Integrated correlators and the bootstrap

Silviu S. Pufu, Princeton University

Based on arXiv:2312.12576 with S. Chester and R. Dempsey, earlier work with D. Binder, S. Chester, N. Agmon, Y. Wang, and work in progress with S. Chester and R. Dempsey

CERN, June 6, 2024

Introduction

- **Strings 2024** conference $\Longrightarrow$ talk about (solving) string theory!
- Via AdS/CFT, string/M-theory in $AdS \times X \longleftrightarrow$ (S)CFT.
- **This talk:** Study $\mathcal{N} = 4$ SYM (and ABJM) theory using bootstrap.
- Various approaches
  - weak coupling expansion (small $\lambda = g_{\text{YM}}^2 N$)
  - integrability (large N)
  - holography (large N, large λ)
  - supersymmetric localization (only protected observables)
  - numerical bootstrap
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- Vector multiplet: A_μ, X_a, λ^i, λ_i, with $a = 1, \ldots, 6$ and $i = 1, \ldots, 4$.
- Conformal manifold param. by g_{YM} and θ. Define $\tau \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}$.
- Anomaly coeffs. $c = a = \frac{N^2-1}{4}$.
- **Question:** What is the low-lying spectrum?
  - **Lowest protected operator**: $S_{ab} = \operatorname{tr}\left(X_a X_b - \frac{1}{6}\delta_{ab} X_c X_c\right)$ has $\Delta = 2$, in $\mathbf{20}'$ of $SO(6)_R$, same multiplet as stress energy tensor
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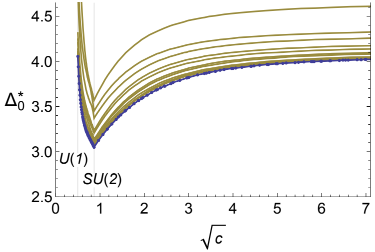
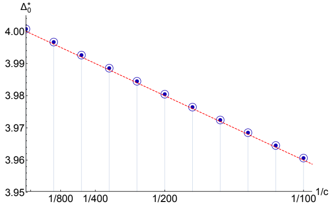
- Plots from [Beem, Rastelli, van Rees '16] showing upper bounds on the dimension of the lowest unprotected scalar operator.





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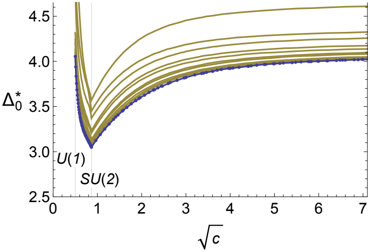

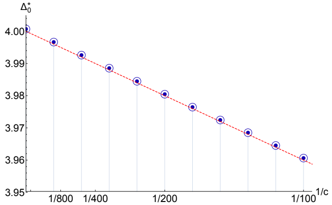

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Integrated correlators

- Supersymmetric localization gives $Z_{S^4}(\tau, \bar{\tau}, m)$ [Pestun '07] where m is $\mathcal{N} = 2$-preserving mass (mass for hypermultiplet).
- Derivs of $Z_{S^4}(\tau, \bar{\tau}, m)$ at $m = 0 \to$ ***integrated* correls** in SCFT (see [Closset, Dumitrescu, Festuccia, Komargodski, Seiberg '12] for 2-pt fns in 3d).
- On S^4 of radius r, the action is (recall $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}$)

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- Here, $\mathcal{O}$, $\overline{\mathcal{O}}$ are $\Delta = 4$ ops (Lagrangian $\pm F\tilde{F}$-term)

  $J(x)$ is a $\Delta = 2$ op. (scalar bilinear, specific component of S_{ab})

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- Derivatives w.r.t. τ, $\bar{\tau}$, m bring down **integrated** ops

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- We consider two **integrated** 4-point functions

$$\left.\frac{\partial^4 \log Z_{S^4}}{\partial\tau\partial\bar{\tau}\partial m^2}\right|_{m=0}\,, \qquad \left.\frac{\partial^4 \log Z_{S^4}}{\partial m^4}\right|_{m=0}$$

(Each equals an integrated 4-pt fn + integrated lower-pt functions that are fixed by SUSY)

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$$\frac{\partial}{\partial\tau} \to 32\pi^2 A(N)\,, \qquad \frac{\partial}{\partial\bar{\tau}} \to 32\pi^2 \overline{A}(S)$$

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To use this in bootstrap, follow 3 steps:

1. Relate derivs of Z_{S^4} to integrated 4-pt functions of S_{ab}. Need Ward ids. Done in [Binder, Chester, SSP, Wang '19; Chester, SSP '20] . For example:

$$\partial_\tau \partial_{\bar\tau} \partial_m^2 \log Z\big|_{m=0} = \int dU\, dV\, \mu(U, V)\, T(U, V)$$

   where $\mu(U, V)$ is a SUSY-preserving measure; $T(U, V)$ appears in $\langle SSSS \rangle$.

2. Calculate $\frac{\partial^4 \log Z}{\partial\tau\partial\bar\tau\partial m^2}$, $\frac{\partial^4 \log Z}{\partial m^4}$ from Pestun's matrix model for any N, τ, $\bar\tau$.
   - Need instanton partition function → interesting $SL(2, \mathbb{Z})$ properties [Chester, Green, SSP, Wang, Wen '19, 20']
   - $(N - 1)$-dimensional integral, so explicit evaluations only for small N
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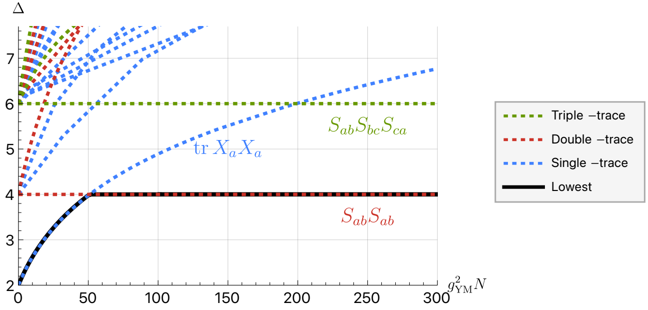
- At large N, **planar** result from integrability [Gromov, Hegedus, Julius, Sokolova '23] :



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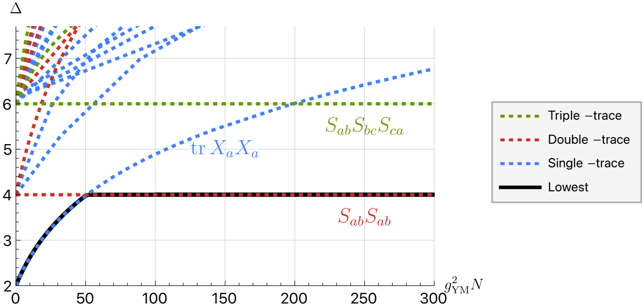

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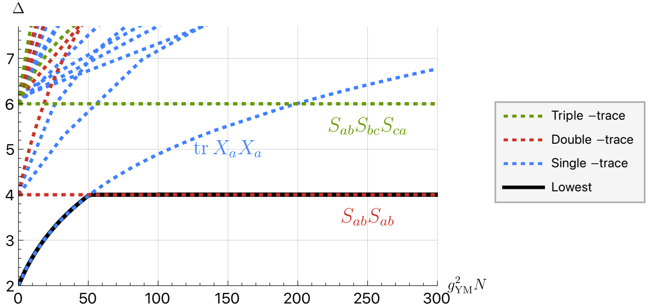

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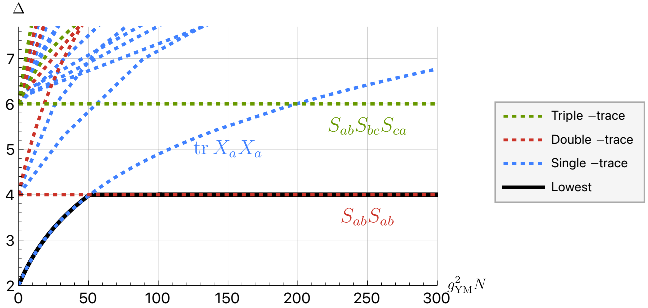

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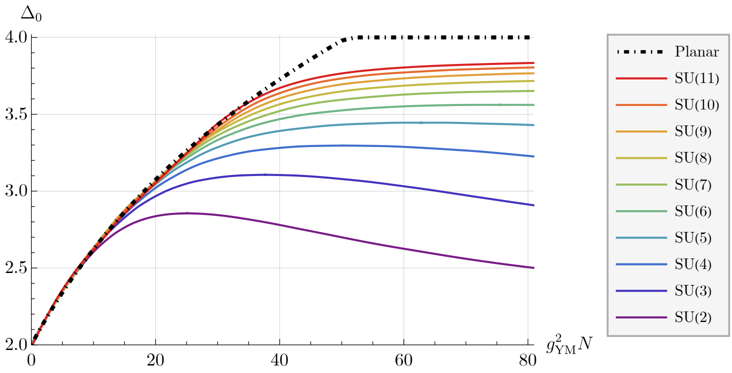
Results

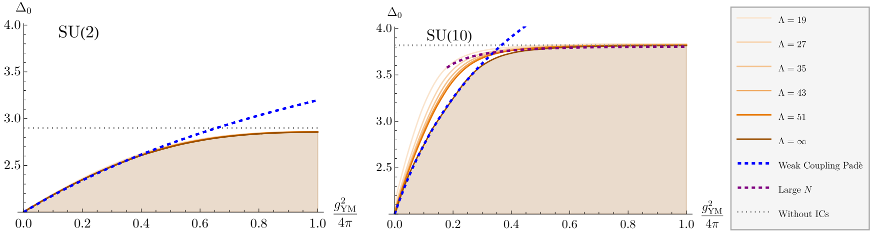
- **Upper bounds** on the dimension of the lowest unprotected scalar for various values of N and g_{YM}, at $\theta = 0$ [Chester, Dempsey, SSP '23]



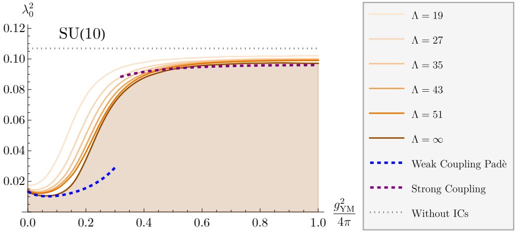
- The max is at $g^2_{\text{YM}} = 4\pi$ (self-dual point). Each curve is invariant under the S-duality transformation $g_{\text{YM}} \to 4\pi/g_{\text{YM}}$.

Results

- Comparison b/w upper bounds for $SU(2)$ and $SU(10)$ and weak coupling and large N expansions.

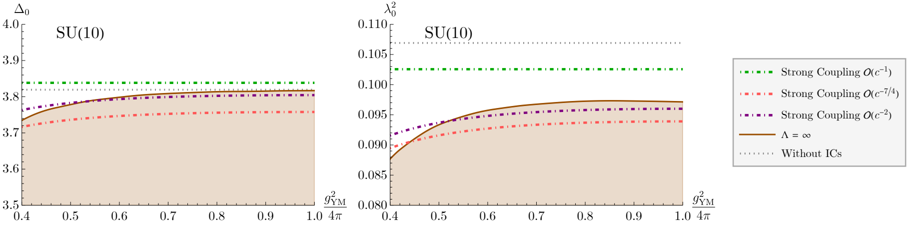


- Bound w/o integrated constraints is never saturated.
- Crossover for OPE coeffs



Results

- Comparison with large N expansions shows bounds are sensitive to NNNLO (beyond tree level SUGRA) correction

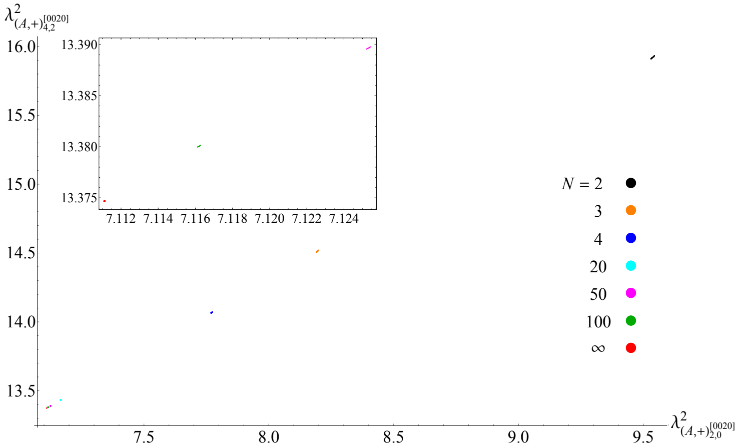


ABJM theory

- In 3d $\mathcal{N} = 8$ SCFTs, the analog of the $\mathbf{20}'$ operator S_{ab} is a $\Delta = 1$ operator S_{AB} in $\mathbf{35}_c$ of $SO(8)_R$, with $A, B = 1, \ldots, 8$.
- One can compute $Z_{S^3}(m_+, m_-)$ where $m_\pm$ are $\mathcal{N} = 4$-preserving masses [Kapustin, Willett, Yaakov '09] .
- The derivatives $\frac{\partial^4 \log Z_{S^3}}{\partial m_+^4}$, $\frac{\partial^4 \log Z_{S^3}}{\partial m_-^4}$ give directly an OPE coefficient [Dedushenko, SSP, Yacoby '16] .
- The mixed derivative $\frac{\partial^4 \log Z_{S^3}}{\partial m_+^2 \partial m_-^2}$ gives an integrated correlator of S_{AB} [Binder, Chester, SSP '18] .

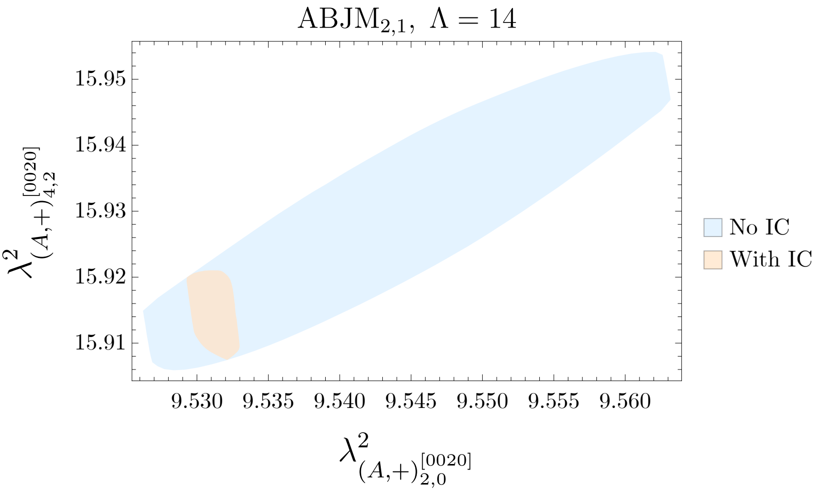
M-theory archipelago

- Bounds for **OPE coefficients** in $\mathcal{N} = 8$ SCFTs, with input about OPE coefficient obtained from 4th mass derivative for the $U(N)_1 \times U(N)_{-1}$ ABJM theory [Agmon, Chester, SSP '19] :



Integrated constraints in ABJM

- Integrated constraint from $\frac{\partial^4 \log Z_{S^3}}{\partial m_+^2 \partial m_-^2}$ reduces size of the island by a factor of ~ 10. For $N = 2$ [Chester, Dempsey, SSP, work in progress] :

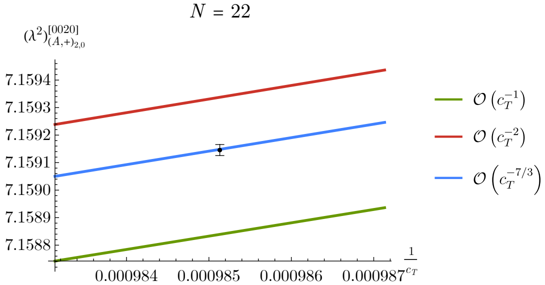


Comparison with large N expansion

- OPE coefficient in terms of $c_T \sim N^{3/2}$:

$$\lambda^2 = \frac{64}{9} + \underbrace{\frac{20480(1-\frac{\pi^2}{3}+\frac{5}{2})}{9\pi^2} c_T^{-1}}_{\text{tree-level SG}} + \underbrace{513.492359 c_T^{-2}}_{\text{one-loop SG}} - \underbrace{\frac{917504(2/3)^{1/3}}{9\pi^{10/3}} c_T^{-7/3}}_{\text{contact } D^6R^4} + \cdots$$

- Numerics are sensitive to NNNLO correction [Chester, Dempsey, SSP, work in progress] :



- With more precision, we would be sensitive to unprotected contributions in M-theory effective action!

A comment on integrated correlators

- In d=4 $\mathcal{N}=4$ SYM, $\mathcal{I}=\partial_\tau\partial_{\bar\tau}\partial_m^2\log Z_{S^4}$,

$$\mathcal{I}\propto\left\langle\left[\int d^4x_1\sqrt{g}\left(\frac{i}{r}J+K\right)\right]\left[\int d^4x_2\sqrt{g}\left(\frac{i}{r}J+K\right)\right]\overline{A}(S)A(N)\right\rangle$$

can be written using ($J=S_{11}+S_{22}-S_{33}-S_{44}$, $A=S_{55}+2iS_{56}-S_{66}$)

$$\langle J(x_1)J(x_2)\overline{A}(x_3)A(x_4)\rangle=\frac{1}{x_{12}^4x_{34}^4}F(u,v)\,.$$

- The integrated correl. is [Binder, Chester, SSP, Wang '19]

$$\mathcal{I}\propto\int_0^\infty\frac{dR}{R}\int_0^\pi d\theta\,\sin^2\theta\,\frac{v}{u^2}F(u,v)\Big|_{\substack{u=1+R^2-2R\cos\theta\\ v=R^2}}$$

- Interpretation: Send $x_3\to\infty$, $x_4\to 0$ and map to $S^3\times\mathbb{R}$:

$$\langle J(\tau,\hat n)J(\tau',\hat n')\rangle_{A\bar A}=\langle\overline{A}|J(\tau,\hat n)J(\tau',\hat n')|A\rangle=\frac{v}{u^2}F(u,v)\,,$$

with $R=e^{\tau-\tau'}$, $\cos\theta=\hat n\cdot\hat n'$.

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- $\mathcal{I}$ is an integral over $S^3 \times \mathbb{R}$ with a flat measure!

$$\mathcal{I} \propto \int_{S^3 \times \mathbb{R}} d\tau \, d\Omega_3 \, \langle J(\tau, \hat{n}) J(\tau', \hat{n}') \rangle_{A\bar{A}}$$

- Easier to prove Ward ids on $S^3 \times \mathbb{R}$ [Dempsey, Offertaler, SSP, Wang, in progress] :

$$\langle K(\tau, \hat{n}) K(\tau', \hat{n}') \rangle_{A\bar{A}} \propto (\Box - 1) \left[e^{-(\tau - \tau')} \langle J(\tau, \hat{n}) J(\tau', \hat{n}') \rangle_{A\bar{A}} \right]$$

- Generalization to half-maximal SCFTs in 4d, 3d, and even to defects.

- For 1/2-BPS line defect in $\mathcal{N} = 2$ SCFTs, (e.g. circular Wilson loop in $\mathcal{N} = 4$ SYM), we have [Dempsey, Offertaler, SSP, Wang '24]

$$\left. \frac{\partial^2 \log \langle \mathbb{W} \rangle}{\partial m^2} \right|_{m=0} \propto \int_{\mathbb{H}^2 \times S^2} d^4x \, \sqrt{g(x)} \, \langle J(x) J(x') \rangle_{\mathbb{W}} \, ,$$

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Conclusion

- Integrated correlators are very powerful when combined with numerical bootstrap.
- For $\mathcal{N} = 4$ SYM they allow us to find coupling-dependent bounds.
- For ABJM theory they allow a more detailed comparison with M-theory.
- Related topics: analytic bootstrap at large N, defects, $SL(2, \mathbb{Z})$, etc.
- For the future: more precision for bootstrap numerics, constraints involving squashing, less supersymmetric theories, etc.