

# 3d gravity as a random ensemble

Gabriel Wong

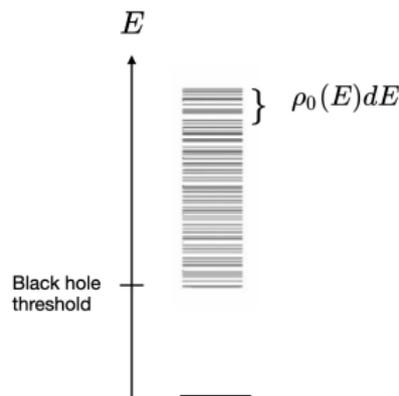
Oxford Math Institute

To appear soon with Dan Jafferis and Liza Rozenberg

- Semi-classical gravity averages over UV microstates. This is due to black hole solutions, whose horizon area gives the high energy density of states:

$$\rho_0(E) = \exp \frac{\text{Area}(E)}{4G}$$

- This density of states is continuous because it is averaged over a high energy window. In AdS3, it corresponds to the Cardy density of the dual CFT.



# Gravity and averaging II: Wormholes

- Gravity also computes ensemble averages. This is because of wormholes.

$$Z_{grav} \left( \beta_1 \text{ (circle)} \quad \text{(circle)} \beta_2 \right) = \beta_1 \text{ (cup)} \quad \text{(cup)} \beta_2 + \beta_1 \text{ (hourglass)} \beta_2 + \dots$$

$$Z_{CFT} \left( \beta_1 \text{ (circle)} \quad \text{(circle)} \beta_2 \right) = Z_{CFT}(\beta_1) Z_{CFT}(\beta_2)$$

- The inclusion of wormhole topologies provides a coarse grained description in terms of an ensemble of quantum systems (see Monday's review talk).

$$\beta_1 \text{ (cup)} \quad \text{(cup)} \beta_2 + \beta_1 \text{ (hourglass)} \beta_2 + \dots = \overline{Z_{CFT}(\beta_1) Z_{CFT}(\beta_2)}$$

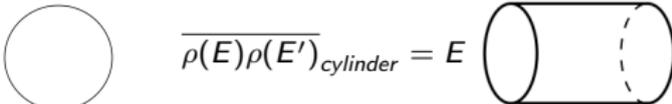
- These two types of coarse graining are related. For a chaotic system like a black hole, averaging over a high energy window is indistinguishable from averaging over an ensemble.

- Just like in statistical physics, the ensemble interpretation of quantum gravity provides the “best” description of a system given a set of constraints (de Boer/Liska/Post/Sasieta, Di Ubaldo, Perlmutter)
- For instance, the micro-canonical ensemble maximizes entropy subject to the constraint of fixed energy.
- A random matrix model, describing a random Hamiltonian  $H$ , is the maximum entropy ensemble for a given average density of states.

$$\mathcal{Z} = \int dH \quad e^{-V(H)} \quad V(H) \longleftrightarrow \overline{\rho(E)}_{\text{ensemble}}$$

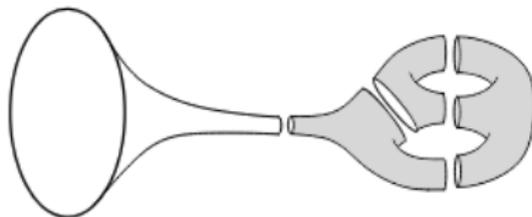
# JT gravity as a maximum ignorance ensemble

- Saad, Shenker, Stanford (SSS) showed that two dimensional JT gravity is equivalent to a double scaled matrix model.
- Inputs:

$$\overline{\rho(E)}_{\text{disk}} = E \quad \overline{\rho(E)\rho(E')}_{\text{cylinder}} = E \quad E' \quad (1)$$


The cylinder determines the eigenvalue statistics associated to the symmetry class of the ensemble.

- This determines the gravity path integral on all 2D geometries via the machinery of topological recursion



[figure from SSS].

- We consider a generalization of the SSS model and study an ensemble of CFT2's dual to AdS3 gravity (Belin/de Boer/Jafferis/Nayak/Sonner)
- CFT2 data is given by the Dilatation operator  $\Delta_s$  graded by spin, and the OPE coefficients  $C_{ijk}$ . Assuming integers spins, we get a set of random matrices and a random tensor:

$$(\Delta_s, C), \quad s \in \mathbb{Z}$$

# AdS3 gravity as a maximum ignorance ensemble

- We consider a generalization of the SSS model and study an ensemble of CFT2's dual to AdS3 gravity (Belin/de Boer/Jafferis/Nayak/Sonner)
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$$(\Delta_s, C), \quad s \in \mathbb{Z}$$

This random data satisfy locality constraints given by the modular bootstrap. This is generated by 4 point crossing equation on the sphere

$$\sum_p C_{12p} C_{p34} \left| \begin{array}{c} 2 \\ \diagdown \\ 1 \end{array} \right| \begin{array}{c} p \\ \text{---} \\ 4 \end{array} \left| \begin{array}{c} 3 \\ \diagup \\ 4 \end{array} \right| \left| \begin{array}{c} 2 \\ \diagdown \\ 1 \end{array} \right| \begin{array}{c} \bar{p} \\ \text{---} \\ 4 \end{array} \left| \begin{array}{c} 3 \\ \diagup \\ 4 \end{array} \right| - \sum_q C_{23q} C_{q41} \left| \begin{array}{c} 2 \\ \diagdown \\ 1 \end{array} \right| \begin{array}{c} q \\ \text{---} \\ 4 \end{array} \left| \begin{array}{c} 3 \\ \diagup \\ 4 \end{array} \right| \left| \begin{array}{c} 2 \\ \diagdown \\ 1 \end{array} \right| \begin{array}{c} \bar{q} \\ \text{---} \\ 4 \end{array} \left| \begin{array}{c} 3 \\ \diagup \\ 4 \end{array} \right| = 0,$$

and torus modular invariance.

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and torus modular invariance.

- We give further evidence that the sum over topologies in AdS3 gravity produces the maximum ignorance ensemble consistent with the modular bootstrap.

- 1 Ensemble of approximate CFT's
- 2 The tensor potential and Virasoro TQFT
- 3 The matrix potential and  $SL(2, \mathbb{Z})$  modular invariance
- 4 Sum over all manifolds and the Schwinger Dyson equation

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# Ensemble of approximate CFT's

- An ensemble of exact CFT's would have a potential  $V_0$  defined by the Cardy density, and a delta function imposing the bootstrap constraints:

$$\mathcal{Z}_0 \equiv \sum_{s \in \mathbb{Z}} \int D[\Delta_s] D[C] e^{-V_0(\Delta_s)} \delta(\text{constraints})$$

- For irrational CFT's with only Virasoro symmetry, we don't know how to impose these constraints. Following (Belin/de Boer/Jafferis/Nayak/Sonner), we relax the constraints by smearing the delta function with a parameter  $\hbar$

$$\mathcal{Z}_\hbar \equiv \sum_{s \in \mathbb{Z}} \int D[\Delta_s] D[C] \exp\left(-V_0(\Delta_s) - \frac{1}{\hbar} V(\Delta_s, C)\right)$$

- $V(\Delta_s, C)$  is a “constraint squared” potential that is minimized on solutions of the bootstrap. This defines an ensemble of *approximate* CFT's, with  $\hbar$  parametrizing the violation of the bootstrap.
- Since  $\hbar = 0$  localizes to exact CFT's, computing  $\mathcal{Z}_\hbar$  and then taking  $\hbar \rightarrow 0$  should teach us something about the space of CFT's

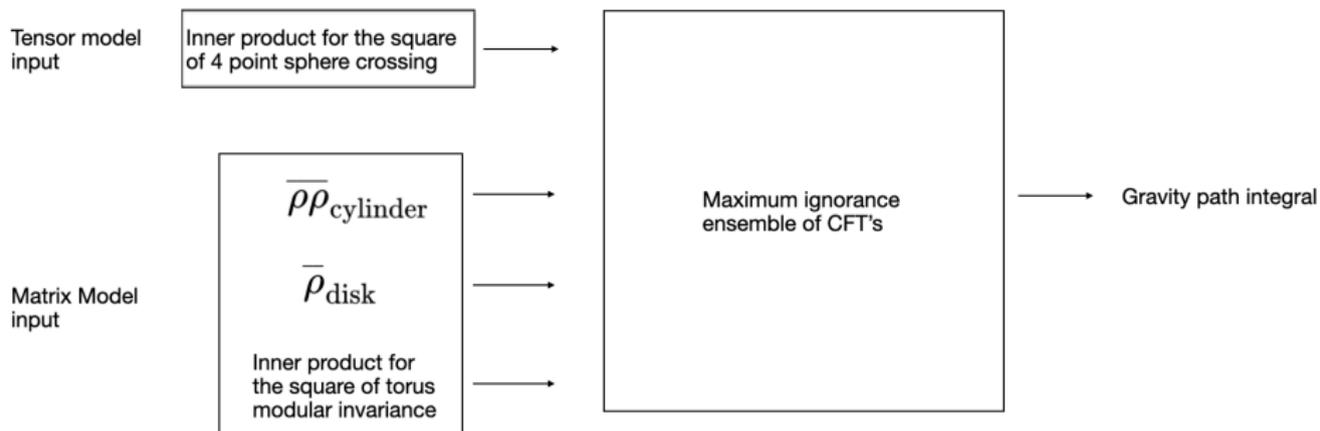
# 3d gravity from the ensemble of approximate CFT's

- We further develop the program presented in (Belin/de Boer/Jafferis/Nayak/Sonner):
- First define the ensemble as an integral over a finite tensor and matrices by truncating to  $N$  primaries. Then take a double scaling limit of  $\Delta_s$  that sends  $N \rightarrow \infty$
- Apriori, we want  $\hbar \rightarrow 0$  at fixed central charge  $c$ . However, we will first re-organize the Feynman diagrams of the ensemble into an  $e^{-c}$  expansion, then take  $\hbar \rightarrow 0$
- We will argue that the  $e^{-c}$  expansion of the ensemble reproduces the topological expansion of 3d gravity.

$$\overline{\rho(\Delta, s)} = \text{[diagram: sphere with a small circle attached]} + \text{[diagram: sphere with a handle]} + \dots$$
$$\overline{C_{ijk} C_{kji}} = \text{[diagram: cylinder with labels i, j, k]} + \text{[diagram: cylinder with a handle and labels i, j, k]} + \dots$$
$$\sum_i \overline{C_{ijk} C_{kji}} \longrightarrow \text{surgery on the manifolds above}$$

- Unifies many recent ideas in 3d gravity and random ensembles: Mertens/Turiaci, Collier/Maloney, Maxfield/Tsaires, Cotler/Jensen, Belin/de Boer, Anous/Belin/de Boer/Liska, Chandra/Collier/Hartman/Maloney, Jafferis/Kolchmeyer/Mukhametzhanov/Sonner, Yan, Di Ubaldo, Perlmutter, Verlinde/Collier/Eberhardt/Mengyang

# A flow chart



- The tensor model story follows (Belin/de Boer/Jafferis/Nayak/Sonner) closely. We offer a new perspective that clarifies the connection to VTQFT (Verlinde/Collier/Eberhardt/Mengyang)

- 1 Ensemble of approximate CFT's
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# Verlinde inner product and 4 point crossing

- 4 point crossing = the vanishing of a vector.

$$\sum_p C_{12p} C_{p34} \left| \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} p \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array} \right| \left| \begin{array}{c} 3 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 4 \end{array} \right| - \sum_q C_{23q} C_{q41} \left| \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} q \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array} \right| \left| \begin{array}{c} 3 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 4 \end{array} \right| = 0$$

We want an inner product to take its square.

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- The space of Virasoro conformal blocks forms a Hilbert space equipped with the Verlinde inner product and a representation of crossing transformations.

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$$\left| \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} p \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array} \right| \left| \begin{array}{c} 3 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 4 \end{array} \right| = \frac{\delta^{(2)}(P_q - P_p)}{S_{\mathbb{1}P_q}} \left| \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \text{---} q \text{---} \\ \diagup \quad \diagdown \\ 1 \end{array} \right| \left| \begin{array}{c} 3 \\ \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \\ 4 \end{array} \right| = \begin{Bmatrix} q & 4 & 1 \\ p & 2 & 3 \end{Bmatrix}$$

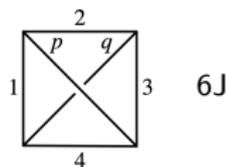
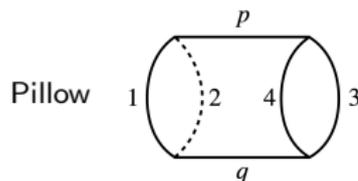
- This is the Hilbert space structure of Virasoro TQFT (Verlinde/Collier/Eberhardt/Mengyang).

# Constraint squared potential

- The square of 4 point crossing defines a quartic potential (Belin/de Boer/Jafferis/Nayak/Sonner)

$$V_4 = \sum_{i_1 \dots i_4} \sum_{p,q} (C_{i_1 i_2 p} C_{p i_3 i_4} C_{q i_2 i_1} C_{i_4 i_3 q}) \left| \left\langle \begin{array}{c} \diagup 2 \\ \diagdown 1 \end{array} \begin{array}{c} p \\ \text{---} \\ \diagup 3 \\ \diagdown 4 \end{array} \right| \left| \begin{array}{c} \diagup 2 \\ \diagdown 1 \end{array} \begin{array}{c} q \\ \text{---} \\ \diagup 3 \\ \diagdown 4 \end{array} \right\rangle \right|^2$$

$$- (C_{i_1 i_2 p} C_{p i_3 i_4} C_{i_1 i_4 q} C_{i_3 i_2 q}) \left| \left\langle \begin{array}{c} \diagup 2 \\ \diagdown 1 \end{array} \begin{array}{c} p \\ \text{---} \\ \diagup 3 \\ \diagdown 4 \end{array} \right| \left| \begin{array}{c} \diagup 2 \\ \diagdown 1 \end{array} \begin{array}{c} q \\ \text{---} \\ \diagup 3 \\ \diagdown 4 \end{array} \right\rangle \right|^2$$



# Constraint squared potential

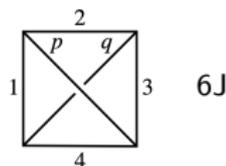
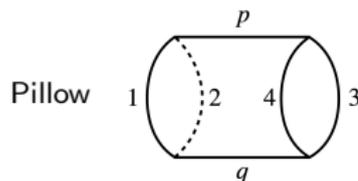
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$$- (C_{i_1 i_2 p} C_{p i_3 i_4} C_{i_1 i_4 q} C_{i_3 i_2 q}) \left| \left\langle \begin{array}{c} \text{Diagram 3} \end{array} \right| \left\langle \begin{array}{c} \text{Diagram 4} \end{array} \right\rangle \right|^2$$

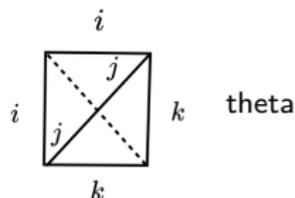
The diagrams in the equation are:

- Diagram 1: A horizontal line with four vertices labeled 1, 2, 3, 4 from left to right. A line labeled 'p' connects vertices 2 and 3.
- Diagram 2: A horizontal line with four vertices labeled 1, 2, 3, 4 from left to right. A line labeled 'q' connects vertices 2 and 3.
- Diagram 3: A horizontal line with four vertices labeled 1, 2, 3, 4 from left to right. A line labeled 'p' connects vertices 2 and 3.
- Diagram 4: A vertical line with four vertices labeled 1, 2, 3, 4 from bottom to top. A line labeled 'q' connects vertices 2 and 3.



- The kinetic term comes from a special case of the 6J:

$$C_{ii1} = 1 \longrightarrow V_2 = - \sum_{ijk} C_{ijk} C_{jik}$$



# Triple line Feynman diagrams and their gravity interpretation

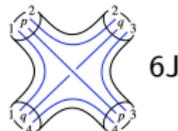
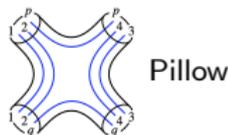
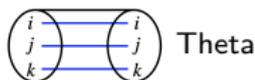
- The triple line Feynman rules are defined by removing the vertices from the graphs

$$\begin{array}{c} i \text{ --- } i \\ j \text{ --- } j \\ k \text{ --- } k \end{array} = \hbar \delta_{il} \delta_{jm} \delta_{kp},$$

$$\begin{array}{c} \text{---} 1 \text{---} \\ \text{---} 2 \text{---} \\ \text{---} 4 \text{---} \\ \text{---} 3 \text{---} \end{array} \begin{array}{c} p \\ q \end{array} = \frac{1}{\hbar} \frac{\delta(P_p - P_q)}{S_{\perp P_q}},$$

$$\begin{array}{c} \text{---} 2 \text{---} \\ \text{---} 1 \text{---} \\ \text{---} 3 \text{---} \\ \text{---} 4 \text{---} \end{array} \begin{array}{c} p \\ q \end{array} = \frac{1}{\hbar} \begin{Bmatrix} q & 4 & 3 \\ p & 2 & 2 \end{Bmatrix}$$

- We interpret these diagrams as multiboundary wormholes with Wilson lines inserted:

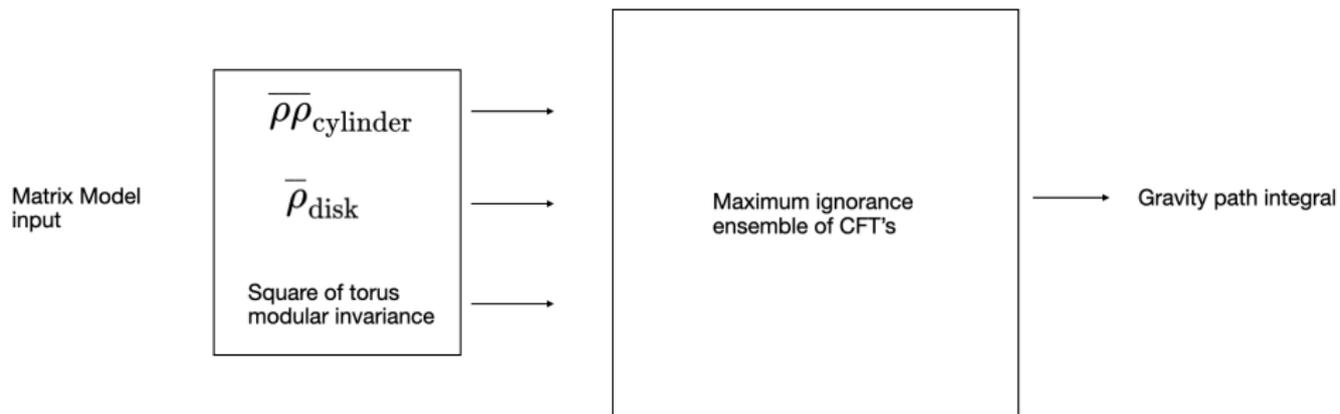


- They come from Wilson line networks in the three sphere, with solid balls removed around the junctions. (Collier/Eberhardt/Mengyang)
- Once the lines are framed, the Feynman rules generate (braided) connect sums of the pillow and 6J manifolds that generates a sum over 3-manifolds. On a fixed manifold they agree with 3d gravity as defined by 2 copies of Virasoro TQFT .

$$Z_{\text{tensor}}(M) = |Z_{\text{Vir}}(M)|^2$$

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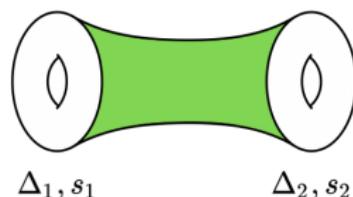


- The random Hamiltonians  $\Delta_s$  belong to the **GOE ensemble** (Yan). The GOE measure exponentiates to the Vandermonde potential

$$K(\Delta_1, s_1; \Delta_2, s_2) = \delta(s_1 - s_2) \text{Log}|\Delta_1 - \Delta_2|^1$$

- The “cylinder” contribution to  $\overline{\rho\rho}$  is the inverse of the Vandermonde kernel. It is the **propagator for  $\rho$** .

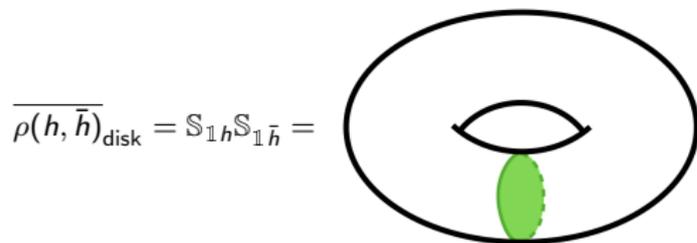
$$\overline{\rho(\Delta_1, s_1)\rho(\Delta_2, s_2)}_{cylinder} = K^{-1}(\Delta_1, s_1; \Delta_2, s_2) =$$



It agrees with the 3d gravity path integral on the torus wormhole computed in (Cotler/Jensen 20), up to an extra factor of 2 due to extra wormholes needed to get agreement with the GOE ensemble (Yan, Jensen) .

# Cardy density and the BTZ black hole

- The disk density for  $\Delta_s$  is given by the Cardy formula. This is a product of Virasoro S matrix elements in the left and right moving sector.

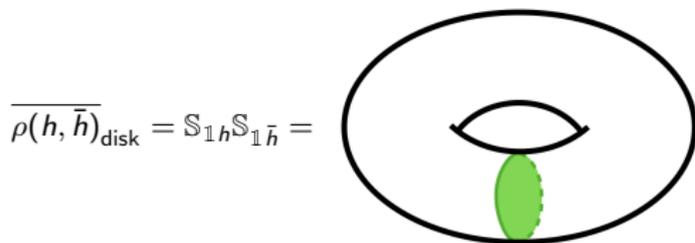


$$\Delta = h + \bar{h}, \quad s = h - \bar{h} \in \mathbb{Z}$$

$$\Delta \geq s + \frac{c-1}{12}$$

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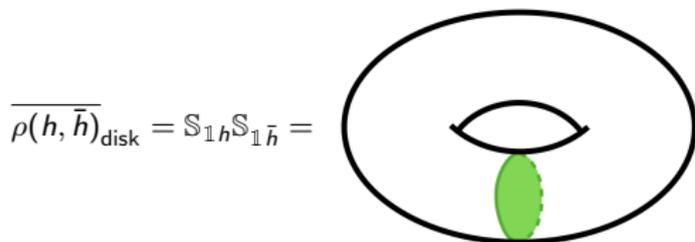
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$$Z_{\text{BTZ}}(\tau, \bar{\tau}) = \chi_1 \left( -\frac{1}{\tau} \right) \chi_1 \left( -\frac{1}{\bar{\tau}} \right) = \int_{\frac{c-1}{24}}^{\infty} \int_{\frac{c-1}{24}}^{\infty} dh d\bar{h} \mathbb{S}_{1h} \mathbb{S}_{1\bar{h}} \chi_h(\tau) \chi_{\bar{h}}(\bar{\tau})$$

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$$\overline{\rho(h, \bar{h})}_{\text{disk}} = S_{1h} S_{1\bar{h}} =$$

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- To get all spins, write  $\sum_{s \in \mathbb{Z}} = \int ds \sum_{n=-\infty}^{\infty} e^{2\pi i n s}$

$$\mathcal{Z}_0 \equiv \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} ds e^{2\pi i n s} \int D\Delta_s D[C] \exp(-V_0(\Delta_s))$$

T transform

# Constraint squared potential for S-modular invariance I:

- The (approximate) CFT partition function is a vector on the Hilbert space  $\mathcal{H}_{T^2} \otimes \mathcal{H}_{T^2}$  of Virasoro characters

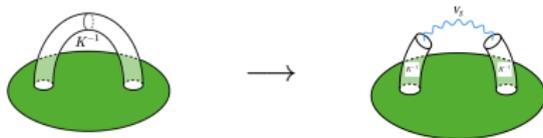
$$|Z(\tau, \bar{\tau})\rangle = \sum_i |h_i\rangle |\bar{h}_i\rangle$$

- Given an inner product, we can define a double trace potential for modular invariance

$$V_S \equiv |(\mathbb{1} - \mathbb{S})|Z(\tau, \bar{\tau})\rangle|^2 = \sum_{i,j} \langle h_i, \bar{h}_i | \hat{V}_S | h_j, \bar{h}_j \rangle$$

- We then expand in powers of the potential  $\frac{1}{\hbar} V_S$ :

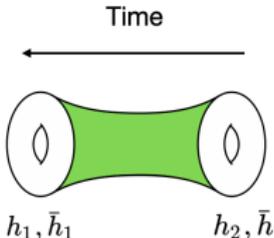
$$\mathcal{Z} \equiv \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} ds e^{2\pi i n s} \int D\Delta_s D[C] \exp(-V_0(\Delta_s) - \frac{1}{\hbar} V_S)$$



- We need to find the gravitational inner product: **it is not the one defined by VTQFT.**

- The GOE Vandermonde  $K$  is the gravity inner product on  $\mathcal{H}_{T^2} \otimes \mathcal{H}_{T^2}$

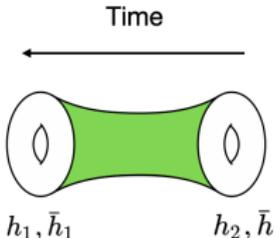
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$$\overline{\rho(h_1, \bar{h}_1)\rho(h_2, \bar{h}_2)}_{cylinder} \equiv \langle h_1, \bar{h}_1 | \hat{K}^{-1} | h_2, \bar{h}_2 \rangle =$$


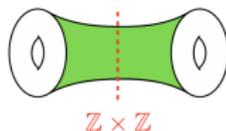
The diagram shows a cylinder with a green interior. The left boundary is labeled  $h_1, \bar{h}_1$  and the right boundary is labeled  $h_2, \bar{h}_2$ . Above the cylinder, a horizontal arrow points to the left and is labeled "Time".

- Because 3d gravity is topological, this propagator is the identity on  $\mathcal{H}_{T^2} \otimes \mathcal{H}_{T^2}$ .  
 $K^{-1}$  gives the resolution of identity in an **non-orthogonal basis** of Virasoro characters

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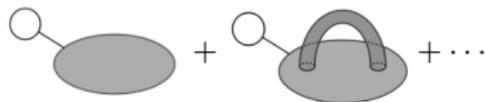
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- The non orthogonality comes from the the gauging of the **bulk mapping class group**.

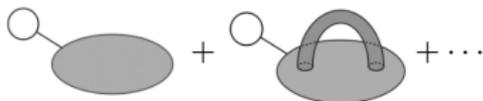


- The VTQFT inner product  $\rightarrow$  orthogonal Virasoro characters. VTQFT has different random matrix statistics than gravity.

- Before adding  $V_S$ , we have a simple set of 3 manifolds obtained from the genus expansion of the double scaled matrices  $\Delta_S$ . The expansion parameter the level spacing  $e^{-c}$ .

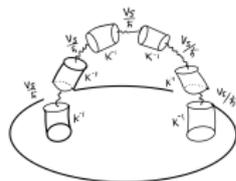


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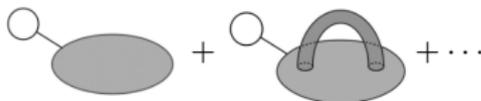
$$K^{-1} \frac{V_S}{\hbar} K^{-1} = \frac{1}{\hbar} K^{-1} - K^{-1} \mathbb{S}$$



In the  $\hbar \rightarrow 0$  limit this produces a projector onto the  $\mathbb{S} = 1$  states  $\Rightarrow S$  modular invariance.

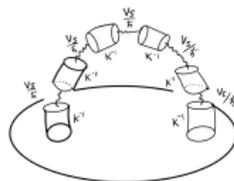
# $SL(2, \mathbb{Z})$ modular invariance from sum over topologies

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- Full  $SL(2, \mathbb{Z})$  modular invariance is achieved when combined with the sum over  $T$  transforms.
- These new wormholes produces Seifert manifolds that are needed to cure the negative density of states that would arise from the  $SL(2, \mathbb{Z})$  sum over BTZ black holes (Maxfield-Turiaci).

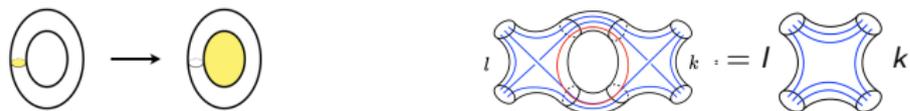
- 1 Ensemble of approximate CFT's
- 2 The tensor potential and Virasoro TQFT
- 3 The matrix potential and  $SL(2, \mathbb{Z})$  modular invariance
- 4 Sum over all manifolds and the Schwinger Dyson equation

# Surgery = averaging over tensor model Loops

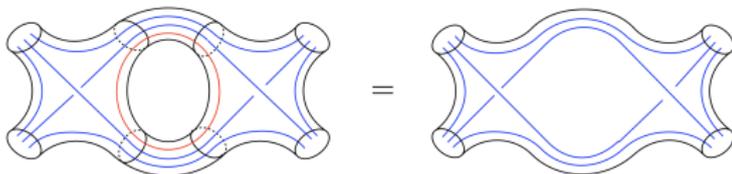
- Loops in the tensor model diagrams  $\leftrightarrow$  insertion of the Cardy density

$$= \int d^2 p S_{1p} S_{1\bar{p}}$$

- This implements toroidal surgery, which is equivalent to Moore-Seiberg identities



- Gluing 6J's with a relative rotation creates arbitrary braids



This allows us to build all 3 manifolds (Lickorish)

- Due to surgery relations, we obtain many tensor model diagrams with the same topology, weighted by different powers of  $\hbar$ .

$$\langle C \cdots C \rangle = \sum_M Z_{\text{grav}}(M, c) f(M, \hbar). \quad Z_{\text{grav}}(M, c) \sim e^{-c \text{Vol} M}$$

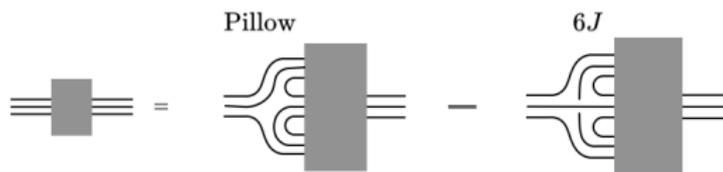
- To match with 3d gravity, we want  $f(M, \hbar) \rightarrow 1$  as  $\hbar \rightarrow 0$

# Schwinger Dyson and the sum over topologies

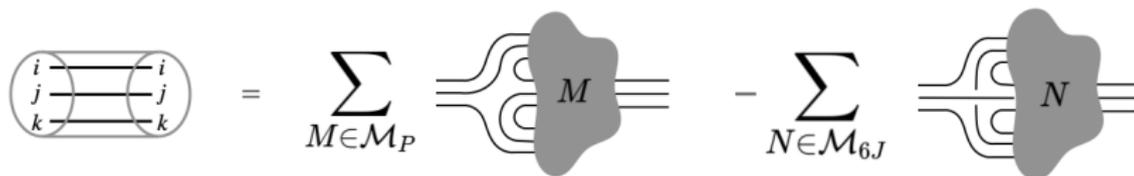
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- To match with 3d gravity, we want  $f(M, \hbar) \rightarrow 1$  as  $\hbar \rightarrow 0$
- We can check this conjecture using the  $\hbar \rightarrow 0$  limit of the Schwinger Dyson equation (Belin/de Boer/Jafferis/Nayak/Sonner), e.g.



- For a single manifold on the LHS, we have a finite counting problem



- We showed how the topological expansion of 3d gravity arises from a random ensemble of approximate CFT's.
- The sum over topologies implements the bootstrap constraints order by order in the  $e^{-c}$  expansion
- A successful non-perturbative completion of the sum would solve the bootstrap equations exactly, and land us in the space of exact CFT's. We expect this is a  $e^{-e^c}$  question.
- The matrix model sums over a simpler class of 3 manifolds: perhaps we should start by understanding the non-perturbative completion of this sum.

- A matrix model prediction: 3 boundary torus wormhole
- Some loose ends:
  - Solve the Schwinger–Dyson equation : generalization of topological recursion?
  - renormalization /cancellation of  $S^2$  handle divergences
  - Regularization of the  $SL(2, \mathbb{Z})$  accumulation point = hyperbolic cusp
- Future work:
  - Random BCFT and EOW branes
  - Adding matter
  - Random ensemble for dS3?
  - Random ensemble of RCFT's