l_p^3}{r^3}$,
which diverges in the BFSS limit, $N \to \infty, \quad \frac{r}{l_p} = \text{fixed}.$

Note:

The BFSS limit involves lower energies and stronger couplings than the usual ’t Hooft limit.

There is a similar story for the next order in velocity, v^6.

Becker, Becker ; Becker, Becker, Polchinski; Paban, Sethi, Stern

Now we discuss the simplest amplitude

The three point amplitude

- Trivial in M-theory → fixed by Lorenz invariance + SUSY.
- In the matrix model we have less symmetries → non-trivial computation.
  - It can be done using supersymmetry (the 3 point amplitude preserves some supersymmetry).
  - Agrees with the gravity one.
  - It is important for the soft theorems. (Combined with other assumptions → Lorentz symmetry).

JM , Herderschee

Tropper, Wang , JM, Herderschee(2)

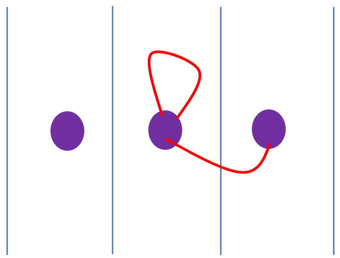
Compactification

- There is a nice story when we compactify some of the transverse dimensions.
- We go from Matrix quantum mechanics → Yang Mills theories in more dimensions.
- We now have massive BPS states.
- They all match between the two descriptions.
- We can compactify up to five dimensions, but not more (using this method).

We will discuss only the simplest case:

Compactifying one dimension.

$$x^9 \sim x^9 + 2\pi R_9.$$



D0 branes + images → 1 + 1 dimensional U(N) Yang Mills theory on a circle (T-dual circle).

W. Taylor

M-theory on a circle of size R_9

$$=$$

Limit of U(N) 1+1 dimensional super Yang mills on a circle.

The two dimensional Yang Mills theory

$$S = \frac{1}{R^-} \int dt \int_0^{2\pi \tilde{R}_9} \frac{d\tilde{x}^9}{2\pi \tilde{R}_9} \mathrm{Tr} \left\{ \frac{1}{2} (D_a X^I)^2 + \frac{1}{4} \frac{R^{-2}}{(2\pi)^2 l_p^6} [X^I, X^J]^2 + \frac{1}{4} \frac{(2\pi)^2 l_p^6}{R^{-2}} F^2 + \mathrm{fermions} \right\}$$

$$\tilde{R}_9 = \frac{l_p^3}{R^- R_9}$$

This is just giving the parameters of the theory in terms of the radii.

The limit is the same as before.
Again we consider a scattering problem where matrices are block diagonal, etc.

$$R^-, \quad N \to \infty , \qquad \frac{N}{R^-} = \mathrm{fixed} , \qquad R_9 = \mathrm{fixed}$$

An example of a BPS state that we can match:

The D0 brane of type IIA

This excitation corresponds to one unit of electric flux for the field theory.

The electric flux part of the action

$$S = \frac{N}{R^-} \int dt \int_0^{2\pi \tilde{R}_9} \frac{d\tilde{x}^9}{2\pi \tilde{R}_9} \frac{(2\pi)^2 l_p^6}{R^{-2}} \frac{1}{2} F_{01}^2 + \cdots$$

leads to:

$$E = -p_+ = \frac{R^-}{N} \frac{1}{2} \frac{n^2}{R_9^2} = \frac{M^2}{2(-p_-)} \qquad \rightarrow \qquad M = \frac{n}{R_9} = \frac{n}{g_s \sqrt{\alpha'}}$$

as expected.

We got a match!

An interesting limit is the case where $R_9 \rightarrow 0$.

We are supposed to get weakly coupled IIA string theory.

One can see that we get the IIA string theory in light cone gauge

Matrix strings → perturbative strings in the light cone

Motl ; Banks, Seiberg ; Dijkgraaf, Verlinde, Verlinde
Motl, Dijkgraaf

Recall that $\tilde{R}_9 = \dfrac{l_p^3}{R^- R_9}$, so that small R_9 means that $\tilde{R}_9$ is large.

This implies that the 1+1 theory is defined on a very large circle.

We can consider the low energy limit → Moduli space approximation: a CFT on $\mathrm{Sym}(R^8)^N = (R^8)^N/S_N$

This is the same as N strings in the light cone gauge.

Leading irrelevant deformation = Twist operator → String interactions in light cone gauge — Mandelstam

↓

Dimension = (3/2, 3/2) → right dependence on string coupling. — Dijkgraaf, Verlinde, Verlinde

↓

Recover usual 4pt string amplitude — Arutyunov, Frolov

Return to the 11 dimensional case

A comment

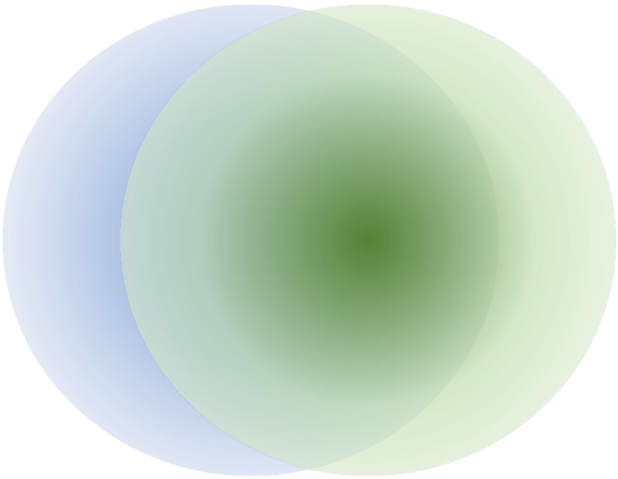
The size of the bound state

The bound state has a size of the order $\ell = N^{\frac{1}{3}}\, l_p$

This can be shown rigorously by a Bootstrap argument Polchinski ; X. Han, Hartnoll, Kruthoff ; Henry Lin

The scaling is expected, it is set by the value of the ‘t Hooft coupling and dimensional analysis.

But in the M-theory limit we are interested in distances that are kept fixed in the large N limit.



The bound states are highly overlapping in the regime of interest.

But we are also at low velocities.

SUSY is crucial…

Why this regime is hard…

Other backgrounds

Two simple examples

Longitudinal M5 branes:

Berkooz, Douglas

k M5 branes along $t, x^-, y_1, y_2, y_3, y_4$

This is gives N D0 and k D4 branes.

Same matrix model but with k fundamental hypermultiplets.

A plane wave

$$ds^2 = -2dx^+dx^- + d\vec{y}^{\,2} + (dx^+)^2\mu^2\left[4(y_1^2+y_2^2+y_3^2)+(y_4^2+\cdots+y_9^2)\right] \ , \qquad F_{+123} \propto \mu$$

Same matrix model plus some mass terms. Berenstein, JM, Nastase

Same number of supersymmetries → SU(2|4) supergroup.

Many nice results about the spectrum of susy branes and susy observables.

Can be studied using localization. Asano, Ishiki, Okada, Shimasaki ; Asano, Ishiki, Shimasaki, Terashima; …

For recent a discussion see: Komatsu, Martina, Penedones, Suchel, Vuignier 2024

Now we will mention something that is not directly related to the BFSS conjecture for 11 dimensional M-theory.

The matrix model is sometimes called the BFSS matrix model.

But we can do other things with this matrix model, we can take other limits that are not directly related to the BFSS conjecture.

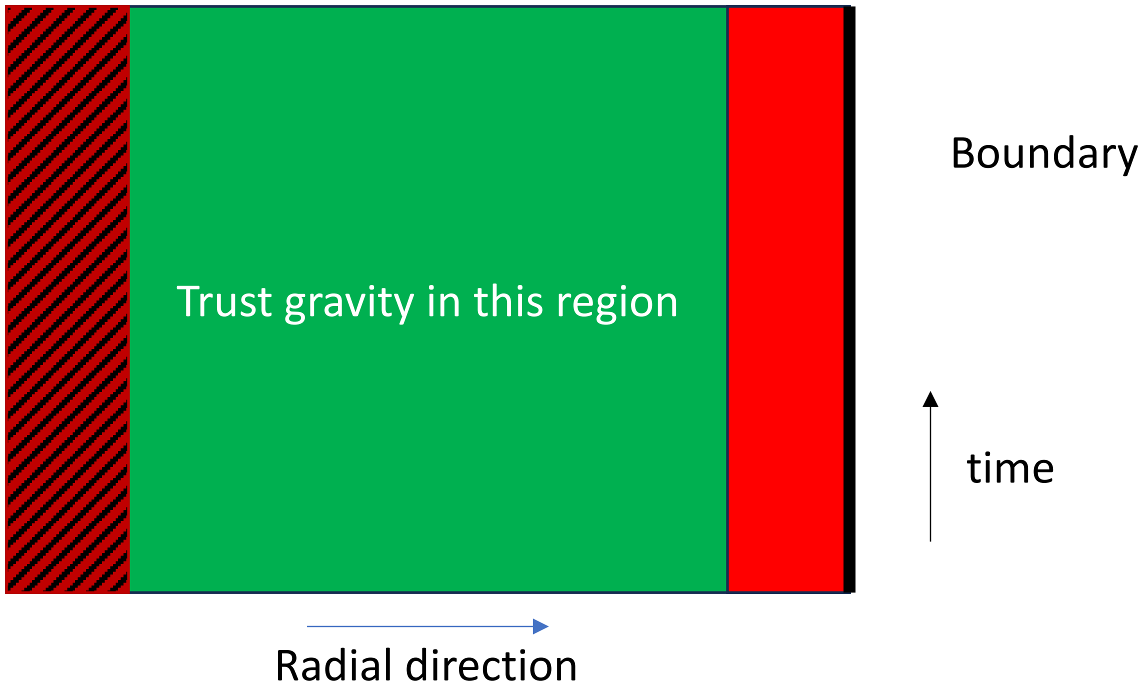
In particular, we can take the standard ‘t Hooft limit.

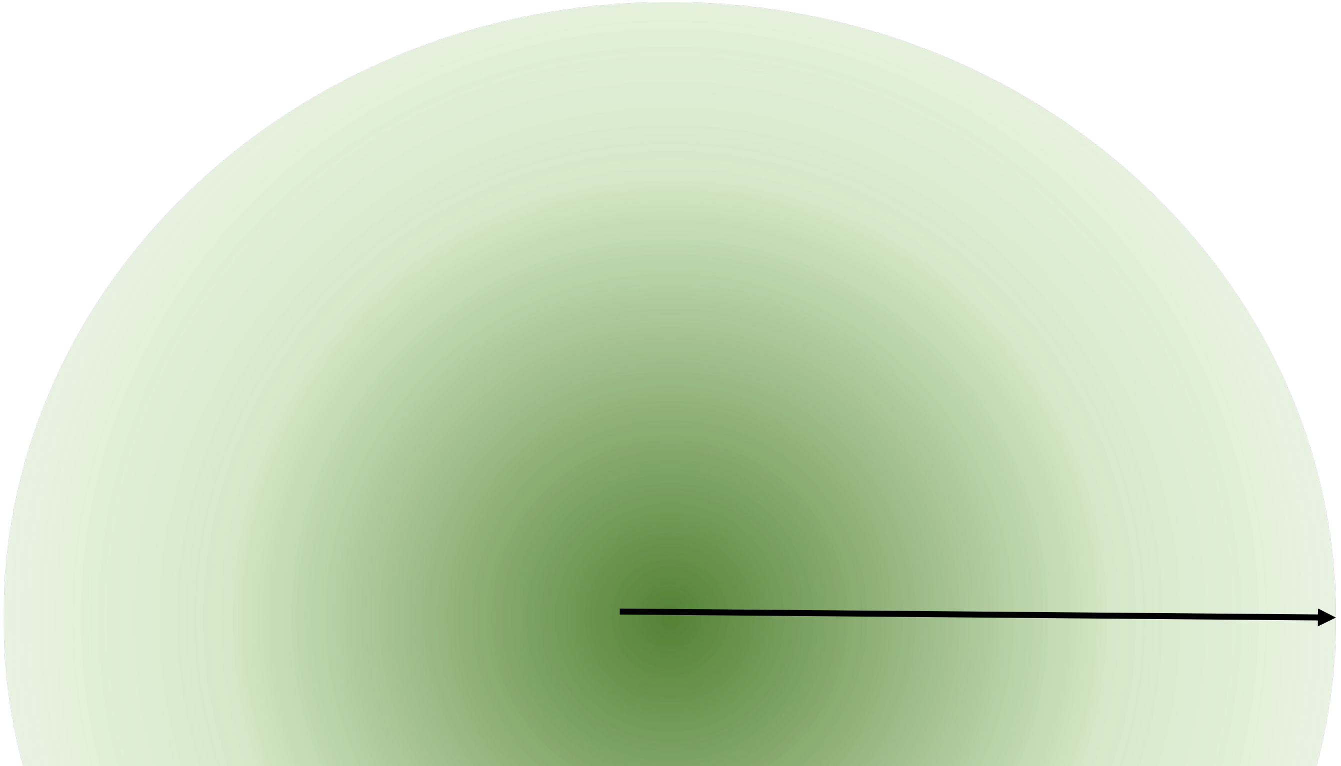
In this limit it becomes a non-conformal example of the gauge gravity duality.

The matrix model is dual to the near horizon region of N D0 branes in type IIA theory.

This is the near horizon geometry of a charged black hole in IIA supergravity.

Itzhaki, JM, Sonnenschein, Yankielowycz



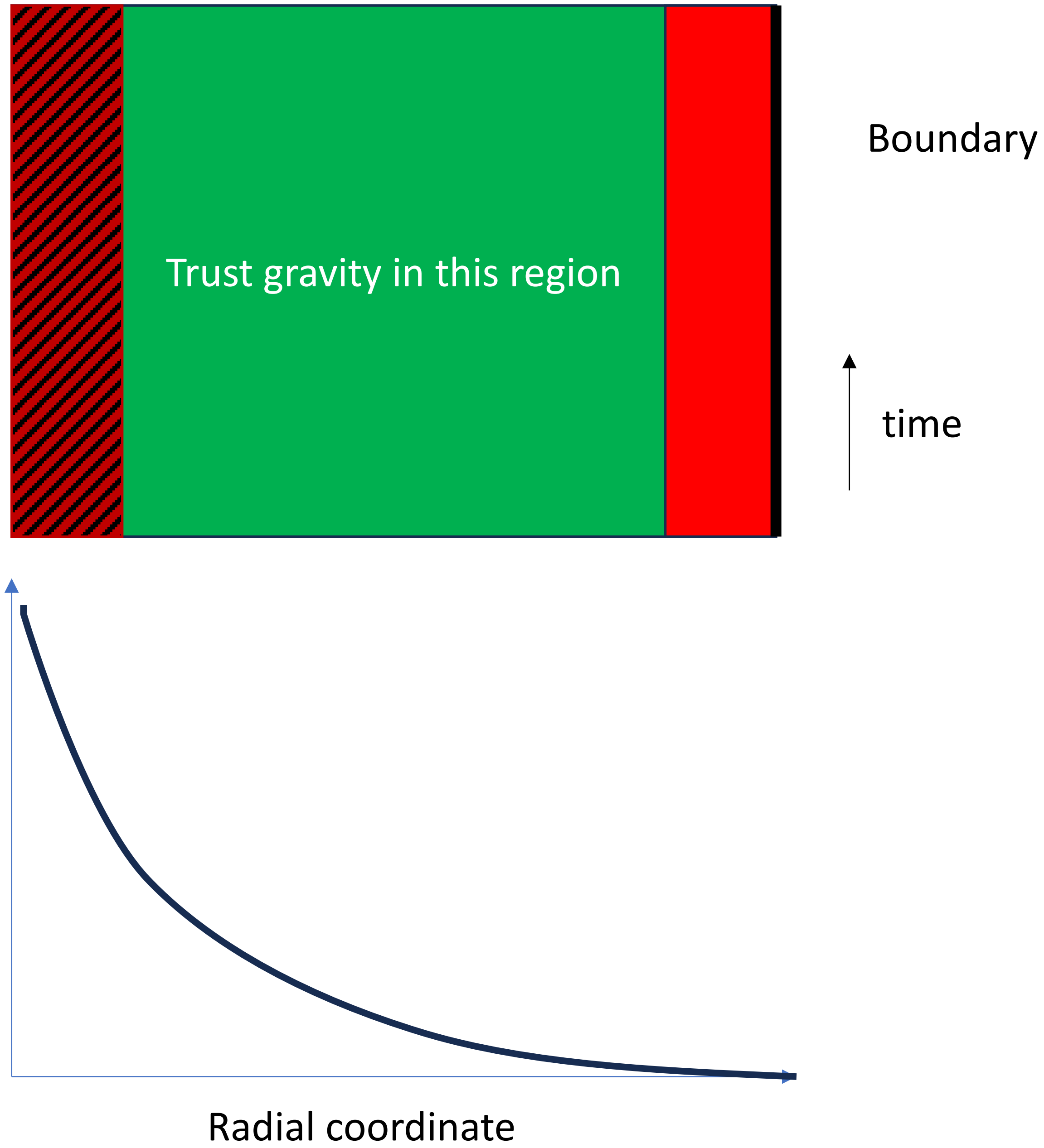


The size of the bound state covers the whole green region in gravity.

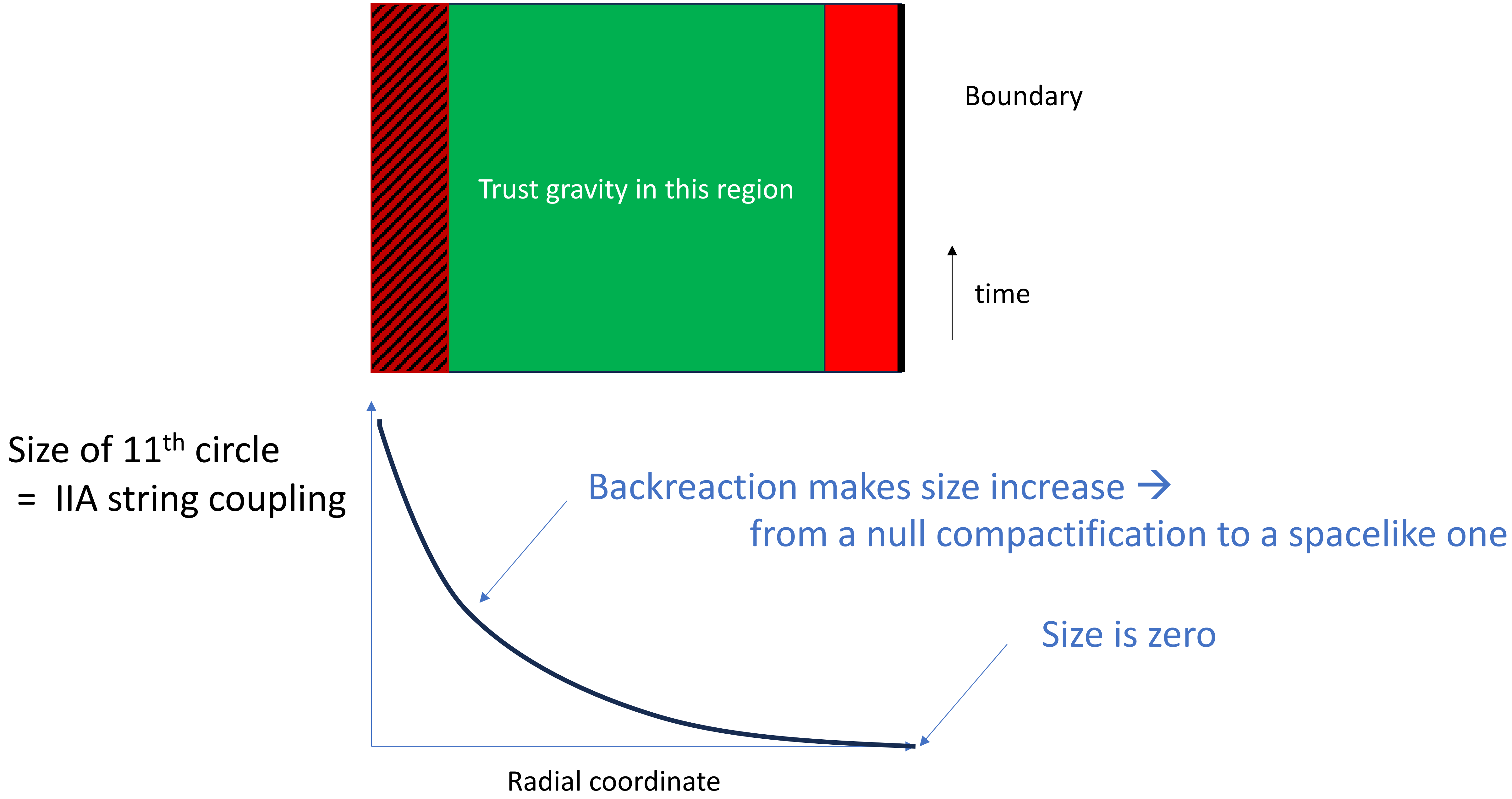
Other observables are natural in this other limit

- Correlation functions of operators. Sekino, Yoneya
- Finite temperature. Berkowitz, Rinaldi, Hanada, Ishiki, Shimasaki, Vranas; Pateloudis, Bergner, Hanada, Rinaldi Schaefer, Vranas, Watanabe, Bodendorfer
- The model with non-zero μ is simpler to think about conceptually.
- There are many other things one can say about this other limit…

An interesting feature of the geometry



Size of 11^{th} circle
= IIA string coupling


Trust gravity in this region
Boundary
time
Size of 11th circle
= IIA string coupling
Backreaction makes size increase →
from a null compactification to a spacelike one
Size is zero
Radial coordinate

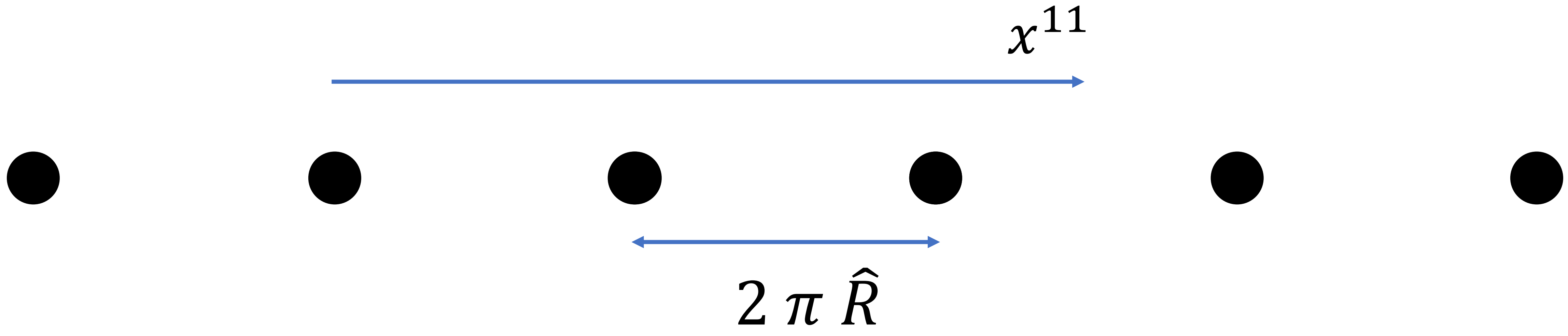
A related point

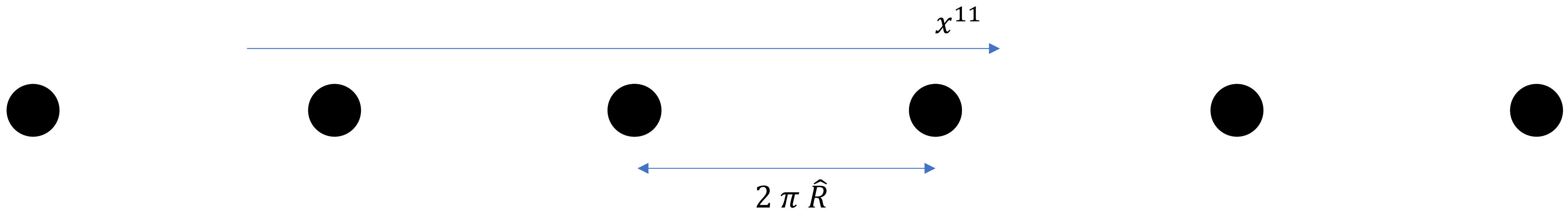
Consider an eleven dimensional black hole

$$ds^2 = -dx^+dx^- + d\vec{y}^{\,2} + (dt - dr)^2 \frac{r_s^8}{r^8} \;, \qquad r_r^8 \propto l_p^9 m$$

$$x^{\pm} = (t \pm x_{11}) \;, \qquad r^2 = x_{11}^2 + \vec{y}^{\,2}$$

In preparation for compactification, consider a periodic array of 11-d black holes.





$$ds^2 \sim -dx^+dx^- + d\vec{y}^{\,2} + \sum_{n=-\infty}^{\infty} (dt - dr_n)^2 \frac{r_s^8}{r_n^8} \ , \qquad x^{\pm} = t \pm x^{11}$$

$$\text{with} \quad r_n^2 = y^2 + (x^{11} - 2\pi\hat{R}n)^2 \ , \qquad \text{for} \quad r_s \ll \hat{R}$$

$$x^{11} \sim x^{11} + 2\pi\hat{R} \ , \qquad t \sim t - 2\pi\hat{R}, \qquad \longrightarrow \qquad x^- \sim x^- + 2\pi R \ , \qquad R = 2\hat{R}$$

Physical size of x^- grows as we approach the black holes.
We can neglect the x^- dependence for $y \gg R$, but not for $y \sim R$.

Near the black hole we have an eleven dimensional black hole and we do not notice the compactification.

In the limit we were taking the black hole has fixed p_+, p_-, and r_s.

But $R \to \infty$, so $R \gg r_s$

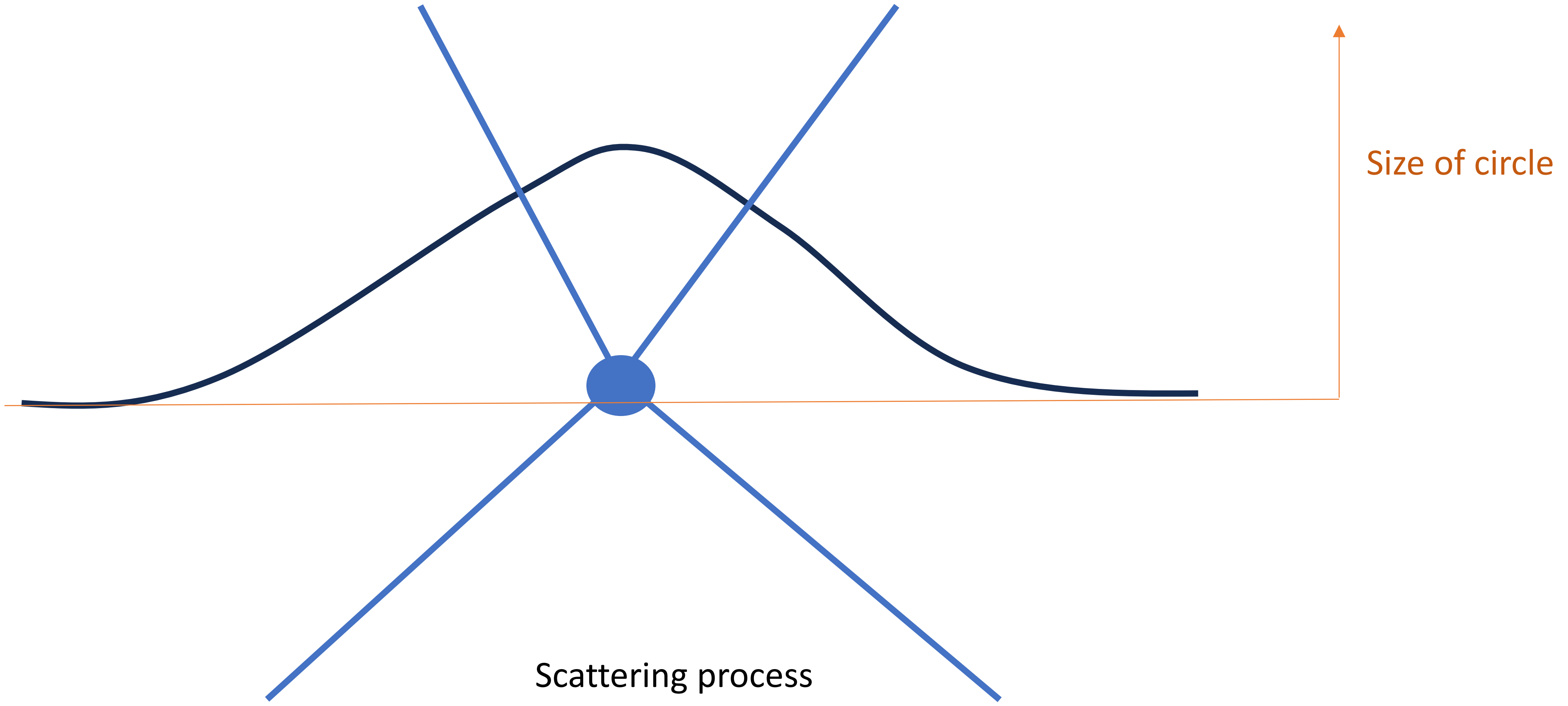
As long as we are at a finite distance from the black hole, the size of the x^- circle becomes large.

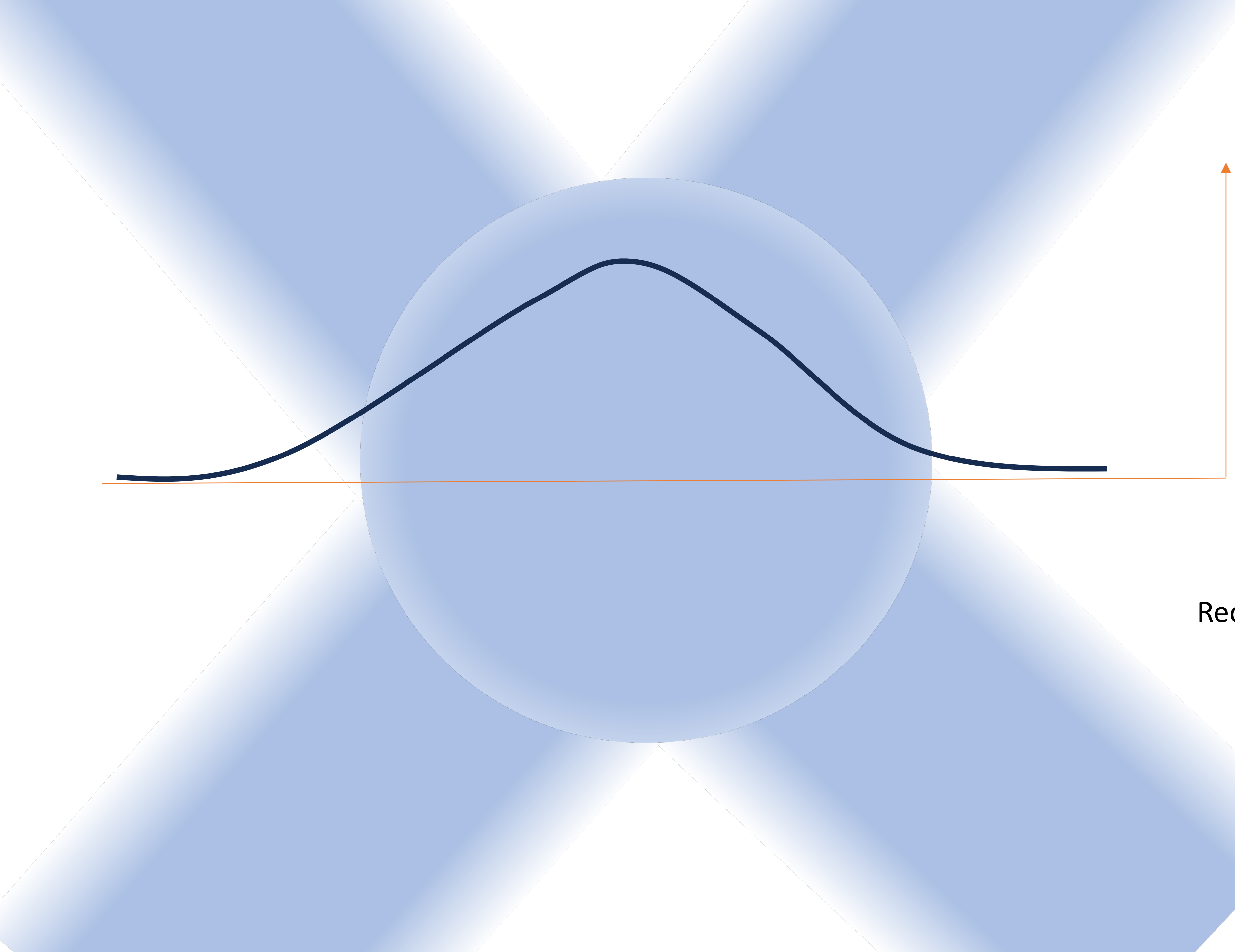
$$(\text{proper size of } x^-) \propto \frac{r_s^8}{r^8} R \to \infty$$

We expect something similar when we consider a scattering amplitude

At large N, the scattering is happening where the circle is spacelike and large.

Polchinski





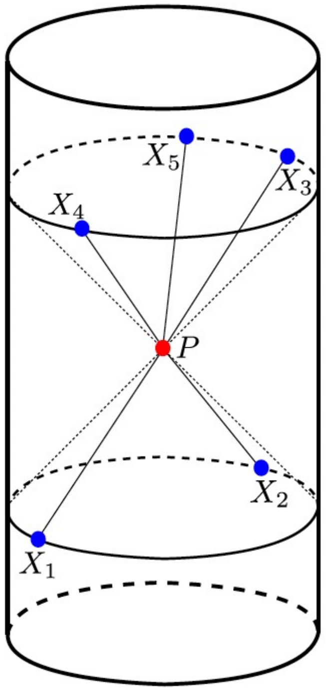
Size of circle

Recall that bound states are large

Comparison with other ways to obtain the flat space limit.

We can start from the ABJM theory that describes $AdS_4 \times S^7$ and then take the large N limit to obtain the flat space amplitudes.

Polchinski , Susskind , Penedones



This is conceptually similar to BFSS, since amplitudes are obtained through a large N limit.

It also involves a strongly coupled theory, away from the ‘t Hooft limit.

(BFSS obtains flat space through a different logic that these AdS examples!.).

We could say that this point of view is further developed.

The 3-pt amplitude is simple, and even the higher derivative corrections to the four point amplitude were found

R^4: Chester, Pufu, X. Yin

D^4R^4: Binder, Chester, Pufu

Celestial holography

- It is supposed to give us the flat space S-matrix.
- The BFSS conjecture gives us the S-matrix.
- It does not realize all the symmetries explicitly.

- Is there some other way to define it that realizes all the symmetries?
- Should it always involve some limit?

Something we have not discussed:

The IKKT matrix integral

Ishibashi, Kawai, Kitazawa, Tsuchiya

Conclusions

- We reviewed the BFSS conjecture for the non-perturbative S-matrix.
- It is an important window into flat space non-perturbative physics.

Thank you

Extra slides with more details

Slightly more precise statement

More precise relation between amplitudes

Matrix model amplitudes are more natural in the canonical normalization, as opposed to the relativistic normalization.

$$|p^c\rangle = \frac{|p\rangle_{rel}}{\sqrt{-2p_-}\sqrt{2\pi R}} \qquad {}_c\langle p'|p\rangle_c = (2\pi)^9\delta^9(\vec{p}\,' - \vec{p})$$

Then the n particle amplitudes in the matrix model have the form

$$\mathcal{A}_{mm} = (2\pi)^{10}\delta(\sum_{i=1}^{n} p_+^i)\delta^9(\sum_{i=1}^{n} \vec{p}^{\,i})\mathcal{M}_{mm}$$

The relativistically normalized n particle amplitudes in eleven dimensions have the form

$$\mathcal{A}_{rel} = (2\pi)^{11}\delta^{11}(\sum_i p_\mu^i)\mathcal{M}_{rel} = 2\pi R\delta(\sum_i N_i)(2\pi)^{10}\delta(\sum_i p_+^i)\delta^9(\sum_i \vec{p}^{\,i})\mathcal{M}_{rel}$$

We then get

$$\boxed{\mathcal{M}_{mm} = (2\pi R)^{1-\frac{n}{2}}\frac{\mathcal{M}_{rel}}{\prod_{i=1}^{n}\sqrt{-2p_-^i}}}$$

Just simple $\frac{1}{\sqrt{V}}$ factors in wavefunctions.

In the large N limit,

the matrix model amplitudes should decay like $N^{1-\frac{n}{2}}$ in order to give a finite eleven dimensional amplitude

For example, the low velocity scattering amplitude goes as

$$\int dt N_1 N_2 \frac{v_4 l_p^9}{R^3 r^7} = p_{1-} p_{2-} \frac{1}{R} \int dt \frac{v_4 l_p^9}{r^7}$$

This goes to zero as required by the previous argument, for n=4.

This was already taken into account in previous comparisons for this problem (though it was phrased differently).