QCD evolution equations and splitting functions

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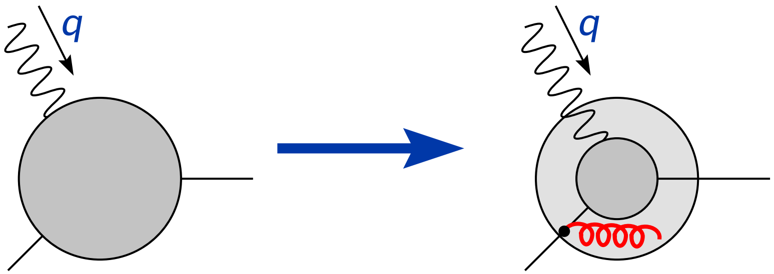
Based on work done in collaboration with:

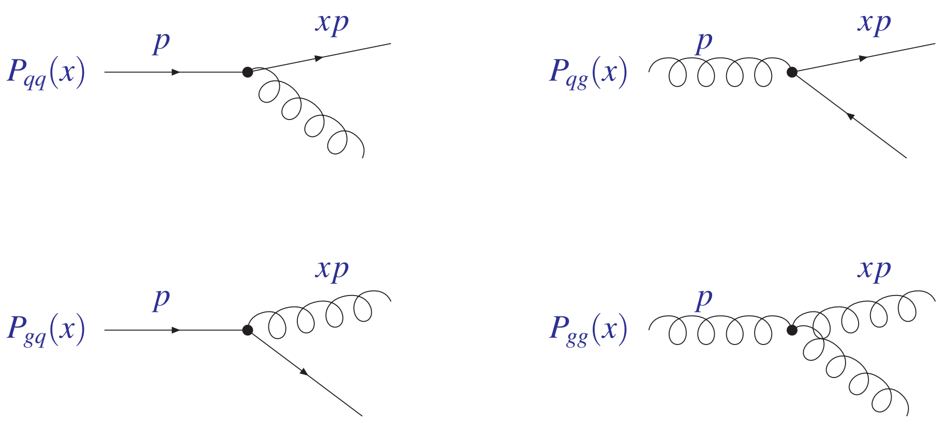
- *Additional moments and x-space approximations of four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:2310.05744`
- *The double fermionic contribution to the four-loop quark-to-gluon splitting function*
  F. Herzog, G. Falcioni, S. M., J. Vermaseren and A. Vogt
  `arXiv:2310.01245`
- *Four-loop splitting functions in QCD – The gluon-to-quark case –*
  F. Herzog, G. Falcioni, S. M., and A. Vogt `arXiv:2307.04158`
- *Four-loop splitting functions in QCD – The quark-quark case –*
  F. Herzog, G. Falcioni, S. M., and A. Vogt `arXiv:2302.07593`
- *Low moments of the four-loop splitting functions in QCD*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:2111.15561`
- *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1805.09638`
- *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*
  S. M., B. Ruijl, T. Ueda, J. Vermaseren and A. Vogt `arXiv:1707.08315`
- Many more papers of MVV and friends . . . `2001 - ...`

QCD evolution at 1% precision

Parton evolution



- Feynman diagrams in leading order



- Proton in resolution $1/Q \longrightarrow$ sensitive to lower momentum partons
- Evolution equations for parton distributions f_i
  - predictions from fits to reference processes (universality)

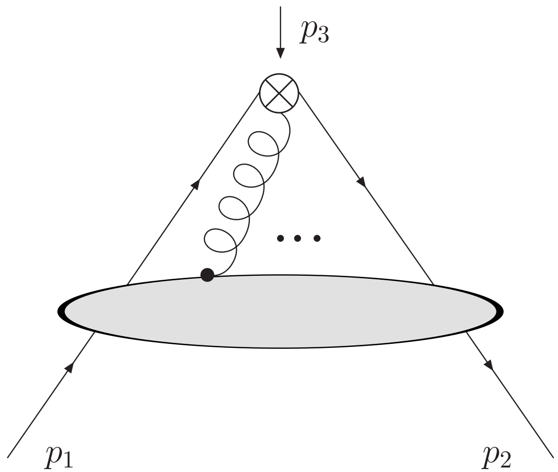
$$\frac{d}{d\ln\mu^2} f_{\rm i}(x,\mu^2) = \sum_{\rm j} \left[P_{\rm ij}(\alpha_s(\mu^2)) \otimes f_{\rm j}(\mu^2) \right](x)$$

- Splitting functions P up to N^{3}LO (work in progress)

$$P_{\rm ij} = \underbrace{\alpha_s P^{(0)}_{\rm ij} + \alpha_s^2 P^{(1)}_{\rm ij} + \alpha_s^3 P^{(2)}_{\rm ij}}_{\text{NNLO: standard approximation}} + \alpha_s^4 P^{(3)}_{\rm ij} + \dots$$

Non-singlet

Operator matrix elements



- Quark operator of spin-N and twist two
  $O_{\rm q}^{\{\mu_1,\ldots,\mu_N\}} = \overline{\psi}\,\gamma^{\{\mu_1} D^{\mu_2} \ldots D^{\mu_N\}}\,\psi$
- N covariant derivatives
  $D_{\mu,ij} = \partial_\mu \delta_{ij} + {\rm i} g_s \, (t_a)_{ij} \, A_\mu^a$
  sandwiched between quark fields ψ, $\overline{\psi}$
- Evaluation of operators in matrix elements $A_{\rm qq}$ with external quark states

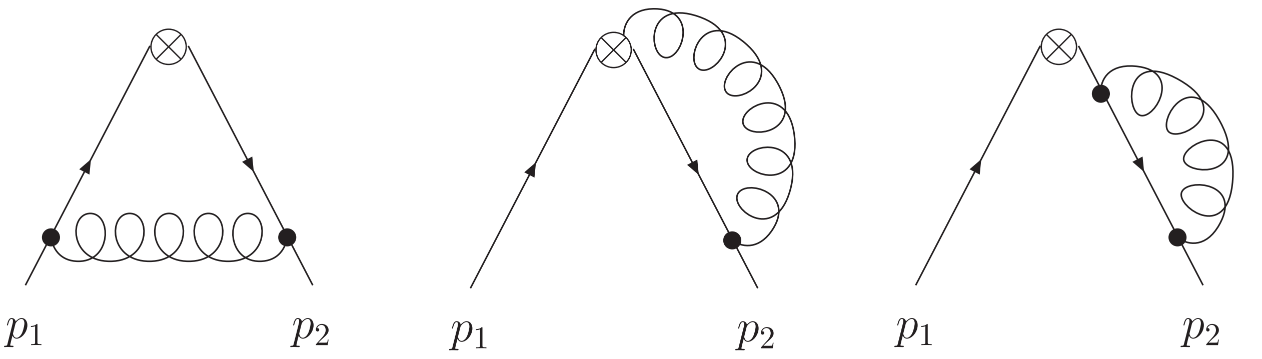
$$A_{\rm qq}^{\{\,\mu_1,\ldots,\,\mu_N\}} = \langle \psi(p_1) | O_{\rm q}^{\{\mu_1,\ldots,\mu_N\}}(p_3) | \overline{\psi}(p_2) \rangle$$

- Anomalous dimensions $\gamma(\alpha_s, N)$ govern scale dependence of renormalized operators

$$\frac{d}{d\ln\mu^2} O^{\rm ren} = -\gamma\, O^{\rm ren} \qquad\qquad \gamma(N) \;=\; -\int_0^1 dx\, x^{N-1} P(x)$$

- Zero-momentum transfer through operator reduces problem to computation of propagator-type diagrams

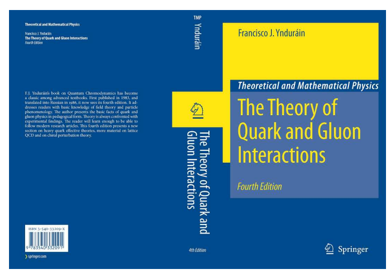
One-loop computation



- Computation of loop integral in $D = 4 - 2\epsilon$ dimensions and expansion in ϵ
  - anomalous dimension $\gamma(N)$ from ultraviolet divergence

$$
\begin{aligned}
\Delta_{\mu_1} \dots \Delta_{\mu_N} \langle \psi(p_1) | O_{\rm q}^{\{\mu_1,\dots,\mu_N\}}(0) | \overline{\psi}(-p_1) \rangle \; &= \\
= \quad 1 + \frac{\alpha_{\rm s}}{4\pi} \frac{1}{\epsilon} \gamma^{(0)}(N) + \mathcal{O}(\alpha_{\rm s}\epsilon^0) + \mathcal{O}(\alpha_{\rm s}^2) & \\
= \quad 1 + \frac{\alpha_{\rm s}}{4\pi} \frac{1}{\epsilon} \left\{ C_F \left(4 S_1(N) + \frac{2}{N+1} - \frac{2}{N} - 3 \right) \right\} + \mathcal{O}(\alpha_{\rm s}\epsilon^0) + \mathcal{O}(\alpha_{\rm s}^2) &
\end{aligned}
$$

- One-loop result with harmonic sum $S_1(N) = \sum_{i=1}^{N} \frac{1}{i}$
- Details in *The Theory of Quark and Gluon Interactions*
  F.J. Yndurain

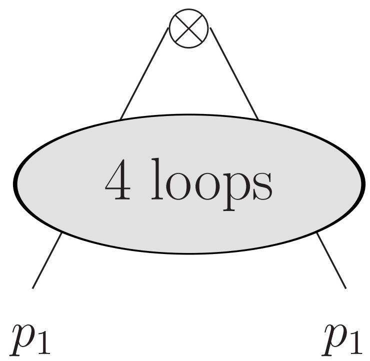


Four-loop computation

- Feynman diagrams for operator matrix elements generated up to four loops with **Qgraf** Nogueira '91
- Parametric reduction of four-loop massless propagator diagrams with integration-by-parts identities encoded in **Forcer** Ruijl, Ueda, Vermaseren '17
- Symbolic manipulations with **Form** Vermaseren '00; Kuipers, Ueda, Vermaseren, Vollinga '12 and multi-threaded version **TForm** Tentyukov, Vermaseren '07
- Diagrams of same topology and color factor combined to meta diagrams
- Non-singlet anomalous dimension
  - 1 one-, 7 two-, 53 three- and 650 four-loop meta diagrams for $\gamma_{\rm ns}^{\pm}$
  - 1 three- and 29 four-loop meta diagrams for $\gamma_{\rm ns}^{\rm s}$

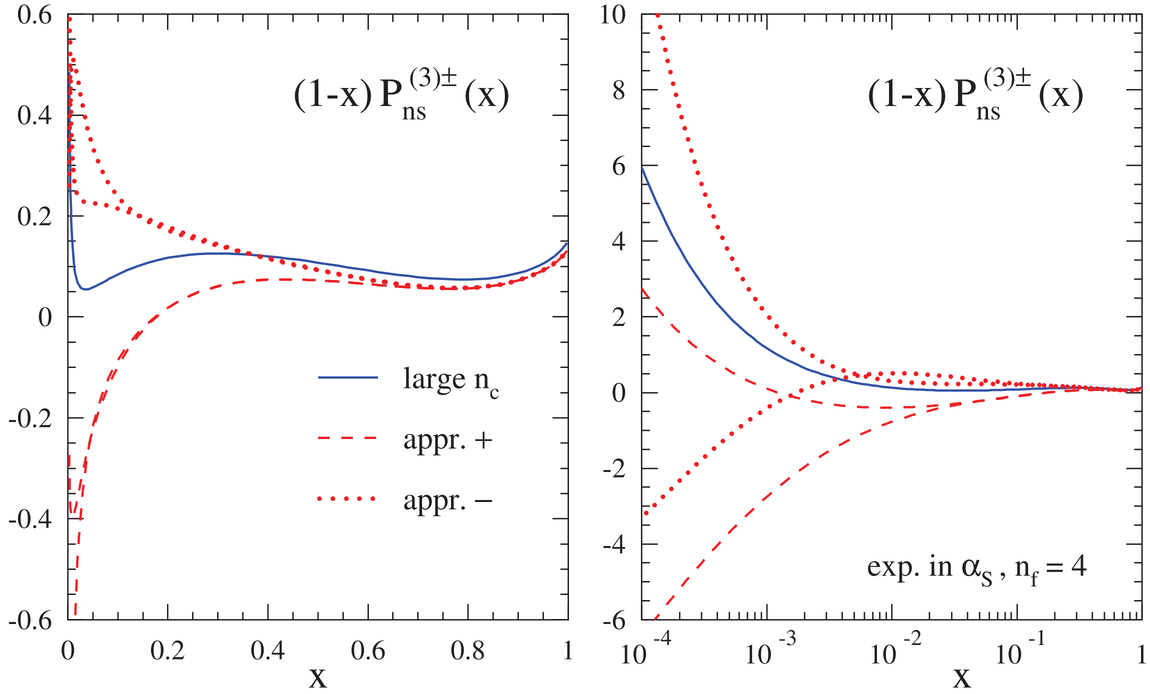
Fixed Mellin moments

- Computation of anomalous dimensions $\gamma(N)$ for Mellin moments mostly up to $N = 18$
  - sometimes higher for complicated topologies ($N = 19$, $N = 20$, ...)
  - much higher for "easy" topologies, e.g., n_f-dependent ($N \simeq 80$, ...)



Four-loop non-singlet splitting functions

- Four-loop $P_{\mathrm{ns}}^{(3)\pm}(x)$ and uncertainty bands beyond large-n_c limit with $n_f = 4$



Analytic results

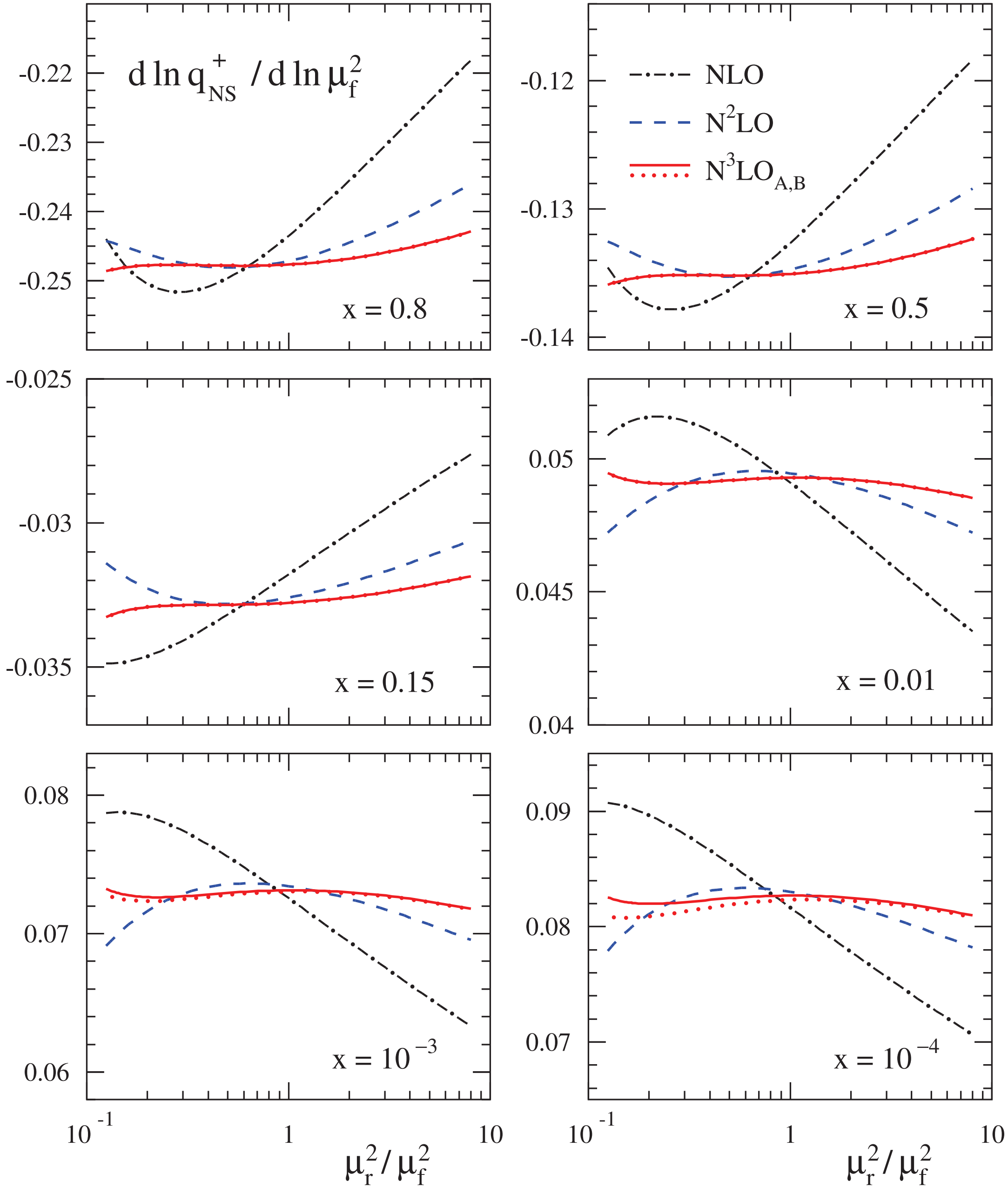
- contributions to non-singlet splitting functions
  - n_f^3 terms Gracey '94
  - n_f^2 terms Davies, Vogt, Ruijl, Ueda, Vermaseren '16;
    Gehrmann, von Manteuffel, Sotnikov, Yang '23
  - leading n_c terms S.M., Vogt, Ruijl, Ueda, Vermaseren '17
  - $n_f C_F^3$ terms Gehrmann, von Manteuffel, Sotnikov, Yang '23

Scale stability of evolution

- Renormalization scale dependence of evolution kernel $d\ln q_{\rm ns}^{+}/d\ln \mu_f^2$
  - non-singlet shape

$$xq_{\rm ns}^{+}(x,\mu_0^2) \;=\; x^{\,0.5}(1-x)^3$$

- NLO, NNLO and N^{3}LO predictions
  - remaining uncertainty of four-loop splitting function $P_{\rm ns}^{(3)+}$ almost invisible

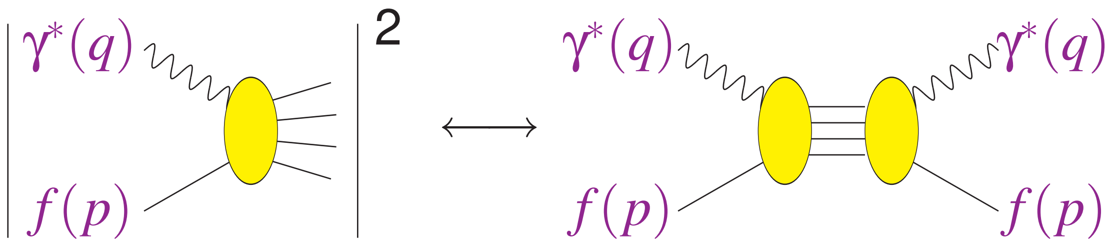


Singlet

Operator product expansion (I)

Optical theorem

- Total cross section related to imaginary part of Compton amplitude
  - momentum transfer $Q^2 = -q^2$
  - Bjorken variable $x = Q^2/(2p \cdot q)$



- Optical theorem relates hadronic tensor $W_{\mu\nu}$ to imaginary part of Compton amplitude $T_{\mu\nu}$

$$W_{\mu\nu} = e_{\mu\nu} \frac{1}{2x} F_L(x, Q^2) + d_{\mu\nu} \frac{1}{2x} F_2(x, Q^2) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} F_3(x, Q^2)$$

- OPE of $T_{\mu\nu}$ for short distances $z^2 \simeq 0$ in Bjorken limit $Q^2 \to \infty$, x fixed
  Wilson '72; Christ, Hasslacher, Mueller '72

$$T_{\mu\nu} \;=\; \mathrm{i} \int d^4 z \, \mathrm{e}^{\mathrm{i}q \cdot z} \langle \mathrm{P} | \, T \left(j_\mu^\dagger(z) j_\nu(0) \right) | \mathrm{P} \rangle$$

Operator product expansion (II)

- Operator product expansion with coefficient functions in Mellin space $C^N_{a,i}$

$$T_{\mu\nu} = \sum_{N,j} \left(\frac{1}{2x}\right)^N \left[e_{\mu\nu}\, C^N_{L,j}\left(\frac{Q^2}{\mu^2}, \alpha_s\right) + d_{\mu\nu}\, C^N_{2,j}\left(\frac{Q^2}{\mu^2}, \alpha_s\right) + \mathrm{i}\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q} C^N_{3,j}\left(\frac{Q^2}{\mu^2}, \alpha_s\right)\right] A^j_{\mathrm{P},N}\left(\mu^2\right) + \text{ higher twists}$$

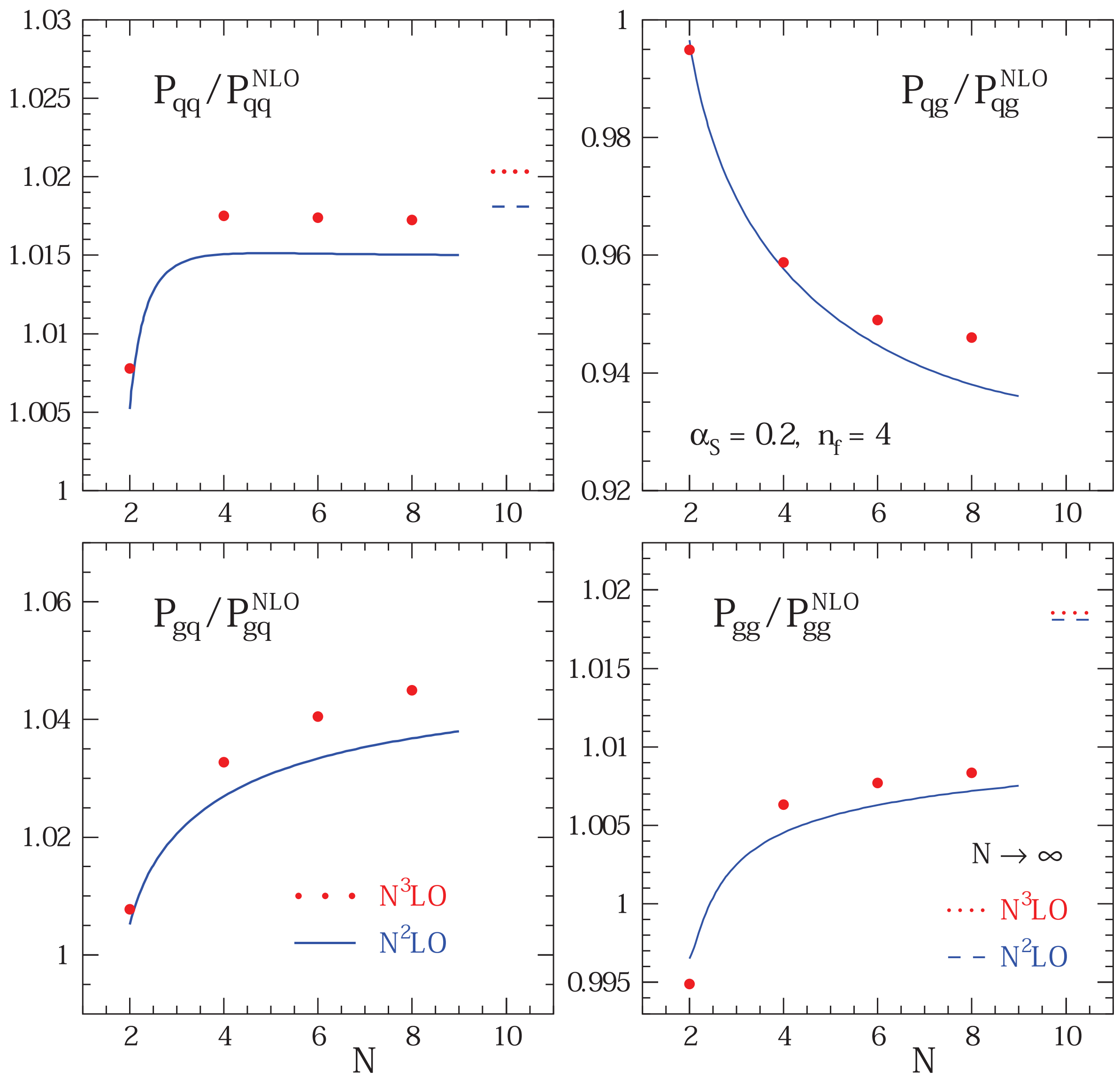
- Operator matrix elements $A^i_{\mathrm{P},N} = \langle P|O^N_{\mathrm{i}}|P\rangle$ in nucleon state
  - leading twist physical partonic operators, e.g., quark operator $O_{\mathrm{q}}^{\{\mu_1,\dots,\mu_N\}} = \overline{\psi}\,\gamma^{\{\mu_1} D^{\mu_2} \dots D^{\mu_N\}}\,\psi$
- Compton amplitude $T_{\mu\nu}$ for parton states yields coefficient functions and anomalous dimensions $\gamma(\alpha_s, N)$ after mass factorization
  - established computational approach through four loops one loop Buras '80; two loops Kazakov, Kotikov '90; S.M., Vermaseren '99 three loops S.M., Vermaseren, Vogt '04; four loops Davies, Vogt, Ruijl, Ueda, Vermaseren '17; S.M., Ruijl, Ueda, Vermaseren, Vogt to appear
  - photon-DIS $\longrightarrow \gamma_{\mathrm{qq}}, \gamma_{\mathrm{qg}}$, Higgs (scalar)-DIS $\longrightarrow \gamma_{\mathrm{gq}}, \gamma_{\mathrm{gg}}$
  - graviton-DIS $\longrightarrow \Delta\gamma_{\mathrm{ij}}$ (polarized quantities) S.M., Vermaseren, Vogt '14

Four-loop singlet Mellin moments

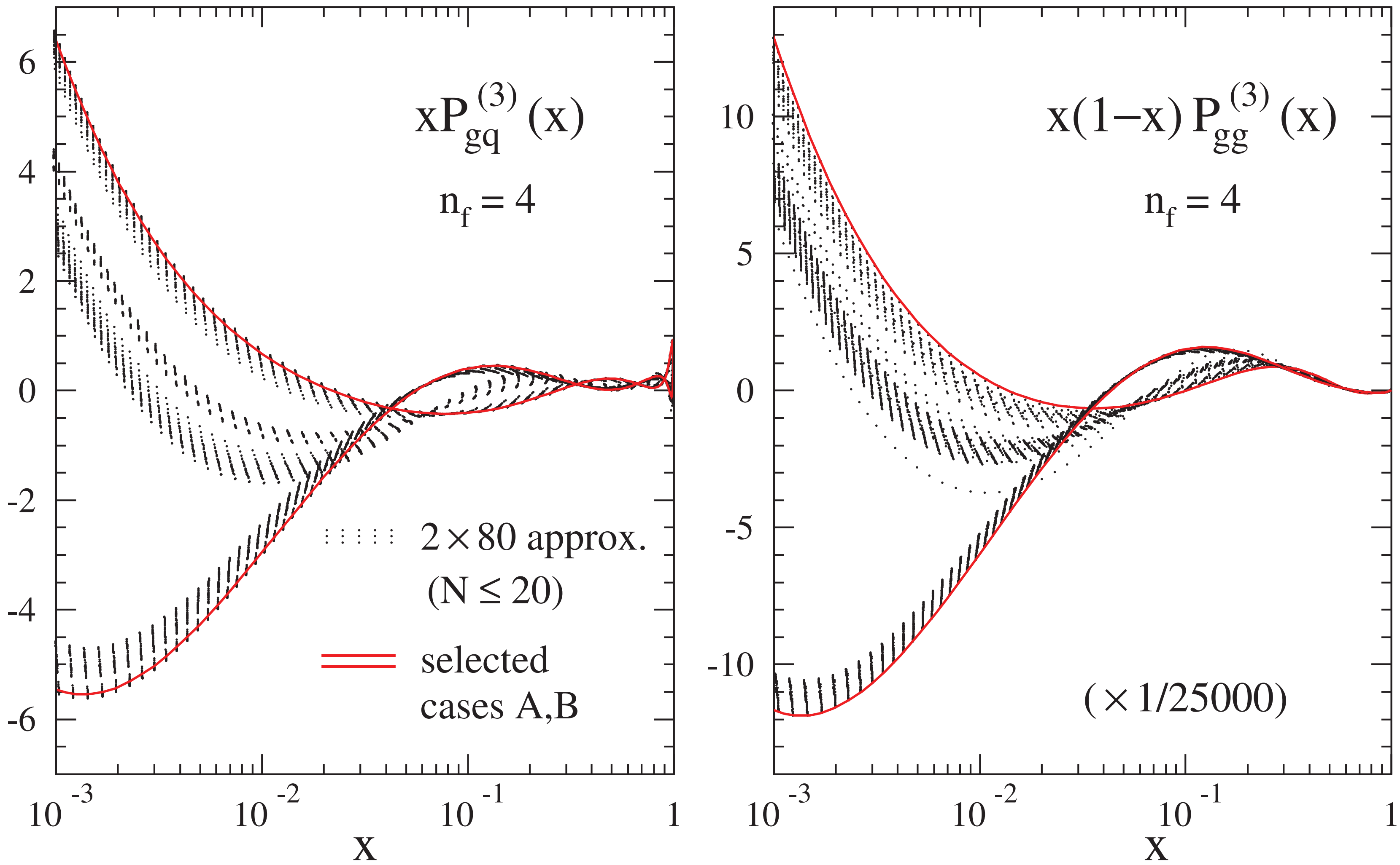
- Singlet moments at NNLO (lines) and N^3LO (even-N points) normalized to NLO results
  S. M., Ruijl, Ueda, Vermaseren, Vogt '21
  - $\alpha_s(\mu_f) = 0.2$ and $n_f = 4$

- Large-N limits in qq- and gg-channel

- Moments up to $N = 10$
  S. M., Ruijl, Ueda, Vermaseren, Vogt '23



Lower row splitting functions

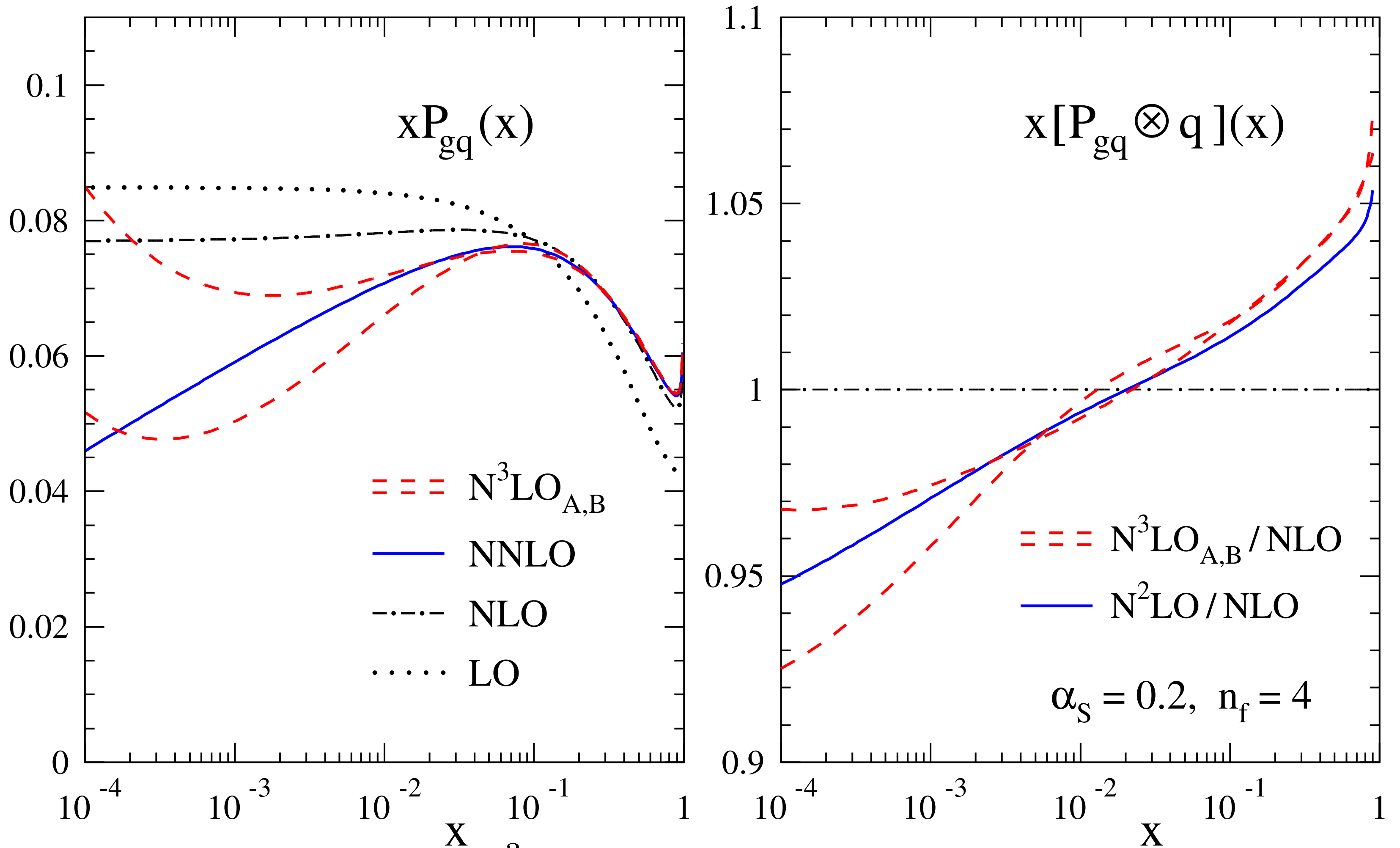


- N^3LO approximations for gluon-quark and gluon-gluon splitting functions

S. M., Ruijl, Ueda, Vermaseren, Vogt '23

  - $P_{gq}^{(3)}(x)$ (left) and $P_{gg}^{(3)}(x)$ (right)

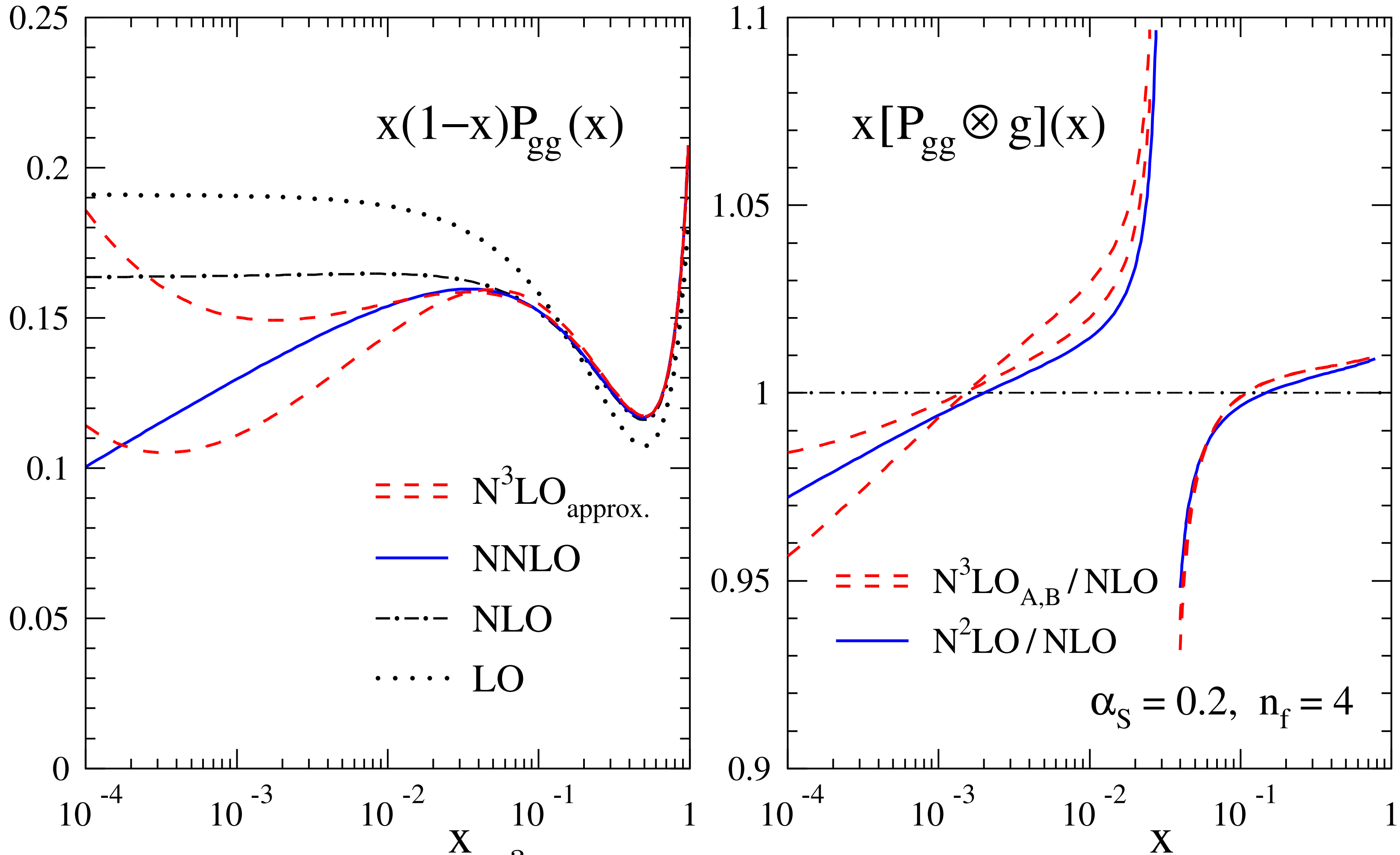
Scale stability of evolution (I)



- Left: NLO, NNLO and N^{3}LO approximations for $P_{gq}(x)$ with $\alpha_s = 0.2$ fixed and $n_f = 4$
- Right: Contribution to evolution kernel $d \ln q_s / d \ln \mu_f^2$ at NLO, NNLO and N^{3}LO for typical gluon shape

$$xg(x, \mu_0^2) = 1.6\, x^{-0.3} (1-x)^{4.5} \left(1 - 0.6\, x^{0.3}\right)$$

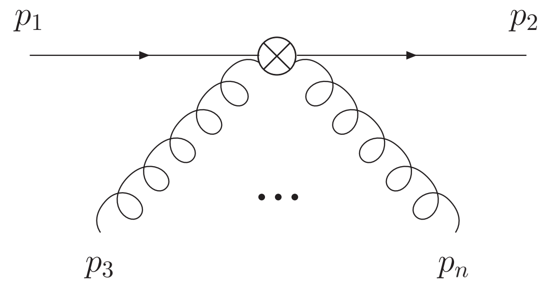
Scale stability of evolution (II)

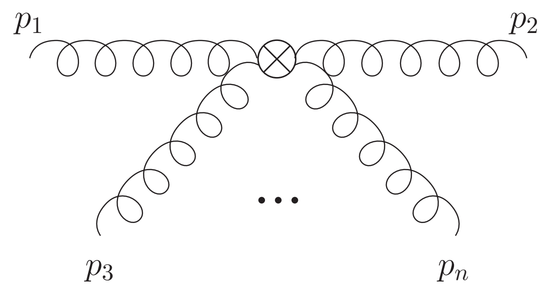


- Left: NLO, NNLO and N^{3}LO approximations for $P_{gg}(x)$ with $\alpha_s = 0.2$ fixed and $n_f = 4$
- Right: Contribution to evolution kernel $d \ln q_s / d \ln \mu_f^2$ at NLO, NNLO and N^{3}LO for typical gluon shape

$$xg(x, \mu_0^2) = 1.6\, x^{-0.3}(1-x)^{4.5}\, (1 - 0.6\, x^{0.3})$$

Operator matrix elements





- Scalar singlet operators of spin-N and twist two from contraction with light-like vector Δ_μ

$$O_{\rm q} = \overline{\psi}\, \not{\Delta}\, D^{N-2}\, \psi$$

$$O_{\rm g} = F_\nu{}^a\, D_{ab}^{N-2}\, F^{\nu;b}$$

  - notation

$$F^{\mu;a} = \Delta_\nu\, F^{\mu\nu;a}\,, \quad A^a = \Delta_\mu\, A^{\mu;a}\,,$$

$$D = \Delta_\mu\, D^\mu\,, \quad \partial = \Delta_\mu\, \partial^\mu$$

- Physical operators mix under renormalization with other operators (equation-of-motion operators and ghost operators)
  - classic work on general theory of renormalization

Dixon, Taylor '74; Kluberg-Stern, Zuber '75; Joglekar, Lee '76

- Applications in QCD
  - at two loops Floratos, Ross, Sachrajda '79 and with correct renormalization of gluon operator in covariant gauge Hamberg, van Neerven '92; Matiounine, Smith, van Neerven '98; Blümlein, Marquard, Schneider, Schönwald '22
  - at three loops Gehrmann, von Manteuffel, Yang '23
  - general procedure (based on BRST invariance) Falcioni, Herzog '22

Alien operators

- Sets of alien operators $O_A^i = O_{\rm q}^i + O_{\rm g}^i + O_{\rm c}^i$ with $i = I, II, \ldots$

$$
\begin{aligned}
O_{\rm q}^{I} &= \eta\, g\, \overline{\psi}\, \not{\Delta}\, t^a\, \psi\, \left(\partial^{N-2} A_a\right) , \\
O_{\rm g}^{I} &= \eta\, (D.F)^a \left(\partial^{N-2} A_a\right) , \\
O_{\rm c}^{I} &= -\eta\, (\partial\, \overline{c}^{\,a}) \left(\partial^{N-1} c_a\right) , \\
O_q^{II} &= g^2\, \overline{\psi}\, \not{\Delta}\, t_a\, \psi \sum_{i+j=N-3} \kappa_{ij}\, f^{\,abc} \left(\partial^{\,i} A_b\right) \left(\partial^{\,j} A_c\right) , \\
O_g^{II} &= g\, (D.F)_a \sum_{i+j=N-3} \kappa_{ij}\, f^{\,abc} \left(\partial^{\,i} A_b\right) \left(\partial^{\,j} A_c\right) , \\
O_c^{II} &= -g \sum_{i+j=N-3} \eta_{ij}\, f^{\,abc}\, (\partial\, \overline{c}_a) \left(\partial^{\,i} A_b\right) \left(\partial^{\,j+1} c\right)
\end{aligned}
$$

- Class O_A^I alien operators with coupling η (function of N and α_s)
- Class O_A^{II} alien operators with couplings η_{ij} and κ_{ij}
  - constraints from (anti-)BRST symmetry Falcioni, Herzog '22
- Class O_A^{III}, O_A^{IV}, etc alien operators with additional polynomials $(\partial^{\,i} A_a)$

Renormalization

- Physical operators $O_{\rm q}$ and $O_{\rm g}$
- Alien operators $O_A^i = O_{\rm q}^i + O_{\rm g}^i + O_{\rm c}^i$
  - gluon and quark equation-of-motion operators and ghost operators

$$\begin{pmatrix} O_{\rm q} \\ O_{\rm g} \\ O_{\rm A} \end{pmatrix} = \begin{pmatrix} Z_{\rm qq} & Z_{\rm qg} & Z_{\rm qA} \\ Z_{\rm gq} & Z_{\rm gg} & Z_{\rm gA} \\ Z_{\rm Aq} & Z_{\rm Ag} & Z_{\rm AA} \end{pmatrix} \begin{pmatrix} O_{\rm q}^{\rm ren} \\ O_{\rm g}^{\rm ren} \\ O_{\rm A}^{\rm ren} \end{pmatrix}$$

- Multiplicative renormalization of operators (Z-factors)

$$\mu^2 \frac{d}{d\mu^2} Z_{\rm ij} = \left(\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} + \gamma_3 \, \xi \frac{\partial}{\partial \xi} \right) Z_{\rm ij} = -\gamma_{\rm ik} Z_{\rm kj}$$

  - gauge parameter ξ with $\xi = 1$ for Feynman gauge
  - QCD β-function and gluon anomalous dimension γ_3 (known to five loops) Baikov, Chetyrkin, Kühn '17; Herzog, Ruijl, Ueda, Vermaseren, Vogt '17; Luthe, Maier, Marquard, Schröder '17; Chetyrkin, Falcioni, Herzog, Vermaseren '17
  - $Z_{\rm ij}$ involving alien operators can be gauge dependent
  - $Z_{\rm Aq}$ and $Z_{\rm Ag}$ have to vanish (alien operators cannot mix into set of physical operators)

Moments of pure-singlet splitting function

- Moments $N = 2, \ldots 20$ for pure-singlet anomalous dimension $\gamma_{\rm ps}^{(3)}(N)$

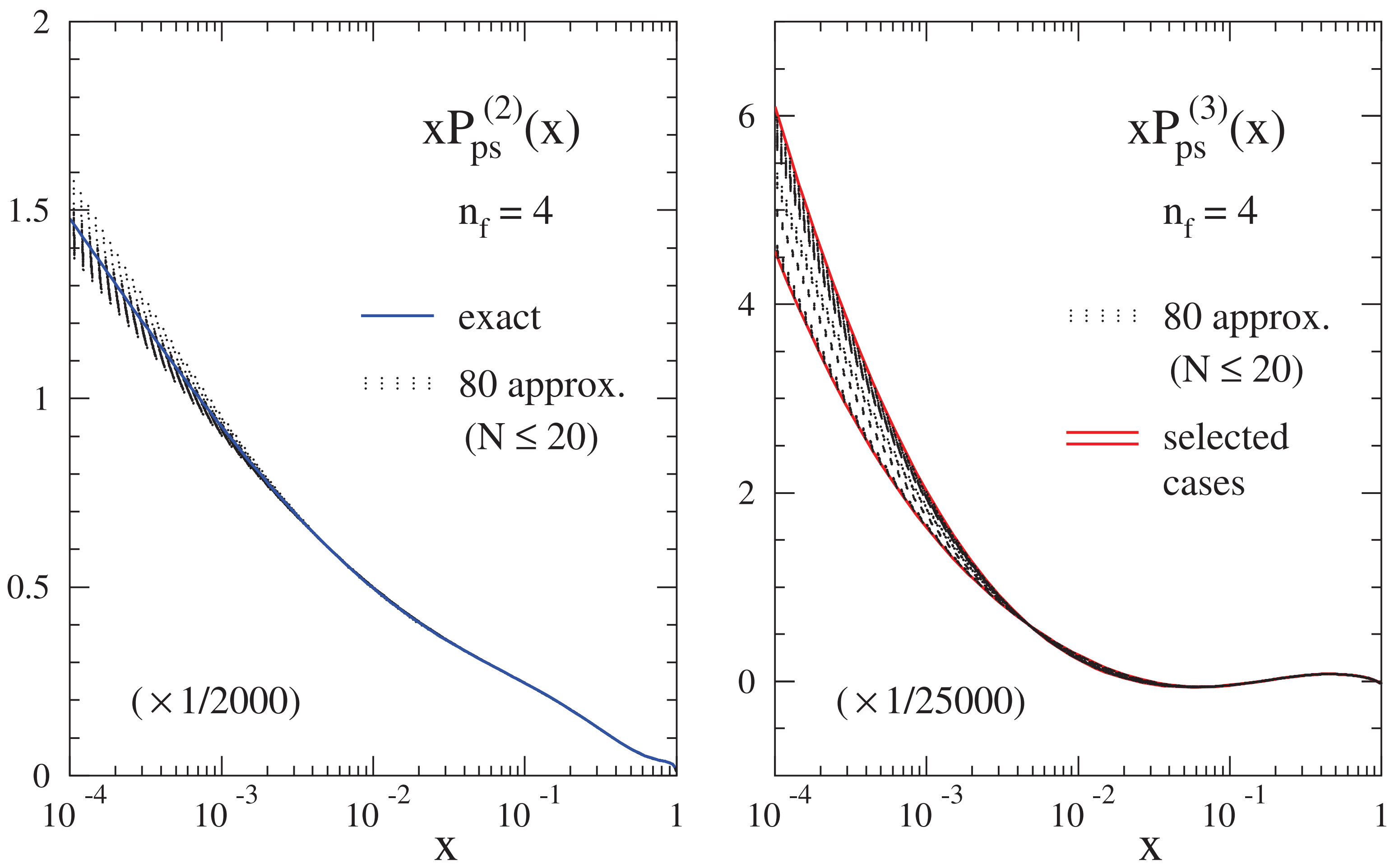
$$
\begin{aligned}
\gamma_{\rm ps}^{(3)}(N\!=\!2) &= -691.5937093\, n_f + 84.77398149\, n_f^2 + 4.466956849\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N\!=\!4) &= -109.3302335\, n_f + 8.776885259\, n_f^2 + 0.306077137\, n_f^3 \,, \\
\gamma_{\rm ps}^{(3)}(N\!=\!6) &= -46.03061374\, n_f + 4.744075766\, n_f^2 + 0.042548957\, n_f^3 \,, \\
&\ldots \\
\gamma_{\rm ps}^{(3)}(N\!=\!20) &= -0.442681568\, n_f + 0.805745333\, n_f^2 - 0.020918264\, n_f^3 \,.
\end{aligned}
$$

- Results $N \leq 8$ agree with inclusive DIS S.M., Ruijl, Ueda, Vermaseren, Vogt '21 (also for $N = 10$ and $N = 12$)
- Quartic color terms $d_R^{abcd} d_R^{abcd}$ agree with S.M., Ruijl, Ueda, Vermaseren, Vogt '18
- Large-n_f parts agree with all-N results Davies, Vogt, Ruijl, Ueda, Vermaseren '17;
- ζ_4 terms in $\gamma_{\rm ps}^{(3)}(N)$ agree with Davies, Vogt '17 based on no-π^2 theorem Jamin, Miravitllas '18; Baikov, Chetyrkin '18
- Renormalization constants involving alien operators (required to three loops) agree with Gehrmann, von Manteuffel, Yang '23
- Checked by n_f^2 terms at all-N Gehrmann, von Manteuffel, Sotnikov, Yang '23

Approximations in x-space

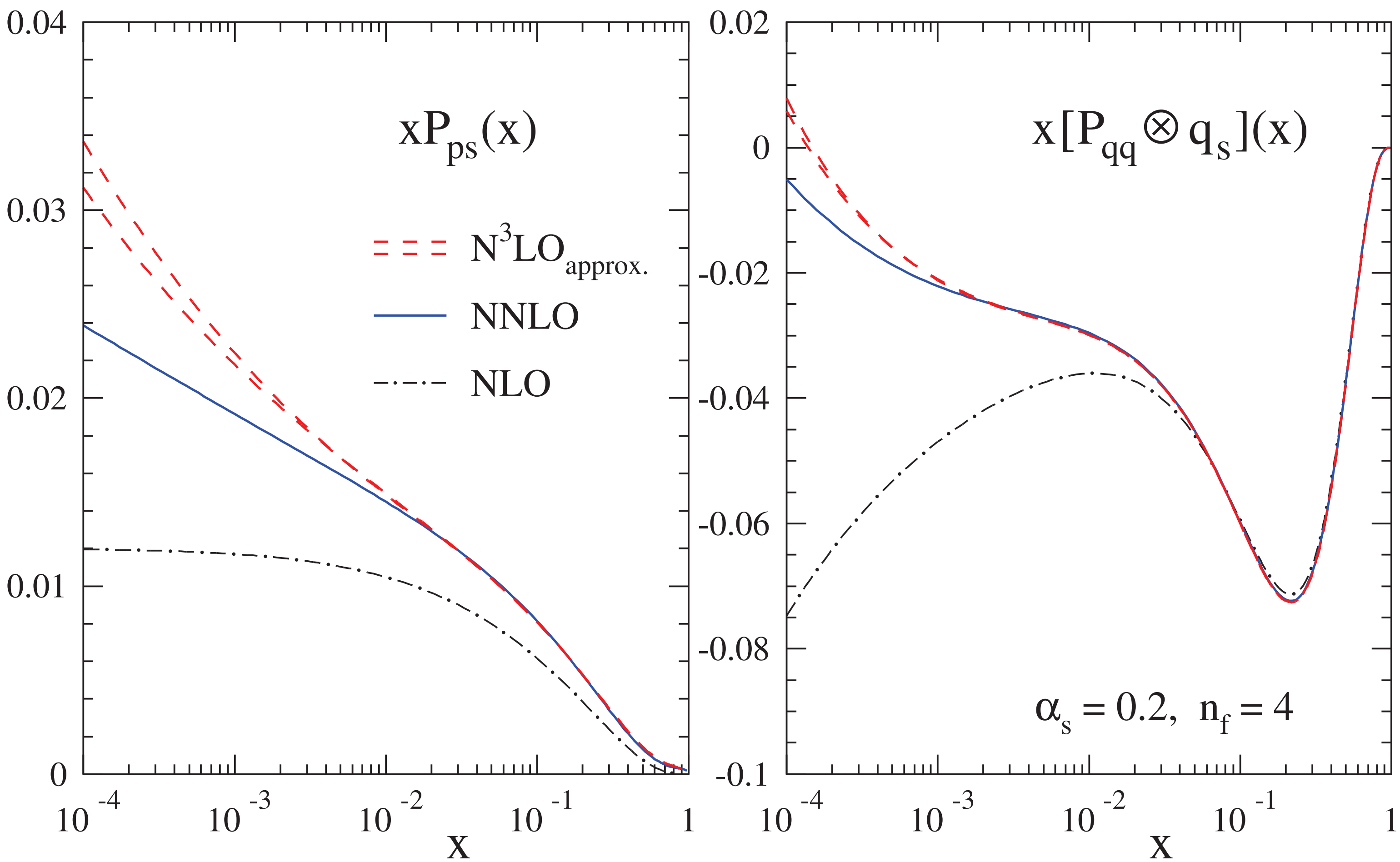
- Large- and small-x information about four-loop splitting function $P_{\rm ps}^{(3)}(x)$
  - leading logarithm $(\ln^2 x)/x$ Catani, Hautmann '94
  - sub-dominant logarithms $\ln^k x$ with $k = 6, 5, 4$ Davies, Kom, S.M., Vogt '22
  - leading large-x terms $(1-x)^j \ln^k(1-x)$ with $j \geq 1$ and $k \leq 4$ with $k = 4, 3$ known Soar, S.M., Vermaseren, Vogt '09
- Approximation of four-loop splitting function $P_{\rm ps}^{(3)}(x)$ with suitable ansatz
  - unknown leading small-x terms: $(\ln x)/x$, $1/x$
  - unknown sub-dominant logarithms: $\ln^k x$ with $k = 3, 2, 1$
  - two remaining large-x terms $(1-x)\ln^k(1-x)$ with $k = 2, 1$
  - different two-parameter polynomials together one function (dilogarithm $\mathrm{Li}_2(x)$ or $\ln^k(1-x)$ with $k = 2, 1$, suppressed as $x \to 1$)

Pure-singlet splitting function



- Approximations to pure-singlet splitting function $P_{\rm ps}^{(n)}(x)$ at $n_f = 4$ with 80 trial functions
  - left: three-loops ($n = 2$) with comparison to known result
  - right: three-loops ($n = 3$) with remaining uncertainty

Pure-singlet splitting function



- Left: NLO, NNLO and N^3LO approximations for $P_{\rm ps}(x)$ with $\alpha_s = 0.2$ fixed and $n_f = 4$
- Right: Contribution to evolution kernel $d \ln q_{\rm s} / d \ln \mu_f^2$ at NLO, NNLO and N^3LO for typical quark-singlet shape

$$xq_{\rm s}(x, \mu_0^2) = 0.6\, x^{-0.3} (1-x)^{3.5} \left(1 + 5.0\, x^{0.8}\right)$$

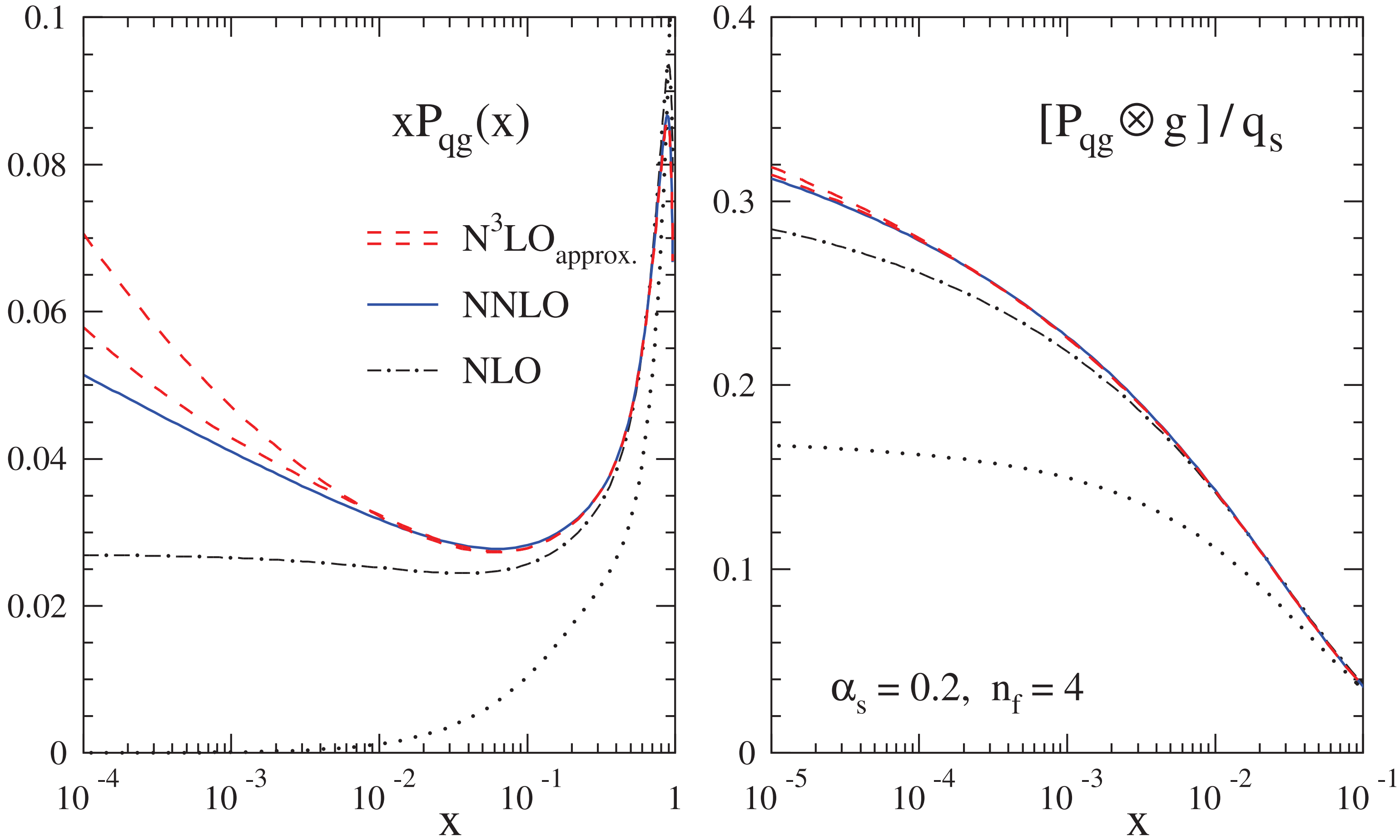
Moments of quark-gluon splitting function

- Moments $N = 2, \ldots 20$ for quark-gluon anomalous dimension $\gamma_{\rm qg}^{(3)}(N)$

$$
\begin{aligned}
\gamma_{\rm qg}^{(3)}(N\!=\!2) &= -654.4627782\, n_f + 245.6106197\, n_f^2 - 0.924990969\, n_f^3 \,, \\
\gamma_{\rm qg}^{(3)}(N\!=\!4) &= 290.3110686\, n_f - 76.51672403\, n_f^2 - 4.911625629\, n_f^3 \,, \\
\gamma_{\rm qg}^{(3)}(N\!=\!6) &= 335.8008046\, n_f - 124.5710225\, n_f^2 - 4.193871425\, n_f^3 \,, \\
\gamma_{\rm qg}^{(3)}(N\!=\!8) &= 294.5876830\, n_f - 135.3767647\, n_f^2 - 3.609775642\, n_f^3 \,, \\
&\ldots \\
\gamma_{\rm qg}^{(3)}(N\!=\!20) &= 52.24329555\, n_f - 109.3424891\, n_f^2 - 2.153153725\, n_f^3 \,.
\end{aligned}
$$

- Approximation of four-loop splitting function $P_{\rm qg}^{(3)}(x)$ again with known large- and small-x information and suitable ansatz

Quark-gluon splitting function



- Left: NLO, NNLO and N^{3}LO approximations for $P_{qg}(x)$ with $\alpha_s = 0.2$ fixed and $n_f = 4$
- Right: Contribution to evolution kernel $d\ln q_s / d\ln\mu_f^2$ at NLO, NNLO and N^{3}LO for typical gluon shape

$$xg(x,\mu_0^2) \; = \; 1.6\,x^{-0.3}(1-x)^{4.5}\;\left(1-0.6\,x^{0.3}\right)$$

Summary

- Experimental precision of $\lesssim 1\%$ motivates computations at higher order in perturbative QCD
  - theoretical predictions at NNLO in QCD nowadays standard
- Push for theory results at N^{3}LO and N^{4}LO
  - evolutions equations expected to achieve percent-level
  - massive use of computer algebra
- Four-loop splitting functions approximated from moments $N = 2, \ldots 20$
  - residual uncertainties negligible in wide kinematic range of x probed at current and future colliders
  - $P_{\rm qq} = P_{\rm ns}^{+} + P_{\rm ps}$ and $P_{\rm qg}$ done
  - $P_{\rm gq}$ and $P_{\rm gg}$ first glimpse, more precise results to come
  - n_f^2 terms for $P_{\rm gq}$ already done at all N