Progress in implementing the kinematical constraint into the small-x JIMWLK evolution equation

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PK, Eur. Phys. J. C 82 (2022) 369, arXiv:2111.07427

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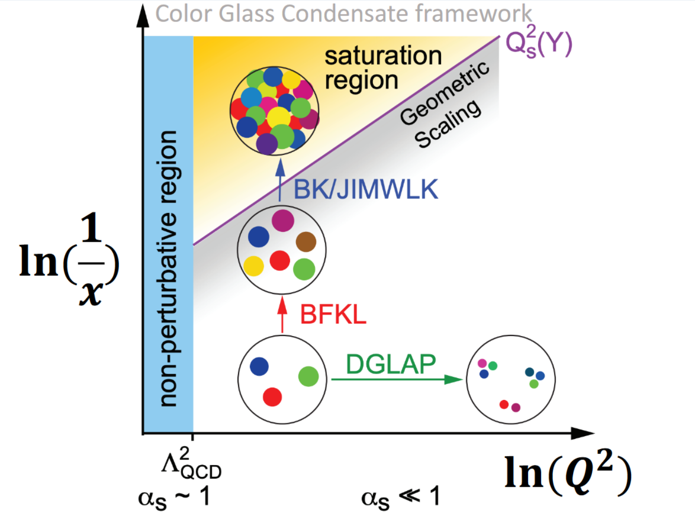
Evolution equations

$\ln Q^2$ and $\ln 1/x$ evolution



Evolution equations

Color Glass Condensate framework

- effective description valid in the saturation regime, where dense and slow gluons (target) are described by classical fields traversed by a fast and energetic probe (projectile),

  [review by Gelis, Iancu, Jalilian-Marian, Venugopalan '10]

- basic degrees of freedom:
  - Wilson lines

$$U(\vec{x})$$

  - dipole correlation function

$$S(\vec{r}) = \left\langle \mathrm{tr}\left[U^\dagger(\vec{x})U(\vec{x}+\vec{r})\right]\right\rangle_{\vec{x}}.$$

- for forward and nearly back-to-back jets, one can apply both the TMD factorization and Color Glass Condensate (CGC) approaches to compute the di-jet cross-section

  [Marquet, Petreska, Roiesnel '16, Caucal, Salazar, Schenke, Stebel, Venugopalan '23]

$\Rightarrow$ previous talks

Evolution equations

Evolution

- assuming a given distribution predict the distribution at larger Q^2
  - DGLAP equation
- assuming a given distribution predict the distribution at small x
  - BFKL (linear) equation
  - JIMWLK (non-linear) equation
  - BK (non-linear at leading color factor N) equation

Precision

- LO: fixed coupling constant, tree-level splitting and recombination amplitudes
- NLO: running coupling constant, NLO splitting and recombination amplitudes
- resummation: LO + all-order resummation of a particular class of contributions
  - kinematical constraint: resummation of contributions with $(\alpha_s \ln x)$

Kinematical constraint

Kinematical constraint

- ordering of dipole lifetimes/sizes
- natural in the language of dipoles
- worked out and implemented for the BFKL and BK equations

[Motyka, Staśto '09]

BK with collinear improvement

[Ducloué, Iancu, Mueller, Soyez, Triantafyllopoulos '19, Ducloué, Iancu, Soyez, Triantafyllopoulos '19]

In summary:

- target rapidity: $\eta \equiv \ln \frac{P^-}{|q^-|} = \ln \frac{2q^+P^-}{Q^2} = \ln \frac{1}{x}$
- dipole rapidity:

$$Y \equiv \ln \frac{q^+}{q_0^+} = \ln \frac{2q^+P^-}{Q_0^2} = \ln \frac{1}{x} + \ln \frac{Q^2}{Q_0^2} = \eta + \rho$$

Q_0 is a soft scale of the unevolved target.

Kinematical constraint

BK with collinear improvement

Evolution equation in the target rapidity η

[Ducloué, Iancu, Soyez, Triantafyllopoulos '19]

$$\frac{\partial \bar{S}_{r=|x-y|}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(x-y)^2}{(x-z)^2 (z-y)^2} \theta\left(\eta - \delta_{xyz}\right) \times$$
$$\times \left[\bar{S}_{xz}(\eta - \delta_{xz,r}) \bar{S}_{zy}(\eta - \delta_{zy,r}) - \bar{S}_{xy}(\eta) \right]$$

Comments:

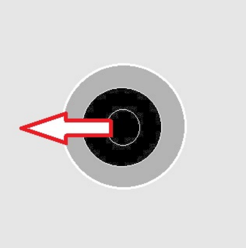
- fixed coupling constant for simplicity
- $r = |x-y|$
- rapidity shifts $\delta_{xz,r} = \max\{0, \ln \frac{r^2}{|x-z|^2}\}$
- $\delta_{xyz} = \max\{\delta_{xz,r}, \delta_{zy,r}\}$
- $\bar{S}_{xy}(\eta) = S_{xy}(Y = \eta + \rho)$

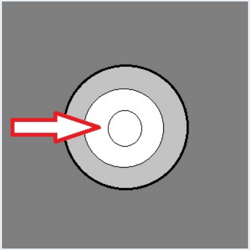
Kinematical constraint

BK with collinear improvement

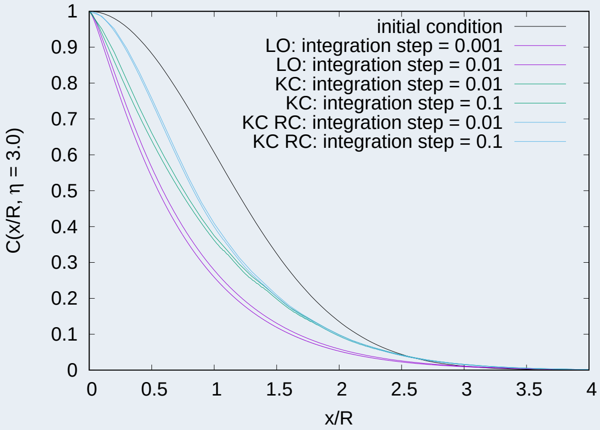
Main differences:

dipole rapidity	target rapidity
$\rho^R_{xz} = \ln \frac{\|x-z\|^2}{R^2}$	$\delta_{xz,r} = \max\{0, \ln \frac{r^2}{\|x-z\|^2}\}$





BK with collinear improvement



JIMWLK evolution equation

Warning notice

Status report: work in progress, no final results yet available.

Beyond the leading N order

- JIMWLK equation describes the non-linear small-x evolution
- it uses Wilson lines as fundamental degrees of freedom
- two-point correlation function $\langle U^\dagger(x)U(y)\rangle$ gives the dipole amplitude
- two-point correlation functions with derivatives provide a basis for small-x TMD structure functions
- initial condition corresponds to a configuration of Wilson lines
- numerically useful reformulation as a Langevin equation

JIMWLK evolution equation

LO JIMWLK: Langevin formulation

$$U(\mathrm{x}, s+\delta s) = \exp\left(-\sqrt{\delta s}\sum_{\mathrm{y}} U(\mathrm{y}, s)(\mathrm{K}(\mathrm{x}-\mathrm{y})\cdot\xi(\mathrm{y}))U^{\dagger}(\mathrm{y}, s)\right)\times$$
$$\times\, U(\mathrm{x}, s)\times\exp\left(\sqrt{\delta s}\sum_{\mathrm{y}} \mathrm{K}(\mathrm{x}-\mathrm{y})\cdot\xi(\mathrm{y})\right).$$

[Rummukainen, Weigert '04, Lappi, Mantysaari '14]

Saturation scale evolution speed

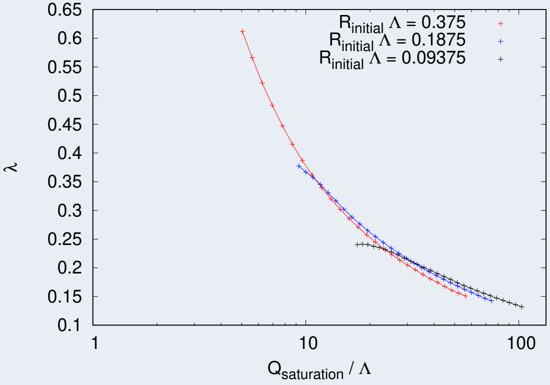


Figure: $R_{\text{initial}}\Lambda$ is the only parameter of the initial condition and of the evolution. Coinciding data from evolution for different values of $R_{\text{initial}}\Lambda$ corresponds to geometrical scaling.

JIMWLK evolution equation with collinear improvement

Collinear improvement

All order resummation of corrections enhanced by kinematical constraints. Known from BFKL studies to be important to correctly describe phenomenology.

Langevin equation formulation

$$U(\mathrm{x}, R, s+\delta s) = \exp\left(-\sqrt{\delta\varepsilon}\sum_{\mathrm{y}}\sqrt{\alpha_s}\theta(s-\rho^R_{\mathrm{xy}})U(\mathrm{y},\hat{R},s-\Delta^R_{xy})\left[\mathrm{K}_{\mathrm{xy}}\cdot\xi(\mathrm{y})\right]U^\dagger(\mathrm{y},\hat{R},s-\Delta^R_{xy})\right)$$

$$\times U(\mathrm{x},R,s)\times$$

$$\exp\left(\sqrt{\delta\varepsilon}\sum_{\mathrm{y}}\sqrt{\alpha_s}\theta(s-\rho^R_{\mathrm{xy}})\mathrm{K}_{\mathrm{xy}}\cdot\xi(\mathrm{y})\right),$$

$$\rho^R_{\mathrm{xy}} = \ln\frac{(\mathrm{x}-\mathrm{y})^2}{R^2},\ \Delta^R_{xy} = \theta\left(|x-y|-R\right)\rho^R_{\mathrm{xy}},\ \hat{R} = \max(|x-y|,R),\ s = \varepsilon\alpha_s.$$

[Hatta, Iancu '16]

Saturation scale evolution speed

JIMWLK with collinear improvement

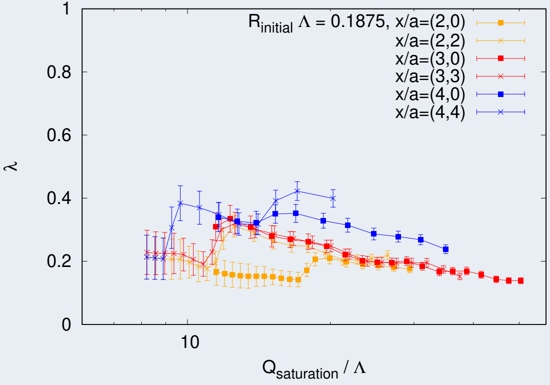


Figure: Preliminary results for the saturation scale evolution speed at $R_{\text{initial}}\Lambda = 0.1875$ for different discretizations.

JIMWLK evolution equation with collinear improvement

Several problems with the previous figure

- gigantic discretization effects
- evolution in $Y \Rightarrow$ we need to translate the equation to η
- do we reproduce the BK equation with the KC for the dipole amplitude?

JIMWLK in η with collinear improvement

Proposal

$$U(\mathrm{x}, R, \eta + \delta\varepsilon) =$$

$$\exp\left(-\sqrt{\delta\varepsilon}\sum_{\mathrm{y}}\sqrt{\alpha_s}\theta(s - P^R_{\mathrm{xy}})U(\mathrm{y}, \hat{R}, s - \Delta^R_{xy})\left[\mathrm{K}_{\mathrm{xy}}\cdot\xi(\mathrm{y})\right]U^\dagger(\mathrm{y}, \hat{R}, s - \Delta^R_{xy})\right)$$

$$\times\, U(\mathrm{x}, R, s) \times$$

$$\exp\left(\sqrt{\delta\varepsilon}\sum_{\mathrm{y}}\sqrt{\alpha_s}\theta(s - P^R_{\mathrm{xy}})\mathrm{K}_{\mathrm{xy}}\cdot\xi(\mathrm{y})\right),$$

$P^R_{\mathrm{xy}} = \ln\frac{R^2}{(\mathrm{x}-\mathrm{y})^2}$.

JIMWLK in η with collinear improvement

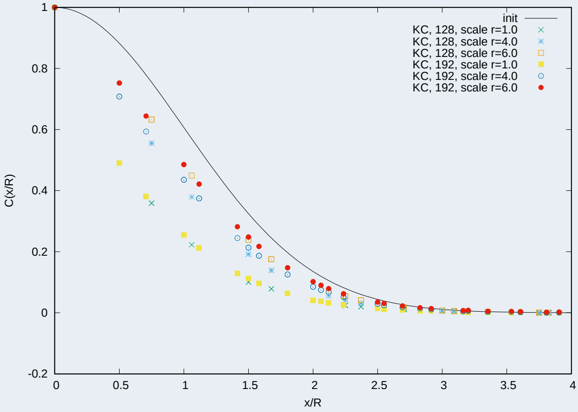


Figure: Preliminary results for the dipole amplitude with KC JIMWLK evolution equation at $\eta = 3.0$.

JIMWLK in η with collinear improvement

Reduction to the BK equation in η

The dipole amplitude is defined as

$$S(\mathrm{x},\mathrm{y}=\mathrm{x}+r,\eta)=\frac{1}{N_c}\langle\mathrm{tr}U^\dagger(\mathrm{x},r,\eta)U(\mathrm{x}+r,r,\eta)\rangle.$$

In order to establish the dependence on η we expand $S(\mathrm{x},\mathrm{y}=\mathrm{x}+r,\eta+\varepsilon)$ in ε,

$$S(\mathrm{x},\mathrm{y}=\mathrm{x}+r,\eta+\varepsilon)=\frac{1}{N_c}\langle\mathrm{tr}U^\dagger(\mathrm{x},r,\eta+\varepsilon)U(\mathrm{x}+r,r,\eta+\varepsilon)\rangle.$$

JIMWLK in η with collinear improvement

Reduction to the BK equation in η

Expand the exponentials

$$\exp\left(i\sqrt{\varepsilon}\alpha^L_{n+1}(x,r)\right) = 1 + i\sqrt{\varepsilon}\alpha^L_{n+1}(x,r) - \frac{1}{2}\varepsilon\left(\alpha^L_{n+1}(x,r)\right)^2,$$
$$\exp\left(-i\sqrt{\varepsilon}\alpha^R_{n+1}(x,r)\right) = 1 - i\sqrt{\varepsilon}\alpha^R_{n+1}(x,r) - \frac{1}{2}\varepsilon\left(\alpha^R_{n+1}(x,r)\right)^2,$$

leading to

$$U_{(n+1)\varepsilon}(x,r) = U_{n\varepsilon}(x,r) + i\sqrt{\varepsilon}\left[\alpha^L_{n+1}(x,r)U_{n\varepsilon}(x,r) - U_{n\varepsilon}(x,r)\alpha^R_{n+1}(x,r)\right] +$$
$$+\varepsilon\left[\alpha^L_{n+1}(x,r)U_{n\varepsilon}(x,r)\alpha^R_{n+1}(x,r) - \frac{1}{2}\left(\alpha^L_{n+1}(x,r)\right)^2 U_{n\varepsilon}(x,r) + \right.$$
$$\left. - \frac{1}{2}U_{n\varepsilon}(x,r)\left(\alpha^R_{n+1}(x,r)\right)^2\right],$$

JIMWLK in η with collinear improvement

Example: one of the cross-terms

$$\mathrm{tr}\langle(\alpha^R)^\dagger_{n+1}(\mathrm{x},r)U^\dagger_{n\varepsilon}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{y},r)\alpha^R_{n+1}(\mathrm{y},r)\rangle_\xi =$$

$$= \frac{1}{\pi^2}\mathrm{tr}\int_{\mathrm{z},\mathrm{z}'}\alpha_s\theta(n\varepsilon-\delta^r_{r_{yz}})\theta(n\varepsilon-\delta^r_{r_{xz'}})U^\dagger_{n\varepsilon-\delta^r_{r_{xz'}}}(\mathrm{z}',r)t^aK^i_{xz'}\times$$

$$\times U_{n\varepsilon-\delta^r_{r_{xz'}}}(\mathrm{z}',r)U^\dagger_{n\varepsilon}(\mathrm{x},r)\times$$

$$\times U_{n\varepsilon}(\mathrm{y},r)U^\dagger_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)t^bK^j_{yz}U_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)\langle\xi^i_{a,n+1}(\mathrm{z}')\xi^j_{b,n+1}(\mathrm{z})\rangle_\xi =$$

$$= \frac{1}{2\pi^2}N_c^2\int_{\mathrm{z}}\alpha_s\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}S_6(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{yz},\eta)+$$

$$-\frac{1}{2\pi^2}S(\mathrm{x},\mathrm{y},\eta)\int_{\mathrm{z}}\alpha_s\theta(n\varepsilon-\delta^r_{r_{xz}})\theta(n\varepsilon-\delta^r_{r_{yz}})K^i_{xz}K^i_{yz}$$

JIMWLK in η with collinear improvement

All the terms yield

$$\frac{\partial S(\mathrm{x},\mathrm{y},\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int_{\mathrm{z}} S(\mathrm{x},\mathrm{y},\eta)$$

$$\Big\{ -\theta(n\varepsilon - \delta^r_{r_{yz}}) K^i_{yz} K^i_{yz} - \theta(n\varepsilon - \delta^r_{r_{xz}}) K^i_{xz} K^i_{xz} + \theta(n\varepsilon - \delta^r_{r_{xz}}) \theta(n\varepsilon - \delta^r_{r_{yz}}) K^i_{xz} K^i_{yz} \Big\} +$$

$$+ \Big\{ \theta(n\varepsilon - \delta^r_{r_{yz}}) K^i_{yz} K^i_{yz} S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{yz},\delta_{yz},\eta) +$$

$$+ \theta(n\varepsilon - \delta^r_{r_{xz}}) K^i_{xz} K^i_{xz} S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{xz},\eta) +$$

$$- \theta(n\varepsilon - \delta^r_{r_{xz}}) \theta(n\varepsilon - \delta^r_{r_{yz}}) K^i_{xz} K^i_{yz} S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{yz},\delta_{yz},\eta) +$$

$$- \theta(n\varepsilon - \delta^r_{r_{xz}}) \theta(n\varepsilon - \delta^r_{r_{yz}}) K^i_{xz} K^i_{yz} S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{xz},\eta) \Big\} +$$

$$+ \theta(n\varepsilon - \delta^r_{r_{xz}}) \theta(n\varepsilon - \delta^r_{r_{yz}}) K^i_{xz} K^i_{yz} S_6(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{yz},\eta)$$

JIMWLK in η with collinear improvement

Recovering KC BK equation in η

Assuming that $\delta_{xz} = \delta_{yz} = \delta$ we have

$$S_6(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{yz},\eta) =$$

$$= \frac{1}{N_c^2}\mathrm{tr}\left[U_{n\varepsilon-\delta^r_{r_{xz}}}(\mathrm{z},r)U^\dagger_{n\varepsilon}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{y},r)U^\dagger_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)\right]\times$$

$$\times\,\mathrm{tr}\left[U^\dagger_{n\varepsilon-\delta^r_{r_{xz}}}(\mathrm{z},r)U_{n\varepsilon-\delta^r_{r_{yz}}}(\mathrm{z},r)\right] =$$

$$= \frac{1}{N_c}\mathrm{tr}\left[U^\dagger_{n\varepsilon}(\mathrm{x},r)U_{n\varepsilon}(\mathrm{y},r)\right] = S(\mathrm{x},\mathrm{y},\eta)$$

and setting

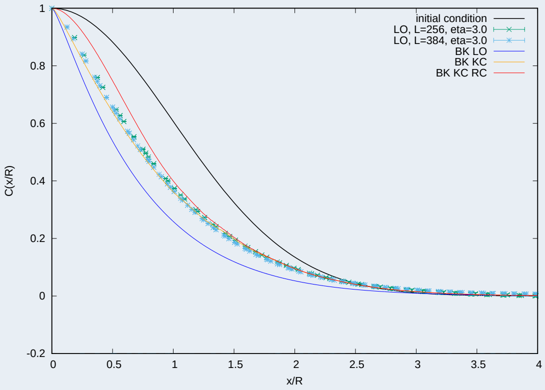
$$S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{xz},\delta_{xz},\eta) = S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta_{yz},\delta_{yz},\eta) \equiv S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta,\eta)$$

in that case the final results reduces to

$$\frac{\partial S(\mathrm{x},\mathrm{y},\eta)}{\partial\eta} = \frac{\bar{\alpha}_s}{2\pi}\int_{\mathrm{z}}\mathscr{K}_{xyz}\theta(n\varepsilon-\delta)\left\{S_2(\mathrm{x},\mathrm{z},\mathrm{z},\mathrm{y},\delta,\eta)-S(\mathrm{x},\mathrm{y},\eta)\right\}$$

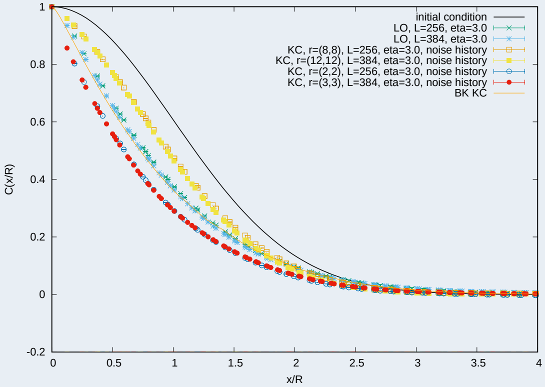
JIMWLK in η with collinear improvement

Preliminary results



JIMWLK in η with collinear improvement

Preliminary results



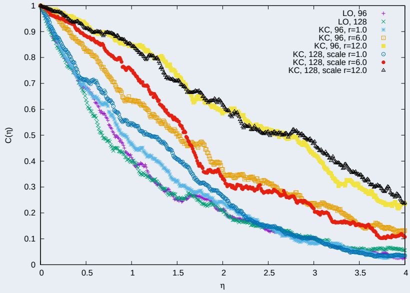
JIMWLK in η with collinear improvement

Recovering KC BK equation in η

In order to diagnoze the dynamics we investigate new correlation functions. The simplest is the correlation in η

$$S(r,\eta) = \frac{1}{VN_c} \langle \mathrm{tr}\ U^{\dagger}(x,r,0)U(x,r,\eta)\rangle_x.$$

Preliminary results



Conclusions

Summary

- JIMWLK equation provides a way to describe DIS data deep in the low-x regime
- numerical implementation and solution possible using the reformulation in terms of Langevin equation
- many systematic effects/ambiguities have to be studied and understood
- collinear resummation for the JIMWLK evolution possible

Outlook

- phenomenological implications/applications will soon be at reach!