# QUANTUM ENTANGLEMENT IN STRING THEORY

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Strings 2024

CERN

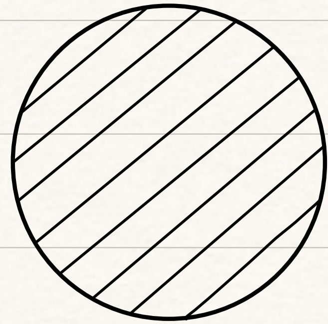
Entanglement Entropy is of fundamental importance in QM & QFT and even more so in Quantum Gravity

Finiteness of entanglement entropy is at the heart of the black Hole information paradox

Can we define a notion of entanglement entropy in string theory given its UV finiteness?

- Motivation
- Difficulties
- A method.

## BLACK HOLE



$$\mathcal{H} \stackrel{?}{=} \mathcal{H}_I \otimes \mathcal{H}_O$$

Tracing over the interior naively gives a density matrix $\rho_O$

Its von Neumann entropy $S = -\mathrm{Tr}\,\rho_O \log \rho_O$ diverges

$\Rightarrow$ infinite q-bits for the black hole

We need a generalization of von Neumann entropy

Because usual notions from local QFT like the algebra of local observables are not available in string theory.

# Quantum Entanglement in QM

Classic example is the Bell pair w/ "spooky" long distance EPR quantum correlations

Two photons in spin 0 state.

$$\xleftarrow{\phantom{xxxx}} \qquad\qquad \xrightarrow{\phantom{xxxx}}$$

$$\mathcal{H}_L \qquad\qquad \mathcal{H}_R$$

$$|\Omega\rangle = \frac{1}{\sqrt{2}}\Big( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \Big)$$

Maximally entangled pure state in the Bipartite Hilbert space $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$

very different from unentangled pure state like $|U\rangle = |\uparrow\uparrow\rangle$

# Entanglement Entropy

State $|\psi\rangle$ $\qquad \langle\psi|\psi\rangle = 1$

Density matrix $\rho = |\psi\rangle\langle\psi|$ $\qquad \mathrm{Tr}\,\rho = 1$

Reduced density matrix $\rho_R = \mathrm{Tr}_{\mathcal{H}_L}\,\rho$

$\mathrm{Tr}_{\mathcal{H}_R}\,\rho_R = 1$ : sufficient to study

correlators like $\langle\psi|\mathcal{O}_R|\psi\rangle = \mathrm{Tr}_{\mathcal{H}_R}\,\mathcal{O}_R$

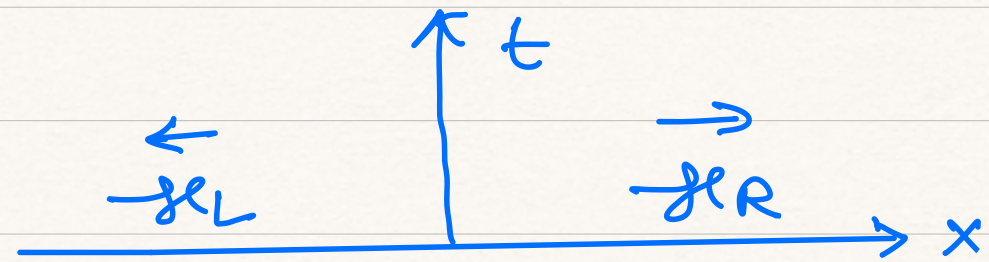
Fine-grained Von Neumann entropy

$S_{EE} = -\mathrm{Tr}_{\mathcal{H}_R}\,\rho_R \log \rho_R$ = Entanglement Entropy

For $|\Omega\rangle$, $\rho_R = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad S_{EE} = \log 2$

# Entanglement Entropy in QFT

$(d+1)$ spacetime, $\vec{y}$ = transverse




$t=0$ Spatial slice

Divide space $x_L = x < 0$ $\quad \mathcal{H}_L$

$\qquad\qquad\quad x_R = x > 0$ $\quad \mathcal{H}_R$

$$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R \quad \text{(Not really true)}$$

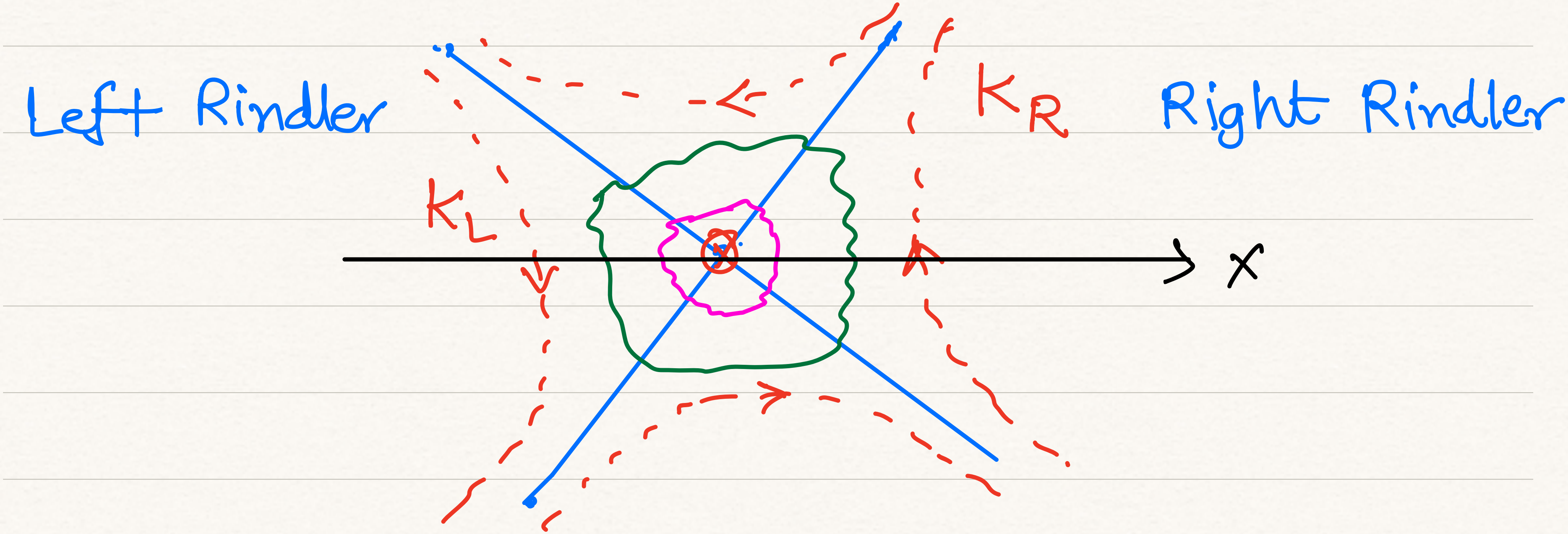
$|\Omega\rangle$ = Minkowski vacuum $\qquad \rho = |\Omega\rangle\langle\Omega|$

$$\rho_R = \operatorname{Tr}_{\mathcal{H}_L} \rho$$

$$S_{EE} = -\operatorname{Tr} \rho_R \log \rho_R = \frac{A(y)}{\varepsilon^{d-1}}$$

Area law with UV divergence $\quad \varepsilon$ = cutoff

# Rindler spacetime




$K_R$ = Right Lorentz boost $\rightarrow J$

$$ds^2 = -dt^2 + dx^2 = -\rho^2 d\tau^2 + d\rho^2$$

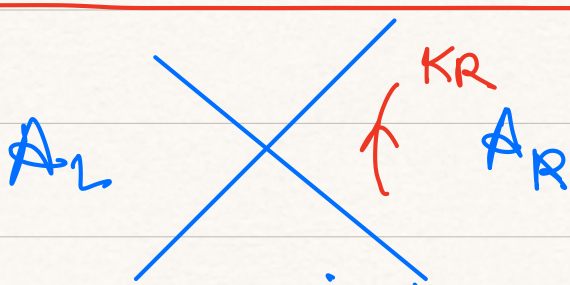
$$T(x) = \frac{1}{2\pi x} \qquad S = \int_\epsilon^\infty dx\, d\vec{y}\, T^d = \frac{A}{\epsilon^{d-1}}$$

$$\rho_R = \exp[-2\pi K_R]$$

# Algebraic QFT

It's not quite correct to assume $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$

Because of strong correlations at the boundary, Hilbert space not really factorized, but the algebra of local observables is factorized.




$[A_R, A_L] = 0$

von Neumann algebra

Minkowski vacuum $|\Omega\rangle$ is a "cyclic separating" state

$K_R =$ Modular Hamiltonian in Tomita-Takesaki theory

UV divergence property of the algebra not of state

Type-III (QFT) does not admit irrep

Type-I (QM) admits irreps. in QG?

Relative entropy $S(\rho|\sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma)$ UV finite

(1) Generalized second law of thermodynamics

$$\Delta S_{gen} = \Delta\left(\frac{A}{4G} + S_{out}\right) \geq 0$$

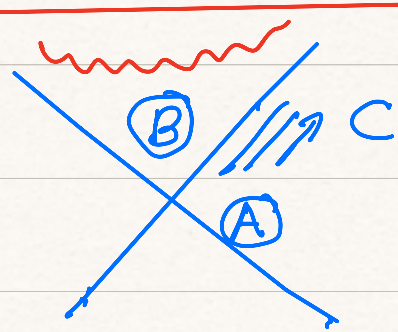
Bekenstein

Proof by Wall follows from monotonicty of relative entropy under inclusion.

Uses null Raichaudhari eqn to relate the change in "energy" to change in Area UV divergent.

$$\Delta S_{rel}(\rho|\sigma) = \Delta H_\sigma - \frac{\Delta S_\sigma}{2\pi} \leq 0$$

(2) Strong Subadditivity Paradox



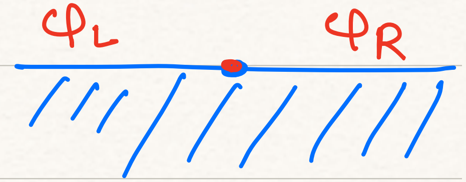

Mathur; (Almheiri, Marolf Polchinki, Sully)

Desirable to have a notion of finite entanglement entropy in many contexts in Quantum Gravity.

# Path Integral

Wave functional of $|\Omega\rangle$ in field basis

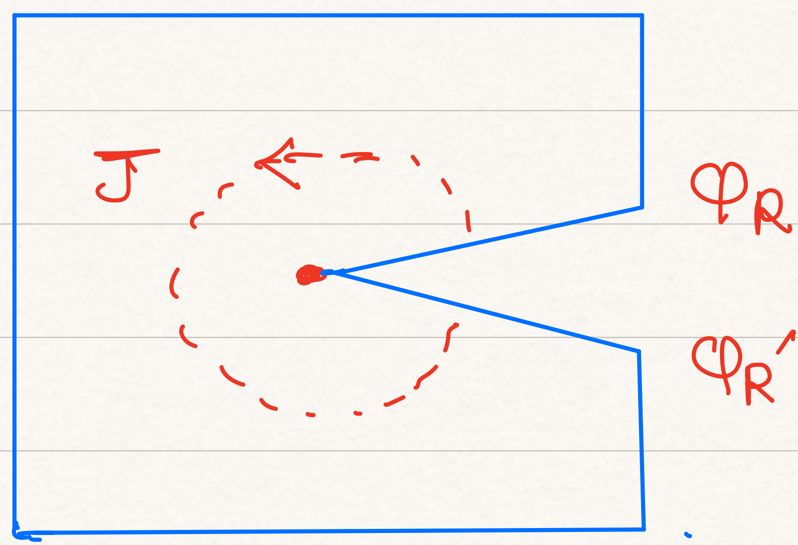
$$\langle \varphi_L, \varphi_R | \Omega \rangle = \Psi_\Omega(\varphi_L, \varphi_R) =$$




Reduced density matrix in field basis

$$\langle \varphi_R' | \rho_R | \varphi_R \rangle = \int D\phi_L \, \langle \varphi_L \, \varphi_R' | \rho | \varphi_L \, \varphi_R \rangle$$

Represented by a path integral on a cut plane




Euclidean Rindler plane

Rotation generator $J$

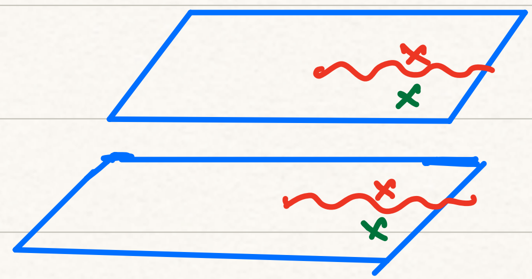
Minkowski vacuum is an ensemble of Bell pairs

# Rényi Entropy

Given the density matrix $\rho_R$, compute

$$\hat{Z}(n) = \mathrm{Tr}\, \rho_R^n \qquad n \text{ integer}$$

path integral over n-sheeted cover (Replica)



$\theta = 2\pi n$

surplus opening angle

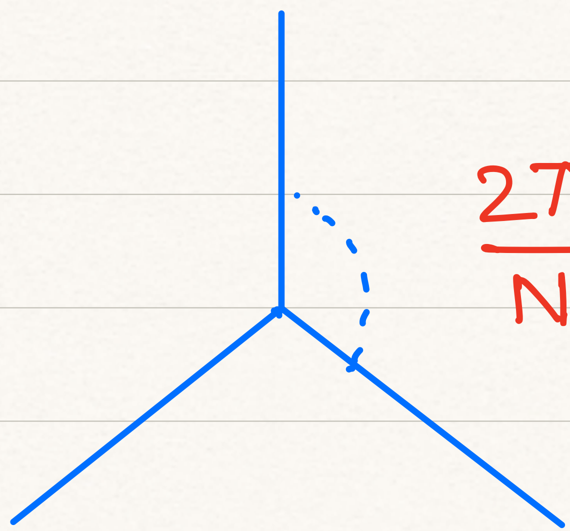
$$S_{EE} = -\frac{d}{dn} \hat{Z}(n) \Big|_{n=1} = -\frac{d}{dn} \mathrm{Tr}\left(e^{n \log \rho_R}\right)\Big|_{n=1}$$

Needs analytic continuation in $n$

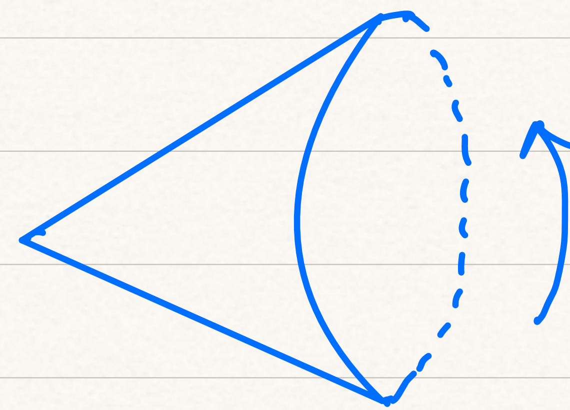
General formula: Much has been learnt in QFT

# Orbifold Method

Instead of an n-fold cover of Euclidean Rindler plane with surplus opening angle consider an orbifold w/ deficit opening angle



$\frac{2\pi}{N}$



$\frac{2\pi}{N}$

Compute $\mathrm{Tr}\, \rho_R^N$ as the partion function of $\mathbb{C}/\mathbb{Z}_N$ orbifold of the Rindler plane

$N = \frac{1}{n}$

# Strings on a Cone

$$\hat{Z}(N) = \mathrm{Tr}\left[\exp\left(-\frac{2\pi}{N} H_R\right)\right]$$

$$S(N) = -\beta \frac{\partial \log \hat{Z}}{\partial \beta} + \log \hat{Z} \qquad \beta = \frac{2\pi}{N}$$

Spacetime partition function

String Field Theory path integral on a cone

Conical curvature singularity at the tip

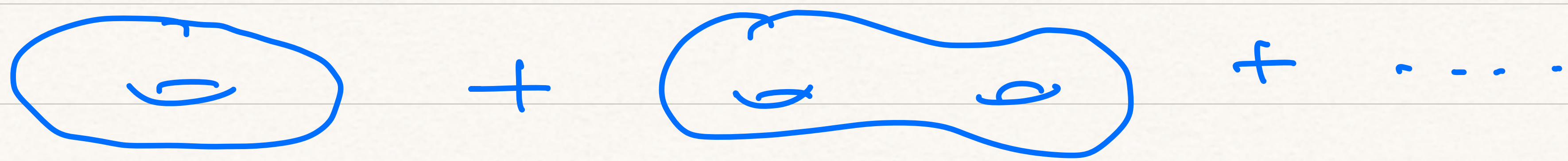
Ricci scalar $\quad R(\vec{x}) = 4\pi\left(1-\frac{1}{N}\right)\delta^{(2)}(\vec{x})$

Deficit angle $\quad \delta_N = 2\pi\left(1-\frac{1}{N}\right)$

$$S = S^{(0)} + S_q$$

$$\boxed{\log \hat{Z}_q(N) = Z_q(N)}$$

$$Z_q(N) = \sum_{g=1}^{N} Z^{(g)}(N)$$



$S^{(0)}$ = Classical Entanglement.

At classical level $\log \hat{Z}^{(0)} \neq Z^{(0)}(N)$

$$S = \frac{d}{dN}\left(N \log \hat{Z}(N)\right)\Big|_{N=1}$$

# $\mathbb{C}/\mathbb{Z}_N$ Orbifold

Type-II in lightcone gauge on $\mathbb{R}^6 \times \mathbb{C}$

Green-Schwarz superstring $(X^i, S^a, \tilde{S}^{\dot{a}})$

$$\mathrm{Spin}(8) \supset \mathrm{Spin}(6) \times \mathrm{Spin}(2)$$

$$X^i \qquad 8_v = 6(0) + \underset{\sim}{1(1)} + \underset{\sim}{1(-1)}$$

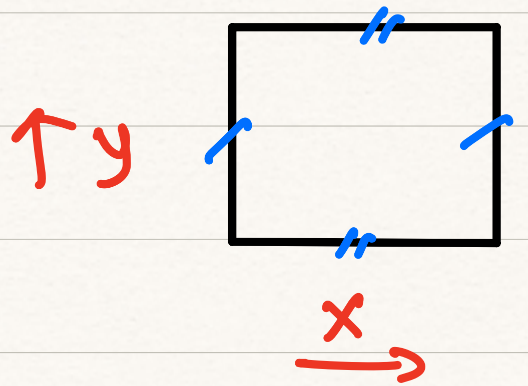
$$S^a \qquad 8_s = \underset{\sim}{4\left(\tfrac{1}{2}\right)} + \underset{\sim}{\bar{4}\left(-\tfrac{1}{2}\right)}$$

$$\tilde{S}^a \qquad 8_c = \underset{\sim}{4\left(-\tfrac{1}{2}\right)} + \underset{\sim}{\bar{4}\left(\tfrac{1}{2}\right)}$$

$$\mathbb{Z}_N = \{1, g, g^2 \ldots g^{N-1}\} \subset U(1) \simeq \mathrm{Spin}(2)$$

$$\boxed{g = \exp\left[\frac{4\pi i}{N} J\right]} \qquad \underline{N \text{ odd}}$$

# One-loop Partition function




$$\phi(x+1, y) = g^k \, \phi(x, y)$$

$$\phi(x, y+1) = g^l \, \phi(x, y)$$

$$\mathcal{Z}(\tau, N) = \sum_{k,l=0}^{N-1} \left| \frac{\vartheta^4\left(\frac{k\tau + l}{N} \middle| \tau\right)}{\eta^4(\tau)\, \vartheta\left(\frac{2k\tau + 2l}{N} \middle| \tau\right)} \right|^2$$

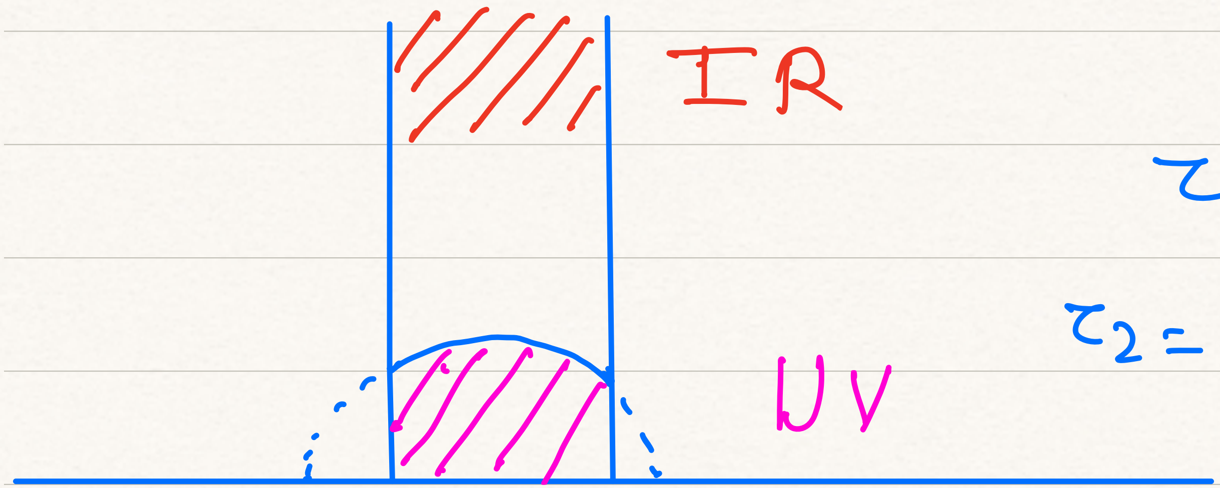
$$Z^{(1)}(N) = \frac{A_H}{N} \int \frac{d^2\tau}{\tau_2^5} \, \mathcal{Z}(\tau, N)$$

$A_H$ = Horizon Area $= \dfrac{V_8}{(2\pi \ell_s^2)^8}$

$\vartheta(z|\tau)$: Jacobi theta $\underline{fn}$ ; $\tau = \tau_1 + i\tau_2$

# UV and IR divergences

String theory provides a natural UV cutoff.
The $\mathbb{Z}_N$ orbifolds break supersymmetry
The spectrum is replete with tachyons.
all localized at the tip of the cone




$\tau$

$$\tau = \tau_1 + i\tau_2$$

$\tau_2$ = Schwinger parameter

$\exp\left[-2\pi m^2 \tau_2\right]$ diverges for $m^2 < 0$

Tachyons need not be a cause for despair

In QM, with $\mathrm{Tr}[\rho_R] = 1$

$\mathrm{Tr}[\rho_R^n]$ must converge.

$\mathrm{Tr}[\rho_R^{1/n}]$ may diverge.

We are computing $\mathrm{Tr}[\rho_R^N]$ with $N = \frac{1}{n}$

We need analytic continuation to the physical region $0 < N \leq 1$ given the data for $N > 1$ with $N$ odd.

Under certain conditions, Carlson Theorem may guarantee uniqueness

| | |
|---|---|
| $f(z) = \sum \frac{z^n f^{(n)}(0)}{n!}$ | $f(N)$ |
| countable data at $z=0$ gives analytic continuation | countable data at $N$ odd gives analytic continuation |
| CAUCHY | CARLSON |

For open strings on the Rindler horizon such an analytic continuation can be found

Witten 2018

# Closed strings on Rindler horizon

Analytic continuation of the closed string partition fn is much harder.

Fortunately, the tachyonic terms have a very specific form and do admit an analytic continuation to a function that is finite for $0 < N \leq 1$

$$Z(N) = \int \frac{d^2\tau}{\tau_2^2} \mathcal{F}(\tau)$$

$$\mathcal{F}(\tau) = \sum_n \exp[2\pi i n \tau_1] \mathcal{F}_n(\tau_2)$$

$$Z(N) = \int \frac{d\tau_2}{\tau_2^2} \mathcal{F}_0(\tau_2)$$

## Tachyonic Terms $T(\tau_2)$

Exponentially growing terms in $\mathcal{F}_0(\tau_2)$

$$T(\tau_2) = \frac{2}{\tau_2^3} \sum_{k=1}^{(N-1)/2} \exp\left[\frac{4\pi\tau_2 k}{N}\right] f_k(\tau_2)$$

$$f_k(\tau_2) = \sum_{n=0}^{\infty} \exp\left[\frac{-2\pi n\tau_2(2N-4k)}{N}\right]$$

Complicated sum but can be resummed by first performing the k-sum.

# Disappearing Tachyons

$$T(\tau_2, N) = \frac{-2}{\tau_2^3} \sum_{n=0}^{\infty} e^{-4\pi n \tau_2} \cdot \frac{1 - e^{\alpha_n (N-1)\tau_2}}{1 - e^{-2\alpha_n \tau_2}}$$

$$\alpha_n = \frac{\pi(4n+2)}{N} > 0 \; ; \; n \geq 0 \; ; \; N > 0$$

Remarkably, $T(\tau_2, N)$ has no "tachyonic" exponentially growing terms for $0 < N \leq 1$

We can write $\mathcal{F}_0(\tau, N) = T(\tau_2, N) + \mathcal{R}(\tau_2, N)$

$\mathcal{R}(\tau_2, N)$ has no divergences and can be easily integrated numerically & extrapolated

The analytic continuation of tachyons is not accidental. It depends on three 'just so' properties of string orbifolds

(1) There are exactly $(N-1)$ twisted sectors, each containing tachyons

(2) In the $k$-twisted sector there is precisely one leading tachyon whose mass-squared is linear in $k$

$$M^2 = -4\pi k/N$$

(3) There are many subleading tachyons but they all have unit multiplicity

# Classical Entanglement Entropy

Classically $\quad \log \hat{Z}(N) \neq Z(N) \quad$ — 1

spacetime $\qquad$ worldsheet

nonzero $\qquad$ zero

Relation 1 true only for bulk action

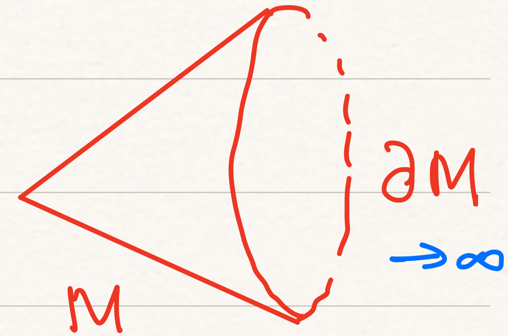
In general there are boundary terms that are not captured by sphere partition

We'll deduce them by spacetime reasoning

$$\log \hat{Z}^{(0)}(N) = I_M(N) + I_{\partial M}(N)$$

$$I_M = \frac{1}{16\pi G}\int_M e^{-2\phi}\left[R + 4(\nabla\phi)^2 - \delta^{2}(x)\, V(T) + \cdots\right]$$

+ Gibbons-Hawking boundary term depending on the extrinsic curvature.




Dilaton equations of motion are exactly satisfied to all orders for CFT background.

$\Rightarrow$ No dilaton tadpoles, $I_M = 0$

But there is a boundary contribution

$$\Rightarrow \quad \log\left(\hat{Z}^{(0)}(N)\right) = \frac{A}{4G\hbar}\left(\frac{1}{N} - 1\right)$$

Analytic continuation in N, $S^{(0)} = \dfrac{A}{4G\hbar}$

# Calabi-Yau Compactifications

Our ability to obtain an analytic continuation that is finite in the physical region $0 < N \leq 1$ depended on a very specific structure of the tachyonic spectrum

May be this is a special feature of 10d superstring that is not more generally valid with less supersymmetry.

For example, in CY orbifolds there can be new tachyons in doubly-twisted sectors with quite different structure

# $T^4/\mathbb{Z}_3$ K3 orbifold

Sectors twisted by $r = 1, 2$ & $k = 1, \cdots N-1$

In addition to 10 d tachyons, there are new tachyons with mass-squared:

$$M^2 = \begin{cases} 0 & 0 \leq \frac{k}{N} \leq \frac{1}{3} \\ 4\pi\left(\frac{1}{3} - \frac{k}{N}\right) & \frac{1}{3} < \frac{k}{N} < \frac{1}{2} \\ -4\pi\left(\frac{2}{3} - \frac{k}{N}\right) & \frac{1}{2} < \frac{k}{N} < \frac{2}{3} \\ 0 & \frac{2}{3} \leq \frac{k}{N} < 1 \end{cases}$$

Not a priori obvious how these tachyons contribute in the physical region.

Fortunately, we again find $T(\tau_2, N)$

$$= \sum_{n=0}^{\infty} \frac{1 - e^{2\pi\tau_2(1+2n)(N-1)}}{1 - e^{-\frac{2\pi\tau_2(1+2n)}{N}}} \cdot e^{\frac{-4\pi\tau_2(3n+1)}{3}}$$

Again, finite for $0 < N \leq 1$ even though exponentially divergent for $N > 1$ as $\tau_2 \to \infty$

This feature continues for several other compactifications we investigated with less supersymmetry $CY_2$, $CY_3$.

# Heterotic Strings

Worldsheet is now chiral with potential for gravitational anomalies. Additional new features: Gauge symmetry must be broken to obtain modular invariant partition functions for $\mathbb{Z}_N$ orbifolds.

For example, $\dfrac{Spin(32)}{\mathbb{Z}_2} \longrightarrow U(16)$

Structure of tachyons is different. New divergences that disappear after imposing level matching ($\tau_1$- integral)

# Entanglement on BH Horizons & Holography.

$$AdS_3 \times S^3 \times T^4$$

NS5 − F1 System

Exact worldsheet theory in bulk

WZW $SL(2, \mathbb{C})$ gauged model level $k$

· BTZ Black Hole Mass $M$, spin $J$

Solvable but nontrivial SCFT

$Z(N, M, J, k)$ can be exactly computed for the $\mathbb{Z}_N$ orbifolds of the horizon

once again tachyonic divergent terms get tamed in the physical region $0 < N \leq 1$

# Exact Analytic Continuation of Z(N)

work in progress with Don Zagier

We need to perform finite sums of the form

$$\sum_{k,l} f(k,l) \quad \text{depending on } e^{\frac{2\pi i l}{N}}$$

root of unity $\omega^N = 1$

where N must be an integer to something where N can be treated as a complex number

## Lemma

$$\frac{1}{N} \sum_{\omega^N = 1} \underbrace{\frac{A+\omega}{A-\omega} \cdot \frac{B+\omega^{-1}}{B-\omega^{-1}}} + 1$$

$$= \frac{1+AB}{1-AB} \left( \frac{1+A^N}{1-A^N} + \frac{1+B^N}{1-B^N} \right)$$

Three one-line proofs. For example, using partial fraction decomposition of $\underbrace{\qquad}$

$$= \frac{1+AB}{1-AB} \left( \frac{1+B\omega}{1-B\omega} - \frac{A+\omega}{A-\omega} \right)$$

then average over $N$th roots of unity.

LHS $N \in \mathbb{Z}$ RHS $N \in \mathbb{C}$

# Open string partition function

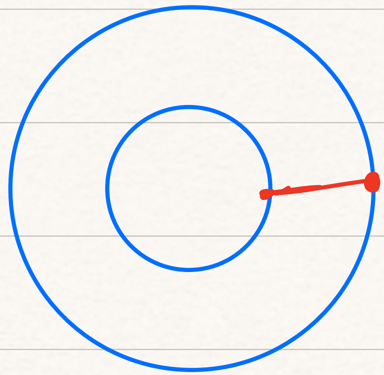
To compute entanglement entropy on the Dp brane worldvolume
Using the lemma we obtain analytic continuation in agreement w/ Witten (18)

$$Z(N) = \int_0^\infty \frac{dT}{T^{\frac{p+1}{2}}} \, Z_W(N, iT)$$

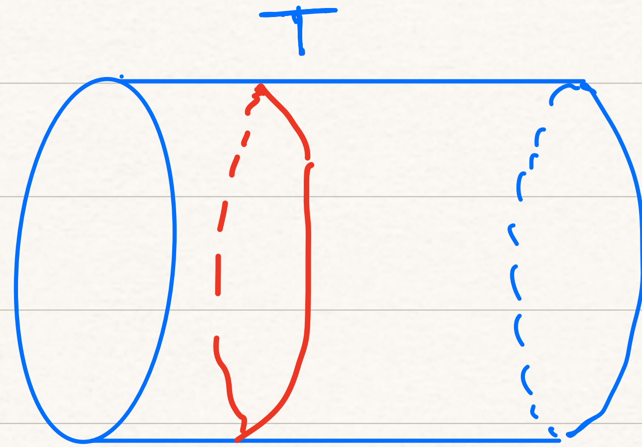
$$Z_W(N,\tau) = \frac{16\,\eta^8(2\tau)}{\eta^{16}(\tau)} \times$$

$$\times \sum_{r \in \mathbb{Z}} \left( \tanh \pi N r T - \frac{1}{N} \tanh \pi r T \right) \frac{1}{\sinh 2\pi r T} + \cdots$$

# Disappearing Tachyons



open channel




closed channel

In the large $\tau$ (IR) limit, one finds exponentially growing tachyonic terms in the closed string channel for $N > 1$

$\boxed{\text{But no tachyons for } 0 < N \leq 1}$

# Summary

- The orbifold method offers a stringy generalization of the replica method.

- We have presented substantial evidence that this method could yield computable entropy that is finite both in UV and IR

- IR divergences have a very specific structure dictated by string theory.

- One obtains a natural order-by-order expansion. A generalization of von Neumann entropy. Tree level term included in the Euclidean calculation w/o statistical interpretation.

Perhaps a more fundamental notion than BH entropy

# Open Problems

- Find the analytic continuation of $Z(N)$ for closed strings on cone.
- Can string field theory be used to compute the boundary terms & the classical entanglement entropy with $\mathbb{C}/\mathbb{Z}_N$ CFT as the starting data?
- Can we map the bulk computation in BTZ background to a quantity in the boundary CFT?

Thank you!