Entropy of Holographic CFTs at large charge and angular momentum

Shiraz Minwalla

Department of Theoretical Physics
Tata Institute of Fundamental Research, Mumbai.

Strings 24, CERN, June 7, 2024

References

Talk based on

- [1] Grey Galaxies as the end point of the Kerr AdS Superradiant instability, ArXiv 2305.08922

  S.Kim, S. Kundu, E, Lee, J. Lee, S. M. and C. Patel

- [2] Giant Dressed Black Holes as the end point of the 'superconducting instability' in $AdS_5 \times S^5$, ArXiv 2406.?????

  S. Choi, D. Jain, S.Kim, E, Lee, S. M. , C. Patel and V. Krishna

- [3] 5 parameter susy $AdS^5 \times S^5$ black holes in from Grey Dual Giants and Galaxies, arXiv 2407.?????

  S. Choi, D. Jain, S.Kim, E, Lee, S. M. , C. Patel and V. Krishna

- [4] Grey Galaxies in $d \geq 4$, ArXiv 2407.?????

  K. Bajaj, S.M, J. Mukherjee, A. Rehman, V. Sharma

- [5] Also ongoing discussions with

  W.A. Hamdan, D.Jain, C. Patel and V. Krishna

Outline

- Introduction
- I: Large Angular Momentum
- II: Large Charge
- III: Large Charge and Angular Momentum
- IV: Supersymmetry

Introduction

- Spectrum of operators: most basic observable of any CFT_d. Same as spectrum of states on S^{d-1} (state operator map).
- In holographic theories, the Hawking Beckenstein entropy of black holes computes the entropy[a] of this spectrum, at energies[b] of order $1/G$.
- The well known AdS Kerr Reisnner Nordstorm black holes yield relatively simple analytic expressions for this entropy as a function of energy and charges. However these black holes are (sometimes) unstable at large values of charge and angular momentum.
- This talk: what is the correct entropy formula for holographic CFTs at large charges? [c]

[a] Coarse grained over intervals much larger than unity

[b] And angular momenta, R charges, etc also of order $1/G$

[c] Equivalently, what is the end point of the large charge black hole instabilities?.

Part I: Large angular momentum

- We begin by studying $S(E, J)$ in any CFT_3 [a] that has with a two derivative dual gravitational description.
- Kerr AdS_4 black holes are parameterized by their energy E and angular momentum J. The entropy of these black holes appears to give a universal [b] prediction for $S(E, J)$ for all CFTs that have a two derivative bulk dual description.
- Unitarity of the dual CFT predicts that these black holes should only exist for $E > J$. Expectation borne out by black hole physics. Black holes at angular momentum J exist only for $E > E_{\rm ext}(J)$, i.e. at energies above extremality.
- Turns out $E_{\rm ext}(J) > J$. This is in agreement with unitarity, but it is a bit surprising that AdS_4 black holes 'oversaturate' the unitarity bound.

[a] CFT_3 is the simplest case because $SO(3)$ angular momental are paremeterized by a single number.

[b] Universal because every two derivative theory of gravity that admits an AdS_d solution also admits a consistent truncation to the d dimensional Einstein equations with a negative cosmological constant.

Existence Plot for Kerr AdS_4 black holes

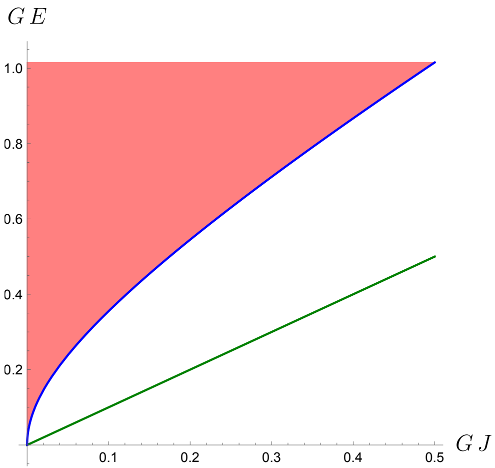


Figure: Dark Blue=Extremal, Green= Unitarity bound

Instability for $\omega > 1$.

- It has been known for 20 years that any black hole with angular velocity $\omega > 1$ is unstable in *AdS* space [a,b] .
- Easy to understand intuitively. A black hole with inverse temperature β and angular velocity ω computes

$$\mathrm{Tr} e^{-\beta(H-\omega J_z)} \tag{1}$$

- Consider the operator $\partial_z^n O$ where O is any single trace operator. When n is large enough, the quantum numbers of this operator are $E \approx n$, $J \approx n$. Consequently $\partial_z^n O$ is Boltzman enhanced rather than suppressed when $\omega > 1$ and n is large enough [c].
- By explicit computation one finds that Kerr-AdS black have $\omega > 1$- and so are unstable -in a band around extremality.

[a] V. Cardoso and O. Dias, hep-th/0405006

[b] S. Green, S. Hollands, A. Ishibashi and R. Wald, ArXiv 1512.02644

[c] As this is true for all n that are large enough, an infinite number of modes are unstable at every $\omega > 1$.

Instability Region of Kerr AdS black holes

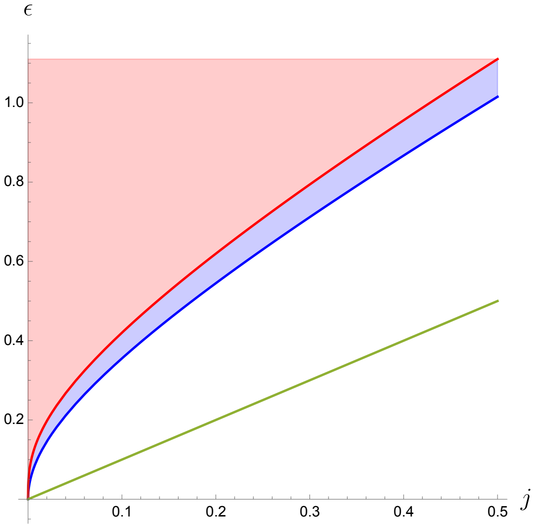


Figure: Red Curve: $\omega = 1$. Blue Curve: extremality. Black hole in the shaded blue region are unstable.

Black hole - graviton gas equilibrium

- In order to understand the end point of the $\omega > 1$ superradiant instability, recall that a black hole in *AdS* lives in equilibrium with a gas of gravitons at the same value of β and ω.
- When working to leading order at large *N* (small *G*) we ignore this gas, because its charges and entropy are subleading compared to those of the black hole. This is correct when $\omega < 1$.
- As $\omega \to 1$, however, the energy, angular momentum etc of the gas diverge. Once ω approaches close enough to unity, the gas contribution to the energy and angular momentum are comparable to the black hole, and can no longer be ignored. Let us understand this quantitatively. In the formulae below we parameterize Newton's constant by $G = \frac{R_{AdS}^2}{N^2}$

Free Gas Thermodynamics I

- The descendents of a scalar primary of dimension Δ take the form

$$(\partial^2)^n \partial_{\mu_1} \ldots \partial_{\mu_l} O$$

and carry quantum numbers

$$E = \Delta + 2n + l, \quad J_z = l - a, \quad n = 0 \ldots \infty, \quad l = 0 \ldots \infty, \quad a = 0 \ldots 2l \tag{2}$$

- Multiparticling over these modes we find

$$\ln Z = \sum_{n,l=0}^{\infty} \sum_{a=0}^{2l} - \ln \left(1 - e^{-\beta(\Delta + 2n + \omega a) - \beta(1-\omega)l} \right) \tag{3}$$

- The summation over n and a is always exponentially damped. However the sum over l is undamped - and so diverges - as $\omega \to 1$. The leading divergence is accurately estimated by replacing the sum over l by an integral.

Free Gas Thermodynamics II

- We find

$$\ln Z = -\frac{1}{\beta(1-\omega)} \sum_{n,a=0}^{\infty} \int_0^{\infty} dx \ln\left(1 - e^{-\beta(\Delta+2n+a)-x}\right)$$
$$= -\frac{C_\Delta(\beta)}{\beta(1-\omega)} \quad (4)$$

where

$$C_\Delta(\beta) = \sum_{n,a=0}^{\infty} \int dx \ln\left(1 - e^{-\beta(\Delta+2n+a)-x}\right) \quad (5)$$

- Consequently

$$\Delta E \sim \Delta L \sim -\frac{C_\Delta(\beta)}{\beta^2(1-\omega)^2} . \quad (6)$$

so the energy and angular momentum in the gas is comparable to that in the black hole when $1-\omega \sim \frac{1}{N}$

Non Interacting Mix

- On the other hand the entropy of the gas is given by

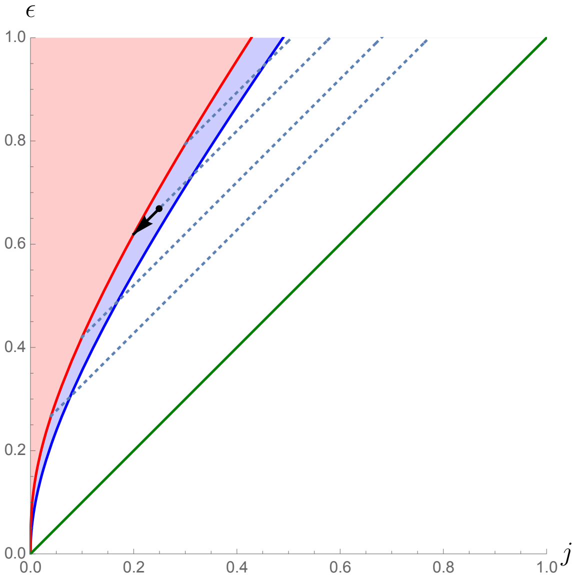
$$S = \frac{C'(\beta)}{(1-\omega)} - \frac{2C(\beta)}{\beta(1-\omega)}$$

  and so is of order N (and so subleading) at this value of ω
- The gas lives in a disk surrounding the black hole. This disk has constant proper thickness, and extends upto a radial distance of $r \sim \sqrt{l} \sim \sqrt{N}$. Though the gas carries energy of order N^2, its density is only of order $N \ll N^2$.
- The black hole and gas are effectively non interacting, because they are widely separated in the radial direction. The gas is of parametrically low density, and so is effectively non self interacting. As a consequence, the thermodynamics of our system is that of a non interacting mix of $\omega = 1$ Kerr AdS black holes and a free gas of gravitons (living in AdS) with $\omega = 1 - \frac{\alpha}{N}$.

Grey Galaxy Gravity Solution

- In [1] we have developed the gravity soluiton for these 'Grey Galaxies' in a mixed asymptotic expansion (small parameter $\frac{1}{N}$). At $r \sim 1$ we have the $\omega = 1$ Kerr-AdS black hole plus + $\mathcal{O}\left(\frac{1}{N}\right)$. At $r \sim \sqrt{N}$, we have the linearized metric backreaction to the stress tensor of the free gas plus fractional corrections $\sim \frac{1}{N}$.
- The free thermal gas is an ensemble of $\sim e^N$ states, so the gravity solution sourced by this gas has statistical fluctuations. At $r \sim \mathcal{O}(1)$, fluctuations $\sim$ mean, but the mean $\sim \mathcal{O}(1/N)$. Intervals of $r \sim \sqrt{N}$, 'contain' $\sim N$ graviton modes and so fluctuations $\sim \frac{\text{mean}}{\sqrt{N}}$. Consequently fluctuations not significant at leading order in large N.
- We conjecture that the superradiant instability of an $\omega > 1$ Kerr AdS black hole evolves to a typical Grey Galaxy ensemble element, and so the Grey Galaxy solution.

New Entropy formula



Boundary stress tensors

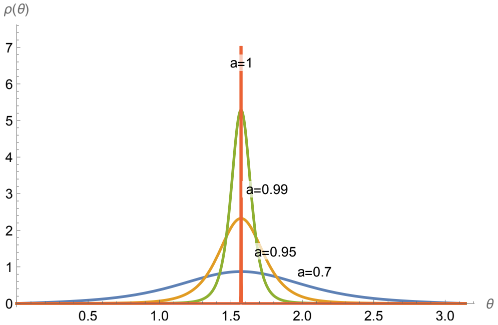


Figure: Plots of the normalized boundary differential energy density, $\rho(\theta)$, as a function of θ: note the smooth black hole contribution and the δ function contribution from the gas.

Grey Galaxies in Higher Dimensions

Grey galaxies also exist in AdS_d for $d \geq 5$ [a]. Main difference from $d = 4$: $SO(d)$ has rank > 1.

Consider e.g. $d = 5$. Two chemical potentials, ω_1 and ω_2, corresponding to orthogonal 'two plane' angular momentum, J_1 and J_2. We have the following three stable phases

- (a): Black holes with $\omega_1 < 1$ and $\omega_2 < 1$.
- (b): Black holes with $\omega_1 \approx 1$ *but* $\omega_2 < 1$, in equilibrium with a gas with large J_1 (or $1 \leftrightarrow 2$)
- (c): Black holes with $\omega_1 = \omega_2 \approx 1$, in equilibrium with a gas with large J_1 and J_2. [b]

Can show that these three phases give a complete (and non overlapping cover) of the unitarily allowed charge space $E \geq J_1 + J_2$.

[a] Currently under investigation in [4]

[b] In phase (b) the gas contribution to the boundary stress tensor is sharply localized on a one dimensional great circle. However in phase (c) the gas contribution to the boundary stress tensor is smooth on S^3.

Part II: Large Charge

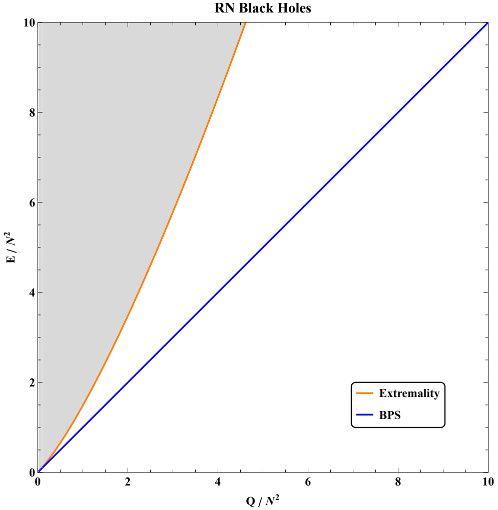
- Gubser pointed out in 2008 that black holes are sometimes unstable at large charge[a].
- Gubser's instability is simplest to understand for small charged black holes. Consider a mode of charge e, and energy ω incident on a flat space black hole with chemical potential μ. Whenever $\mu e > \omega$ this mode exhibits the superradiant phenomenon. The AdS box turns this into an instability. End point is a small black hole with $\mu = \frac{\omega}{e}$, immersed in an *AdS* size cloud of the charge e scalar. Non interacting mix of the black and condensate.
- Picture correct only for very small black holes. Corrected order by order in black hole size[b]. Condition for instability- and nature of final state - extremely complicated when black hole large. 100s of studies. Almost all numerical.

[a] S. Gubser, ArXiv: 0801.2977 [b] P. Basu, S. Bhattacharyya, R. Loganayagam, S.M. , ArXiv: 1003.3232

Large Charge in $\mathcal{N} = 4$ Yang Mills

- With a convenient choice of normalization for gauge fields, small black holes have $\mu \leq 1$, and so are all 'Gubser stable' provided that every bulk field has $e/\Delta < 1$. On the other hand, small charged black holes are unstable in a range of energies around extremality if there exists even one bulk boson with $e/\Delta > 1$.
- Criterion less universal than angular momentum. We turn to the study of a specific theory: $\mathcal{N} = 4$. For simplicity specialize to the case of equal $SO(6)$ Cartans $Q_1 = Q_2 = Q_3 = Q$.
- As in the case of angular momentum, black holes exist only at energies above extremality. Once again, the mass of extremal black holes at charge Q turns out to be strictly greater than the BPS bound $E = 3Q$. Bit odd. Sounds like something should intervene.

Existence Region for RN AdS black holes



Gubser Instabilities in $\mathcal{N}=4$ Yang Mills

- In $\mathcal{N}=4$ Yang Mills, the BPS bound asserts that $e/\Delta \leq 1$. Chiral operators (like $Tr(X^2+Y^2+Z^2)$) saturate this inequality. Small BHs in $\mathcal{N}=4$ lie on the edge.
- Infact it turns out that IIB Sugra admits a consistent truncation to Einstein Maxwell plus a single charged scalar field dual to that mode [a]. A detalied analysis of this consistent truncation (analytical for small black holes [a] but numerical for general black holes [b]) shows that these black holes are infact unstable in a band around extremality. The curve separating stable from unstable BHs lies at $\mu=\mu_c(q)$, where $\mu_c(q)=1+2q+\mathcal{O}(q^2)$. Note $\mu_c(q)>1$. Numerics show inequality true at all values of q.
- End point of this instability is a 'hairy black hole'. This phase extends down all the way to the BPS bound (where it reduces to a 'soliton'). Removes the troubling gap.

[a] S. Bhattacharyya, S.M, K. Papadodimas, ArXiv: 1005.1287 [b] J Marcevicuite , J. Santos , ArXiv:1806.01849

Importance of higher chiral modes

- The Ang. Mom. and charge stories have some differences. First, Ang. Mom. involves an infinite number of modes, while the Gubser instability involves single bulk scalar field. Second, the instaibility condition - and final entropy formula - was extremely simple for Grey Galaxies, but complicated -and only numerically known - in the case of charge. [a]
- Now $\mathcal{N} = 4$ Yang Mills does have an infinite number of chiral modes - $Tr(X^n + Y^n + Z^n)$ - all of which saturate the BPS bound. While it was self consistent to truncate to the lowest of these, its not clear that this truncation captures the dominant instability. Could the chiral operators at large n play an important role? Answer: Yes, in a manner that is surprisingly easy to analyse [2].
- Recall pure AdS hosts dual giants. Duals with charge Q live at radius $r = \sqrt{\frac{Q}{N}}$. Key point: when $Q \gg N$, $r \gg 1$.

[a] Except in the small charge limit, in which the entropy formula for the hairy black hole is reproduced by a non interacting mix of black hole and condensate.

Instability at $\mu > 1$

- It follows that large dual dual giants - those with charge $Q \gg N$ - also exist in the black hole background, and that their properties are essentially unaffected by the black hole. Since dual giants have $E = Q$, it follows immediately that any black hole with $\mu > 1$ in unstable to their emission.
- In the canonical ensemble this follows because $e^{-\beta(E-\mu Q)} > 1$ (for these duals) when $\mu > 1$.
- In the microcanonical ensemble, the duals and BHs are effectively non interacting - except for one effect. The nucleation of each giant reduces the effective black hole value of N by one unit. A short computation shows that the emission of n dual giants of total charge $Nq_{\rm tot}$ changes the entropy of the seed black hole by

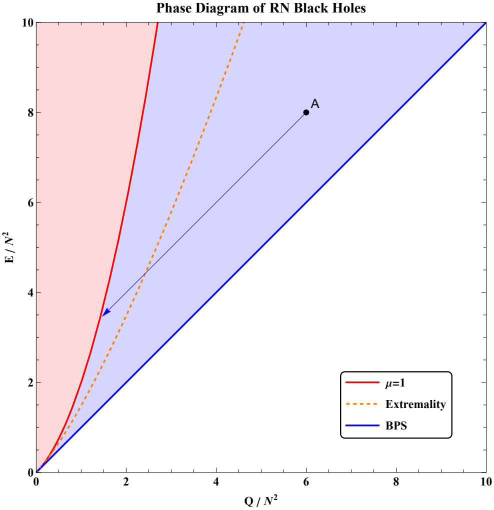
$$\delta S = \beta(\mu - 1)q_{\rm tot} + 2Nn\beta\left(\epsilon_{BH} - \mu q_{BH} - Ts_{BH}\right) \qquad (7)$$

When $\mu > 1$ this is positive for large enough $q_{\rm tot}$. Turns out $(\epsilon_{BH} - \mu q_{BH} - Ts_{BH})$ is always negative, so δS maximum at $n = 1$.

Non Intreacting Mix

- Recall, however, that RNAdS black holes became ‘Gubser Unstable’ only at $\mu > 1$. Consequently the dual giant instability kicks in before the Gubser instability. Morever, it turns out that all the ‘Gubser stabilized’ black holes above have $\mu > 1$. So they are all, in turn, unstable to emission of duals.
- The end point of this instability is a single dual giant surrounding a $\mu \approx 1$ black hole. The entropy of this configuration (see next slide) turns out to be always greater than that of the Gubser stabilized black hole.
- It follows, therefore, that the Gubser mechanism is replaced by ‘grey dual giants; in the phase diagram of $\mathcal{N} = 4$ Yang Mills. We believe that the situation is simiar for, e.g. ABJM theory or the M5 brane theory....

Entropy of Dual Giant Dressed Black Holes



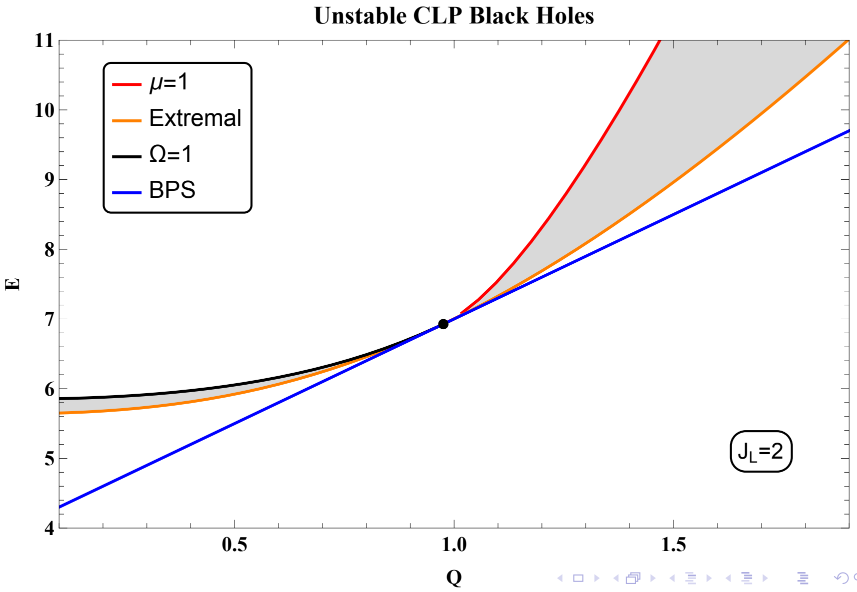
Part III: Angular Momentum and Charge

Staying with $\mathcal{N} = 4$ Yang Mills theory - also with black holes with $Q_1 = Q_2 = Q_3 = Q$, let us turn on angular momentum with $J_1 = J_2 = J$. We have a 3 parameter set of (traditional) black holes with these charges, parameterized by E, Q and J. This space hosts 4 interesting two dimensional surfaces

- (a) The BPS Surface $E = 3Q + 2J$
- (b) The surface of extremal black holes
- (c) The surface $\omega_{BH} = 1$
- (d) The surface $\mu_{BH} = 1$

Remarkably enough, these four surfaces all coincide on a single curve, the curve of Gutowski Reall SUSY black holes.

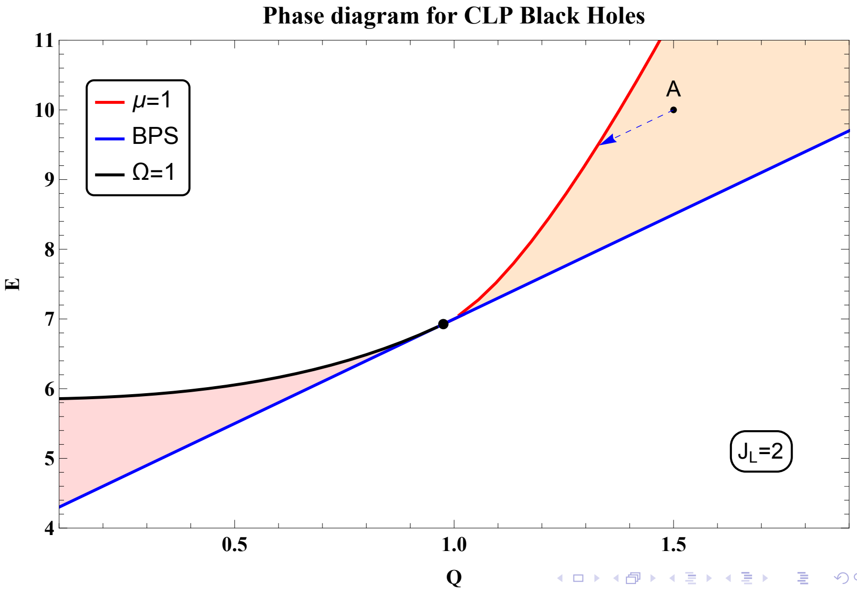
Constant J Slice of BH Configuration Space



End point of the instabilities

- All points below the $\omega = 1$ curve are unstable to the formation of a Grey Galaxy. All points below the $\mu = 1$ curve are unstable to the formation of Grey dual giants. Note that these two conditions are never simultaneously obeyed (no nontrivial competition between grey galaxies and grey dual giants)
- Both phases above extend all the way down to the BPS plane, giving a full cover of all charges allowed by the BPS bound.
- In summary, the phase diagram of $\mathcal{N} = 4$ (with the charges considered) has three distinct phases with sharp phase transitions between them. Black holes with $\mu < 1$ and $\omega < 1$. Grey galaxies. Grey dual giants.
- Note that nonsusy extremal black holes never appear anywhere in this phase diagram.

Constant J Slice of Phase Diagram



Part IV: Supersymmetry

- Atleast naively, the phase diagram described above descends down to the BPS sheet, suggesting that this sheet is divided into to parts separated by the curve of Gutowski Reall SUSY black holes. One side of this curve hosts grey galaxies: the other hosts grey dual giants.
- However, our construction of both grey dual giants and grey galaxies was approximate, with corrections in a power series in $\frac{1}{N}$. Susy, on the other hand, is a yes/no question. Can we be sure that our apparently BPS grey dual giants and grey galaxies are actually susy? We will now give 3 bits of evidence that this is indeed the case [3].

Genuine Supersymmetry

- One can construct dual giant gravitons in the background of a Gutowski Reall black hole, and demonstrate (at the level of probe analysis) that it is exactly supersymmetric - even at finite distances from the black hole [a]. Suggests that Giant Dressed BHs are exactly susy.
- At the linearized level, gravitions at large angular momentum have been argued to be both susy as well as regular at the horizon [b]. There is also direct evidence for gravitons dressing black holes from the direct evaluation of the susy cohomology at $SU(2)$ and $SU(3)$ [b].
- Finally (in the case of angular momentum), there is a second solution - the so called RBH (see [1])- which can be constructed exactly (by quantization of a coset of $SO(D,2)$), is exactly susy, and gives rise to the same phase diagram as Grey Galaxies.

[a] O. Aharony, F. Benini, O. Mamroud, E.Milan, ArXiv: 2104.13932

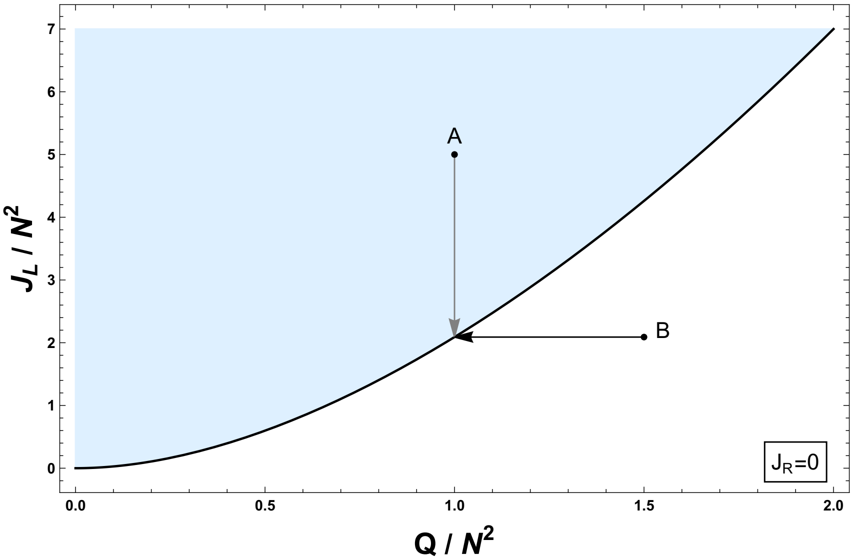
[b] S. Choi, S. Kim, E. Lee, S. Lee, J. Park ArXiv: 2304.10155. Also see S. Kim's 2023 Strings talk

‘5 parameter’ set of susy black holes

- It thus seems that Gutowski Reall black holes can be dressed with either angular momentum or charge in a genuinely supersymmetric manner. This yields a prediction for the entropy of BPS states as a function of all 5 charges. This is a prediction could be taken as a target of the ‘cohomology programme’ we heard about from Chi Ming earlier in this meeting.
- Note that we have two different phases with a phase transition on the GR sheet. The entropy at any point in each of the phases is given by the entropy of the corresponding ‘shadow’ GR black hole, where the shadow rules are illustrated on the next slide.[a]

[a] See S. Lee and F. Larsen, ArXiv: 2405.17648 for a recent field theory ‘derivation’ of the shape of this sheet.

Supersymmetric Entropy when $J_1 = J_2 = J$



$J_1 \neq J_2$

- The last four or five years has seen dramatic progress in the computation of the superconformal index - and its match with susy black holes. In the rest of this talk I will discuss a possible implication of our results for the superconformal Index.
- The potentially interesting application of our results occurs when $J_1 \neq J_2$. I do not have the time to describe the generalization in detail: please see D. Jain's poster. As above, the entropy for any value of the charges is given by a 'shadow'of the given charge point on the sheet of susy black holes. In [3] we work out the rules that determine the shadow of every point in charge space. In the rest of this talk I simply use these rules without derivation. In what follows, I use the notation $J_L = \frac{J_1+J_2}{2}$ and $J_R = \frac{J_1-J_2}{2}$

While our construction covers a large codimension 0 of charge space, it leaves out some allowed region(from CFT) of charge space. We do not have the conjecture of the entropy at these charges.

The Index

- The superconformal index is labelled by $Q + J_L = \alpha$, and J_R. It receives contributions from all states with charges $(\alpha - \frac{n}{2}, \frac{n}{2})$ where n is an integer that ranges from 0 to 2α. The index counts all these states, but with a catch. States at even n are counted with a plus sign, while states at odd n are counted with a minus sign. The index is protected, and relatively easy to compute in the field theory.
- The impressive computations over the last 4-5 years, find a perfect match between the leading order result for the index and susy black hole entropy at the same value of α and J_R (this match is best established when $J_R = 0$, but has also been established when J_R is nonzero, but not too large.

Index and Black Holes

- If one believed that susy states exist only on a codimension one submanifold in the space of charges, then one could explain the match between the index and black hole entropy as follows.
- The index receives contributions from states at all values of n behind. However it just so turns out that a significant number (order e^{N^2}) of black hole states exist only at one value of n - that corresponding to the black hole. For this reason the sum over n is dominated by the value of n corresponding to the black hole.
- However we have argued that large number of susy states exist at generic values of the charge. How then can we explain the matching of the index with black holes? Lets investigate this for different values of (α, J_R).

The Shadow of the Index line at small J_R

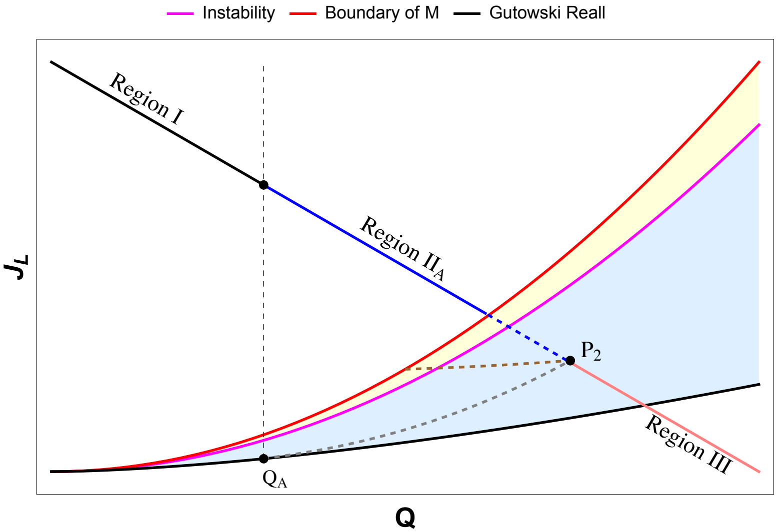


Figure:

The Shadow of the Index line at larger J_R

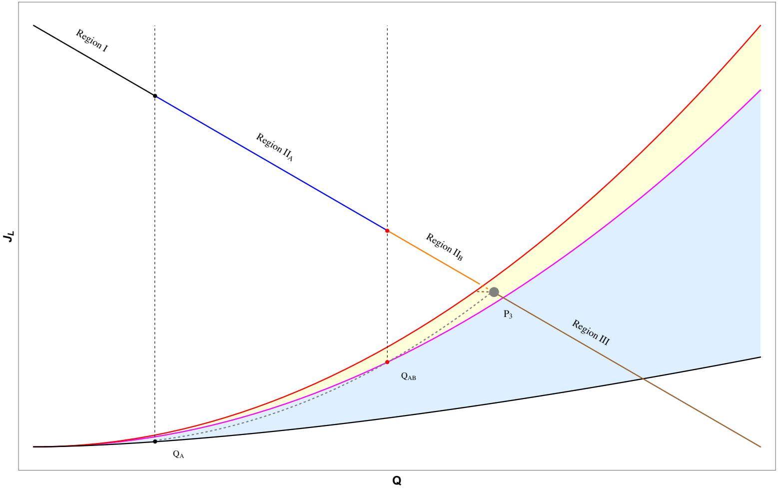


Figure:

Phase Transition in the Index?

- The Index can be written as $\mathrm{Tr}\left((-1)^{2J_L} e^{-\nu_L(Q+J_L)-\nu_R J_R}\right)$. The purple curve occurs when the chemical potentials of the black hole obey $\nu_R = \nu_L$. [a]
- Atleast naively, our analysis suggests that the microcanonical version of the index undergoes a phase transition as a function of J_R. We would predict a large J_R' phase in which the Index is dominated by a supersymmetric grey galaxy (or RBH).
- This prediction is tentative, however, because the effect of the oscillating phase has been ignored. It is possible that the oscillations suppress the contribution of an apparently dominant configuration. Would be great to have direct confirmation from field theory.

[a] In the Yellow region $\nu_R > \nu_L$. It is unclear that the index is well defined in the canonical ensemble with these chemical potentials (because, when $\mathrm{Re}(\nu_R) > \mathrm{Re}(\nu_L)$, the contribution of the derivative with $J_L = J_R = \frac{1}{2}$ seems to diverge): note analogy with Part I. However, it is possible that the rapid osciallations (associated with the imaginary parts of the black hole chemical potential) allow for enough cancellation to permit a defiition via analytic

Conclusions

- New Black Holes dominate microcanonical phase diagram of $\mathcal{N} = 4$ Yang Mills. Many new phases. Sharp phase transitions. Note that extremal non supersymmetric black holes never appear in the phase diagram.
- The solutions are all extremely simple. An effectively non interacting mix of the black hole and gas (for angular momentum) or giant dual gravitons (for charge). Noninteracting nature a consequence of large separation in the radial direction.
- New phases should continue to exist at large but finite values of λ. Possible they exist all the way down to $\lambda = 0$ and can be seen in perturbation theory.
- Construction gives atleast a lower bound for supersymmetric cohomology as a function of all 5 charges. Also suggests a possible phase transition in the Index. Would be great to verify from field theory.

Back Up: $J_1 \neq J_2$: I

- This story generalizes in an interesting way to configurations with $J_1 \neq J_2$, i.e. configurations with $J_R \neq 0$. We must first understand he structure of suprsymmetric black holes with $J_R \neq 0$.
- In the 3d space spanned by J_L, Q and J_R, the space of susy black holes make up a two dimensional sheet. One way to visualize this sheet as a union of curves, one at each fixed value of Q.
- At every fixed Q one finds a finite arc. The arc gets bigger on increasing Q. Entropy decreases on increasing J_R (or J_L) at fixed Q, but increases on increasing Q at fixed J_L

Back Up: $J_1 \neq J_2$: II

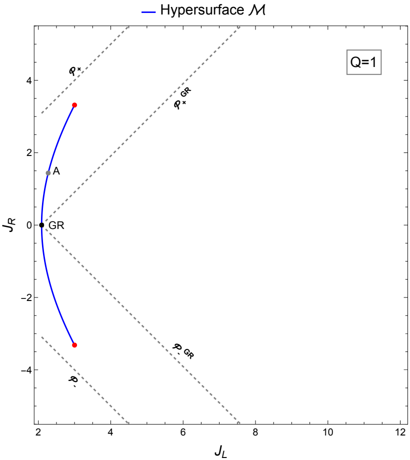
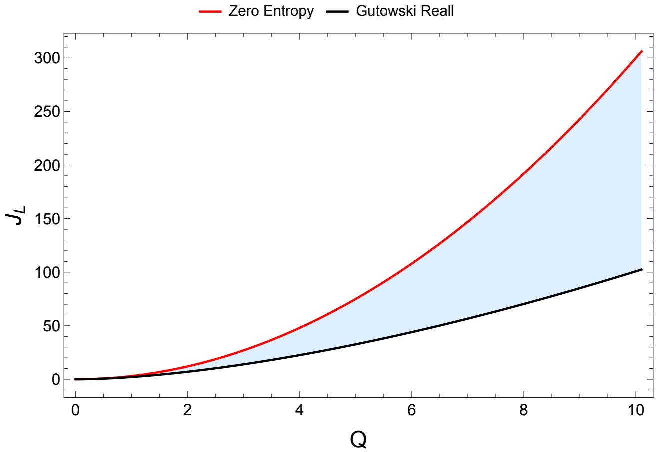


Figure: Constant Q cross section of the three dimensional space (Q, J_L, J_R). Susy black holes exist at all points on the blue arc. Their size goes at the red dot. Entropy maximum on black (GR) dot.

Back Up: $J_1 \neq J_2$: III (Top view)



Back Up: $J_1 \neq J_2$ IV

- Now consider RBHs constructed out of these black holes. These RBH's are given by $\partial_{z_1}^{n_1} \partial_{z_2}^{n_2} P$. The combination of derivatives carries $J_L = \frac{n_1+n_2}{2}$ and $J_L = \frac{n_1-n_2}{2}$. Clearly

$$|J_R| \leq J_L.$$

- For this reason the RBHs built out of any given black hole lie in a wedge with apex at that black hole. The opening angle of the wedge is 90 degrees.
- Consider any point on the plane of the previous figure. This point lies in the wedge of many different black holes. We get the maximum entropy if we choose the apex black hole to have the lowest possible $|J_R|$.

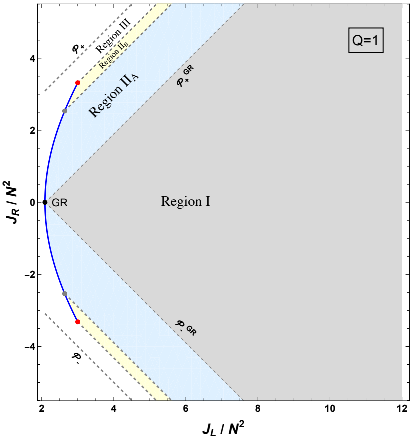


Figure: Phase diagram of RBH soluions at fixed Q. The distiction between regions II_A and II_B will become clear later. Note no solutions exist in region III. Also wedges never intersect the susy BH curve.

Backup: Inclusion dual dressed giants

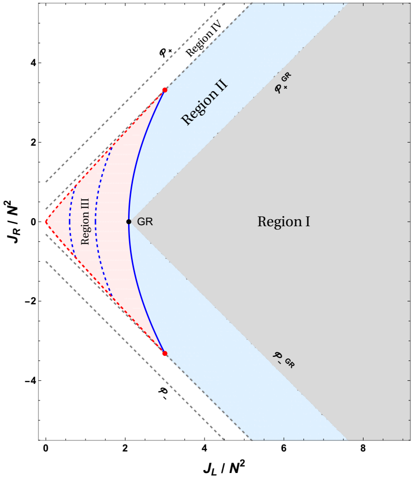


Figure: Constant $Q_1 = Q_2 = Q_3 = Q$ slice of the susy phase diagram. Regions 1 and 2 host distinct susy Grey Galaxies (or RBHs). Region 3 hosts Dual Dressed susy black holes.

Back Up: When the index line misses the black hole sheet

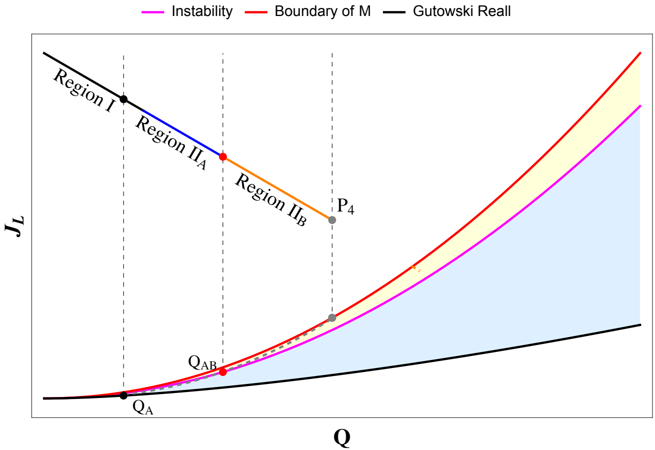


Figure: Once again Q_{AB} gives the dominant contribution.

Back Up: Possibility of Phase Cancellations

- The discussion above suggests the possibility that the index undergoes a phase transition at values of α, J_R that lie on the purble curve. If this suggestion is correct the final situation will be very similar to the discussion in part I, with the index being dominated by the pure black hole at small J_R, but by a susy grey galaxy (or RBH) at larger J_R
- However it is by no means certain that this possibility is indeed borne out. The possible loophole lies in the fact that the summation over the index line has alternating signs. For example consider the function

$$(1+x)^{N^2}$$

The binomial expansion of this function has N^2 terms. The magnitude of these terms is the same when for $x = \pm\alpha$. When $x = \alpha$ the function is well approximated, at large N^2, by the largest term in the series. Clearly, however, the same is not true at $x = -\alpha$.